Global Weak Solutions for Compressible Navier-Stokes-Vlasov-Fokker-Planck System

Hai-Liang Li and Ling-Yun Shou*

*School of Mathematical Sciences, Academy for Multidisciplinary Studies,
Capital Normal University, Beijing 100048, P.R. China.*

Dedicated to Professor C.-J. Xu on the occasion of his 65th birthday

Abstract. The one-dimensional compressible Navier-Stokes-Vlasov-Fokker-Planck system with density-dependent viscosity and drag force coefficients is investigated in the present paper. The existence, uniqueness, and regularity of global weak solution to the initial value problem for general initial data are established in spatial periodic domain. Moreover, the long time behavior of the weak solution is analyzed. It is shown that as the time grows, the distribution function of the particles converges to the global Maxwellian, and both the fluid velocity and the macroscopic velocity of the particles converge to the same speed.

1 Introduction

Fluid-particle models have a wide range applications such as biosprays in medicine, chemical engineering, compressibility of droplets, fuel-droplets in combustion theory, pollution settling processes, and polymers to simulate the motion of particles dispersed in dense fluids [1, 3, 8, 24, 32, 37, 39].

*Corresponding author. *Email addresses:* `shoulingyun11@gmail.com` (L.-Y. Shou), `hailiang.li.math@gmail.com` (H.-L. Li)

In this paper, we consider the initial value problem (IVP) for the one-dimensional compressible Navier-Stokes-Vlasov-Fokker-Planck (NS-VFP) system

$$\begin{cases} \rho_t + (\rho u)_x = 0, & \text{(1.1a)} \\ (\rho u)_t + (\rho u^2)_x + (P(\rho))_x = (\mu(\rho)u_x)_x - \int_{\mathbb{R}} \kappa(\rho)(u-v)f\,dv, & \text{(1.1b)} \\ f_t + vf_x + (\kappa(\rho)(u-v)f - \kappa(\rho)f_v)_v = 0, \quad (x,v)\in\mathbb{T}\times\mathbb{R}, \ t>0, \ \mathbb{T}:=\mathbb{R}/\mathbb{Z} & \text{(1.1c)} \end{cases}$$

with the initial data

$$(\rho(x,0),\rho u(x,0),f(x,v,0)) = (\rho_0(x),m_0(x),f_0(x,v)), \quad (x,v)\in\mathbb{T}\times\mathbb{R}, \tag{1.2}$$

where $\rho=\rho(x,t)$ and $u=u(x,t)$ are the fluid density and velocity associated with the dense phase (fluid) respectively, and $f=f(x,v,t)$ denotes the distribution function associated with the dispersed phase (particles). The system (1.1) can be viewed as the compressible Navier-Stokes equations (1.1a)-(1.1b) for the fluid and the Vlasov-Fokker-Planck equation (1.1c) for the particles coupled each other through the drag force term $\kappa(\rho)(u-v)$. The pressure $P(\rho)$ and the viscosity coefficient $\mu(\rho)$ are given by

$$P(\rho) = A\rho^\gamma, \quad \mu(\rho) = \mu_0 + \mu_1\rho^\beta, \tag{1.3}$$

and the drag force coefficient $\kappa(\rho)$ is chosen to be

$$\kappa(\rho) = \kappa_0\rho, \tag{1.4}$$

where the constants $A,\mu_0,\mu_1,\kappa_0,\gamma$, and β satisfy

$$A>0, \quad \mu_0>0, \quad \mu_1>0, \quad \kappa_0>0, \quad \gamma>1, \quad \beta\geq0.$$

Without loss of generality, we take $A=\mu_0=\mu_1=\kappa_0=1$ in the present paper.

There are a lot of important progress on the analysis of the global existence and dynamical behaviors of solutions for fluid-particle systems [5,10,11,13,14,17–20,22,25,27–31,36]. Among them, for incompressible NS-VFP equations, He [20] and Goudon *et al.* [17] proved the global regularity and exponential decay rate of classical solutions in spatial periodic domain, and Chae *et al.* [10] showed the global existence of weak solutions in spatial whole space. For compressible NS-VFP system with constant drag force coefficient, the global existence of weak solutions to three-dimensional initial boundary value problem with the adiabatic constant $\gamma > \frac{3}{2}$ was obtained by Mellet and Vasseur [36], the global well-posedness of strong solutions to Cauchy problem was established either for two-dimensional large initial data in [22] and for three-dimensional small initial data

in [11, 25] respectively, and the nonlinear time-asymptotical stability of planar rarefaction wave was investigated by Li *et al.* [30]. As for compressible NS-VFP system with density-dependent drag force coefficient, Li *et al.* [31] got the global existence and time-decay estimates of strong solutions around the equilibrium state in both spatial periodic domain and spatial whole space, and Li *et al.* [29] analyzed the Green function and pointwise behaviors of strong solutions to three-dimensional Cauchy problem. In addition, the compressible NS-VFP equations can be approximated by some macroscopical two-phase models as the fluid-dynamical limits, the readers can refer to [9,15,30,35] and references therein.

In the present paper, we study the existence, uniqueness, regularity, and large time behavior of global weak solution to the IVP (1.1)-(1.4) for general initial data, as a continuation of the previous works [27,28].

First, we give the definition of global weak solutions to the IVP (1.1)-(1.4) as follows:

Definition 1.1. (ρ, u, f) *is said to be a global weak solution to the IVP* (1.1)-(1.4) *provided for any* $T > 0$ *that*

$$\begin{cases} \rho \in L^\infty\big(0,T;L^\gamma(\mathbb{T})\big), \\ \sqrt{\rho}u \in L^\infty\big(0,T;L^2(\mathbb{T})\big), \quad \sqrt{\mu(\rho)}u_x \in L^2\big(0,T;L^2(\mathbb{T})\big), \\ f\log f \in L^\infty\big(0,T;L^1(\mathbb{T}\times\mathbb{R})\big), \quad |v|^2 f \in L^\infty\big(0,T;L^1(\mathbb{T}\times\mathbb{R})\big), \end{cases} \quad (1.5)$$

the Eqs. (1.1a)-(1.1c) *are satisfied in the sense of distributions, and the following entropy inequality holds for a.e.* $t \in [0,T]$:

$$\int_{\mathbb{T}} \left(\frac{1}{2}\rho|u|^2 + \frac{\rho^\gamma}{\gamma-1}\right)(x,t)dx + \int_{\mathbb{T}\times\mathbb{R}} \left(\frac{1}{2}f|v|^2 + f\log f\right)(x,v,t)dvdx$$

$$+ \int_0^t \int_{\mathbb{T}} \left(\mu(\rho)|u_x|^2\right)(x,\tau)dxd\tau$$

$$\leq \int_{\mathbb{T}} \left(\frac{1}{2}\frac{|m_0|^2}{\rho_0} + \frac{\rho_0^\gamma}{\gamma-1}\right)(x)dx + \int_{\mathbb{T}\times\mathbb{R}} \left(\frac{1}{2}f_0|v|^2 + f_0\log f_0\right)(x,v)dvdx. \quad (1.6)$$

Denote by n and nw the macroscopical density and momentum related to the moments of the solution f to (1.1c) as

$$n(x,t) := \int_{\mathbb{R}} f(x,v,t)dv, \quad nw(x,t) := \int_{\mathbb{R}} vf(x,v,t)dv \quad (1.7)$$

with the initial values

$$n(x,0) = n_0(x) := \int_{\mathbb{R}} f_0(x,v)dv, \quad nw(x,0) = j_0(x) := \int_{\mathbb{R}} vf_0(x,v)dv.$$

For $p \in [1, \infty]$, $k, l \geq 0$, and $\langle v \rangle := (1 + |v|^2)^{\frac{1}{2}}$, define

$$\begin{cases} L_l^p(\mathbb{T} \times \mathbb{R}) := \{ \langle v \rangle^l g \in L^p(\mathbb{T} \times \mathbb{R}) \mid \|g\|_{L_l^p(\mathbb{T} \times \mathbb{R})} = \|\langle v \rangle^l g\|_{L^p(\mathbb{T} \times \mathbb{R})} < \infty \}, \\ H_l^k(\mathbb{T} \times \mathbb{R}) := \{ \langle v \rangle^l g \in H^k(\mathbb{T} \times \mathbb{R}) \mid \|g\|_{H_l^k(\mathbb{T} \times \mathbb{R})} = \|\langle v \rangle^l g\|_{H^k(\mathbb{T} \times \mathbb{R})} < \infty \}. \end{cases}$$

We have the following global existence of weak solutions to the IVP (1.1)-(1.4) for general initial data:

Theorem 1.1. *Suppose that the initial data (ρ_0, m_0, f_0) satisfies*

$$\begin{cases} 0 \leq \rho_0 \in L^\infty(\mathbb{T}), \\ \dfrac{m_0}{\sqrt{\rho_0}} = 0 \quad in \quad \{ x \in \mathbb{T} \mid \rho_0(x) = 0 \}, \quad \dfrac{m_0}{\sqrt{\rho_0}} \in L^2(\mathbb{T}), \\ 0 \leq f_0 \in L_3^1 \cap L^\infty(\mathbb{T} \times \mathbb{R}). \end{cases} \tag{1.8}$$

Then the IVP (1.1)-(1.4) admits a global weak solution (ρ, u, f) in the sense of Definition 1.1 satisfying for any $T > 0$ that

$$\begin{cases} 0 \leq \rho(x,t) \leq \rho_+, \quad a.e. \quad (x,t) \in \mathbb{T} \times [0,T], \tag{1.9a} \\ 0 \leq f(x,v,t) \leq e^{\rho_+ T} \|f_0\|_{L^\infty(\mathbb{T} \times \mathbb{R})}, \quad a.e. \quad (x,v,t) \in \mathbb{T} \times \mathbb{R} \times [0,T], \tag{1.9b} \\ \operatorname*{ess\,sup}_{t \in [0,T]} \left(t^{\frac{1}{2}} \|u(t)\|_{H^1(\mathbb{T})} + \|f(t)\|_{L_3^1(\mathbb{T} \times \mathbb{R})} \right) + \|\sqrt{\rho} f_v\|_{L^2(0,T;L^2(\mathbb{T} \times \mathbb{R}))} \leq C_T, \tag{1.9c} \end{cases}$$

where $\rho_+ > 0$ is a constant independent of the time $T > 0$, and $C_T > 0$ is a constant dependent of the time $T > 0$.

Moreover, the conservation laws of mass and momentum hold for $t \in [0,T]$:

$$\begin{cases} \displaystyle \int_{\mathbb{T}} \rho(x,t) = \int_{\mathbb{T}} \rho_0(x) dx, \tag{1.10a} \\ \displaystyle \int_{\mathbb{T} \times \mathbb{R}} f(x,v,t) dv dx = \int_{\mathbb{T}} n(x,t) dx = \int_{\mathbb{T}} n_0(x) dx, \tag{1.10b} \\ \displaystyle \int_{\mathbb{T}} (\rho u + n w)(x,t) dx = \int_{\mathbb{T}} (m_0 + j_0)(x) dx, \tag{1.10c} \end{cases}$$

and (ρ, u) converges to $(\overline{\rho_0}, \overline{u}(t))$ as $t \to \infty$:

$$\lim_{t \to \infty} \left(\|(\rho - \overline{\rho_0})(t)\|_{L^p(\mathbb{T})} + \|(\sqrt{\rho}(u - \overline{u}))(t)\|_{L^2(\mathbb{T})} \right) = 0, \quad p \in [1, \infty), \tag{1.11}$$

where the asymptotical states $\overline{\rho_0}$ and $\overline{u}(t)$ are given by

$$\overline{\rho_0} := \int_{\mathbb{T}} \rho_0(x) dx, \quad \overline{u}(t) := \frac{\int_{\mathbb{T}} \rho u(x,t) dx}{\int_{\mathbb{T}} \rho_0(x) dx}. \tag{1.12}$$

Then, we study the regularity and uniqueness of the global weak solution (ρ, u, f) given by Theorem 1.1, and analyze the large time behavior of (ρ, u, f) to the equilibrium state

$$\left(\overline{\rho_0}, u_c, M_{\overline{n_0}, u_c}(v)\right),$$

where u_c is the constant

$$u_c := \frac{\int_{\mathbb{T}}(m_0 + j_0)(x)dx}{\int_{\mathbb{T}}(\rho_0 + n_0)(x)dx}, \tag{1.13}$$

$M_{\overline{n_0}, u_c}(v)$ is the global Maxwellian

$$M_{\overline{n_0}, u_c}(v) := \frac{\overline{n_0}}{\sqrt{2\pi}} e^{-\frac{|v - u_c|^2}{2}}, \tag{1.14}$$

and $\overline{n_0}$ is the constant

$$\overline{n_0} := \int_{\mathbb{T}} n_0(x)dx. \tag{1.15}$$

Theorem 1.2. *Suppose that the initial data (ρ_0, m_0, f_0) satisfies*

$$\inf_{x \in \mathbb{T}} \rho_0(x) > 0, \quad \rho_0 \in H^1(\mathbb{T}), \quad \frac{m_0}{\rho_0} \in H^1(\mathbb{T}), \quad 0 \le f_0 \in L^2_{k_0} \cap L^\infty(\mathbb{T} \times \mathbb{R}), \tag{1.16}$$

where $k_0 > \frac{7}{2}$ is a constant. Then the global weak solution (ρ, u, f) to the IVP (1.1)-(1.4) given by Theorem 1.1 is unique. In addition to the properties (1.9)-(1.11), it holds for any $T > 0$ that

$$\begin{cases} \rho(x, t) \ge \rho_- > 0, \quad a.e. \quad (x, t) \in \mathbb{T} \times [0, T], \quad \sup_{t \in [0, T]} \|\rho(t)\|_{H^1(\mathbb{T})} \le C_0, & (1.17a) \\[2mm] \|u\|_{L^2(0, T; H^2(\mathbb{T}))} + \|u_t\|_{L^2(0, T; L^2(\mathbb{T}))} + \|f_v\|_{L^2(0, T; L^2_{k_0}(\mathbb{T} \times \mathbb{R}))} \le C_T, & (1.17b) \\[2mm] \operatorname*{ess\,sup}_{t \in [0, T]} \left(\|u(t)\|_{H^1(\mathbb{T})} + t^{\frac{1}{2}} \|u(t)\|_{H^2(\mathbb{T})}\right) \le C_T, & (1.17c) \\[2mm] \operatorname*{ess\,sup}_{t \in [0, T]} \left(\|f(t)\|_{L^2_{k_0}(\mathbb{T} \times \mathbb{R})} + t^{\frac{1}{2}} \|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T} \times \mathbb{R})} + t\|f_{vv}(t)\|_{L^2_{k_0-2}(\mathbb{T} \times \mathbb{R})} \right. \\[2mm] \qquad \left. + t^{\frac{3}{2}} \|(f_x, f_{vvv})(t)\|_{L^2_{k_0-2}(\mathbb{T} \times \mathbb{R})}\right) \le C_T, & (1.17d) \end{cases}$$

where $\rho_- > 0$ and $C_0 > 0$ are two constants independent of the time $T > 0$, and $C_T > 0$ is a constant dependent of the time $T > 0$.

Furthermore, the solution (ρ,u,f) satisfies

$$
\begin{cases}
\lim_{t\to\infty} \Big(\|(\rho-\overline{\rho_0})(t)\|_{L^\infty(\mathbb{T})} + \|(u-u_c)(t)\|_{L^2(\mathbb{T})} \Big) = 0, & \text{(1.18a)} \\[2mm]
\lim_{t\to\infty} \Big(\|(n-\overline{n_0})(t)\|_{L^1(\mathbb{T})} + \|\big(n(w-u_c)\big)(t)\|_{L^1(\mathbb{T})} \Big) = 0, & \text{(1.18b)} \\[2mm]
\lim_{t\to\infty} \|(f-M_{\overline{n_0},u_c})(t)\|_{L^1(\mathbb{T}\times\mathbb{R})} = 0. & \text{(1.18c)}
\end{cases}
$$

Remark 1.1. Different from the compressible Navier-Stokes-Vlasov system [28], the Vlasov-Fokker-Planck equation (1.1c) has the regularizing effect (1.17d) due to hypoellipticity of the nonlinear Fokker-Planck operator $v\partial_x - \rho\partial_v(v+\partial_v)$.

Remark 1.2. The distribution function f lacks the uniform-time integrability under the assumptions of Theorem 1.2 since the term $f\log f$ in entropy for (1.1) may not be controlled by its dissipation, which is essentially different from the compressible Navier-Stokes-Vlasov model [27]. To have the large time behavior (1.18c), we employ the ideas by Bouchut and Dolbeault [4] to prove the $L^1(\mathbb{T}\times\mathbb{R})$-convergence of $f^s(x,v,t):=f(x,v,t+s)$ for $t\in(0,1)$ to the unique limit $M_{\overline{n_0},u_c}$ as $s\to\infty$. It should be noted that the compactness lemmas in [4,16] could not be applied here due to the non-smooth coefficients in (1.1c). Indeed, because $(\sqrt{f^s})_v$ is uniformly bounded in $L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))$, we can apply the techniques developed by Arsénio and Saint-Raymond [2] to obtain the strong convergence of f^s in all variables.

Remark 1.3. By (1.17) and Theorem 1.1, one can prove that the IVP (1.1)-(1.4) admits a unique global classical solution subject to regular initial data.

Remark 1.4. By Theorem 1.2 and similar arguments as used in [28], we are able to establish the global well-posedness of the IVP (1.1)-(1.4) in spatial real line.

The rest part of the paper is arranged as follows. In Section 2, we establish the a-priori estimates for the compressible NS-VFP system (1.1). The uniqueness of the weak solution will be proved in Section 3. Then, we analyze the large time behavior of global solutions in Section 4. In Section 5, we show the convergence of approximate sequence to the corresponding weak solution to the IVP (1.1)-(1.4).

2   The a-priori estimates

2.1   Basic estimates

First, by (1.1), we have the following properties:

Lemma 2.1. *Let $T > 0$, and (ρ, u, f) be any regular solution to the IVP (1.1)-(1.4) for $t \in (0,T]$. Then it holds*

$$\frac{d}{dt} \int_{\mathbb{T}} \rho(x,t) dx = 0, \quad \frac{d}{dt} \int_{\mathbb{T} \times \mathbb{R}} f(x,v,t) dv dx = \frac{d}{dt} \int_{\mathbb{T}} n(x,t) dx = 0, \tag{2.1}$$

$$\frac{d}{dt} \int_{\mathbb{T}} (\rho u + nw)(x,t) dx = 0, \tag{2.2}$$

$$\frac{d}{dt} \left(\int_{\mathbb{T}} \left(\frac{1}{2} \rho |u|^2 + \frac{\rho^\gamma}{\gamma - 1} \right)(x,t) dx + \int_{\mathbb{T} \times \mathbb{R}} \left(\frac{1}{2} |v|^2 f + f \log f \right)(x,v,t) dv dx \right)$$

$$= - \int_{\mathbb{T}} \left(\mu(\rho) |u_x|^2 \right)(x,t) dx - \int_{\mathbb{T} \times \mathbb{R}} \left(\rho |(u-v) \sqrt{f} - 2(\sqrt{f})_v|^2 \right)(x,v,t) dv dx, \tag{2.3}$$

where n and nw are defined by (1.7).

Then, applying the similar arguments as used in [28], we can show

Lemma 2.2. *Let $T > 0$, and (ρ, u, f) be any regular solution to the IVP (1.1)-(1.4) for $t \in (0,T]$. Then under the assumptions of Theorem 1.1, it holds*

$$\begin{cases} \rho(x,t) \geq 0, \quad f(x,v,t) \geq 0, \quad (x,v,t) \in \mathbb{T} \times \mathbb{R} \times [0,T], & (2.4a) \\ \sup_{t \in [0,T]} \left(\|(\sqrt{\rho} u)(t)\|_{L^2(\mathbb{T})} + \|f(t)\|_{L^1_{\frac{1}{2}}(\mathbb{T} \times \mathbb{R})} + \|(f \log f)(t)\|_{L^1(\mathbb{T} \times \mathbb{R})} \right) \leq C, & (2.4b) \\ \|\sqrt{\mu(\rho)} u_x\|_{L^2(0,T;L^2(\mathbb{T}))} + \|\sqrt{\rho}((u-v)\sqrt{f} - 2(\sqrt{f})_v)\|_{L^2(0,T;L^2(\mathbb{T} \times \mathbb{R}))} \leq C, & (2.4c) \\ \|u(t)\|_{L^\infty(\mathbb{T})} \leq \|u_x(t)\|_{L^2(\mathbb{T})} + C, \quad t \in [0,T], & (2.4d) \\ \rho(x,t) \leq \rho_+, \quad (x,t) \in \mathbb{T} \times [0,T], & (2.4e) \end{cases}$$

where $C > 0$ and ρ_+ are two constants independent of the time $T > 0$.

Next, we have the time-dependent estimates of the distribution function f:

Lemma 2.3. *Let $T > 0$, and (ρ, u, f) be any regular solution to the IVP (1.1)-(1.4) for $t \in (0,T]$. Then under the assumptions of Theorem 1.1, it holds*

$$\begin{cases} f(x,v,t) \leq e^{\rho_+ T} \|f_0\|_{L^\infty(\mathbb{T} \times \mathbb{R})}, \quad (x,v,t) \in \mathbb{T} \times \mathbb{R} \times [0,T], & (2.5a) \\ \sup_{t \in [0,T]} \|f(t)\|_{L^1_{\frac{1}{3}}(\mathbb{T} \times \mathbb{R})} + \|\sqrt{\rho} f_v\|_{L^2(0,T;L^2(\mathbb{T} \times \mathbb{R}))} \leq C_T, & (2.5b) \\ \sup_{t \in [0,T]} \left(\|n(t)\|_{L^4(\mathbb{T})} + \|nw(t)\|_{L^2(\mathbb{T})} \right) \leq C_T, & (2.5c) \end{cases}$$

where the constant ρ_+ is given by (2.4e), and $C_T > 0$ is a constant dependent of the time $T > 0$.

Proof. First, we multiply (1.1c) by pf^{p-1} for any $p \in [2,\infty)$ and integrate the resulting equation by parts over $\mathbb{T} \times \mathbb{R}$ to obtain

$$\frac{d}{dt}\|f(t)\|_{L^p(\mathbb{T}\times\mathbb{R})}^p + p(p-1)\|(\sqrt{\rho}f^{\frac{p-2}{2}}f_v)(t)\|_{L^2(\mathbb{T}\times\mathbb{R})}^2$$
$$\leq (p-1)\|\rho(t)\|_{L^\infty(\mathbb{T})}\|f(t)\|_{L^p(\mathbb{T}\times\mathbb{R})}^p,$$

which together with (2.4e) and the Grönwall inequality implies

$$\begin{cases} \|\sqrt{\rho}f_v\|_{L^2(0,T;L^2(\mathbb{T}\times\mathbb{R}))} \leq C_T, & (2.6a) \\ \displaystyle\sup_{t\in[0,T]} \|f(t)\|_{L^p(\mathbb{T}\times\mathbb{R})} \leq e^{\rho_+ T}\|f_0\|_{L^\infty(\mathbb{T}\times\mathbb{R})}^{1-\frac{1}{p}}\|f_0\|_{L^1(\mathbb{T}\times\mathbb{R})}^{\frac{1}{p}}. & (2.6b) \end{cases}$$

We get (2.5a) after taking the limit in (2.6b) as $p \to \infty$. Then multiplying Eq. (1.1c) by $\langle v \rangle^3$ with $\langle v \rangle := (1+|v|^2)^{\frac{1}{2}}$, integrating the resulting equation by parts over $\mathbb{T} \times \mathbb{R}$, and applying the Grönwall inequality, we derive

$$\sup_{t\in[0,T]} \|f(t)\|_{L_3^1(\mathbb{T}\times\mathbb{R})} \leq e^{C\rho_+(T+\|u\|_{L^1(0,T;L^\infty(\mathbb{T}))})}\|f_0\|_{L_3^1(\mathbb{T}\times\mathbb{R})} \leq C_T, \quad (2.7)$$

where one has used the estimates (2.4).

We are going to estimate n and nw. For any $R > 0$, it is easy to verify

$$nw(x,t) \leq \left(\int_{\{|v|\leq R\}} + \int_{\{|v|\geq R\}}\right) vf(x,v,t)dv$$
$$\leq 2\|f(x,\cdot,t)\|_{L^\infty(\mathbb{R})}R^2 + \frac{1}{R^2}\int_\mathbb{R} |v|^3 f(x,v,t)dv. \quad (2.8)$$

Choosing

$$R = \left(\int_\mathbb{R} |v|^3 f(x,v,t)dv\right)^{\frac{1}{4}}$$

in (2.8) and making use of (2.5a) and (2.7), we obtain

$$\sup_{t\in[0,T]} \|nw(t)\|_{L^2(\mathbb{T})} \leq \sup_{t\in[0,T]} (2\|f(t)\|_{L^\infty(\mathbb{T}\times\mathbb{R})}+1)^{\frac{1}{2}}\|f(t)\|_{L_3^1(\mathbb{T}\times\mathbb{R})}^{\frac{1}{2}} \leq C_T. \quad (2.9)$$

Similarly, one has

$$\sup_{t\in[0,T]} \|n(t)\|_{L^4(\mathbb{T})} \leq \sup_{t\in[0,T]} (2\|f(t)\|_{L^\infty(\mathbb{T}\times\mathbb{R})}+1)^{\frac{1}{4}}\|f(t)\|_{L_3^1(\mathbb{T}\times\mathbb{R})}^{\frac{1}{4}} \leq C_T. \quad (2.10)$$

The combination of (2.6a), (2.7), and (2.9)-(2.10) gives rise to (2.5b)-(2.5c). The proof of Lemma 2.3 is complete. $\qquad\qquad\square$

The estimates (2.4) and (2.5) are not enough to obtain the compactness of the density ρ in the framework of Lions [34]. To overcome the difficulty caused by the density-dependent viscosity coefficient $\mu(\rho)$, we need to establish the additional $L^1(0,T;L^\infty)$-estimate of the effective viscous flux $\rho^\gamma - \mu(\rho)u_x$:

Lemma 2.4. *Let $T > 0$, and (ρ, u, f) be any regular solution to the IVP (1.1)-(1.4) for $t \in (0,T]$. Then under the assumptions of Theorem 1.1, it holds*

$$\sup_{t \in [0,T]} t^{\frac{1}{2}} \|u(t)\|_{H^1(\mathbb{T})} + \|\rho^\gamma - \mu(\rho)u_x\|_{L^p(0,T;L^\infty(\mathbb{T}))} \leq C_T, \quad p \in [1,\tfrac{4}{3}) \tag{2.11}$$

for $C_T > 0$ a constant.

Proof. Multiplying (1.1b) by u_t and integrating the resulting equation by parts over $\mathbb{T}$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\left(\sqrt{\mu(\rho)}u_x\right)(t)\right\|_{L^2(\mathbb{T})}^2 + \|(\sqrt{\rho}u_t)(t)\|_{L^2(\mathbb{T})}^2 = \sum_{i=1}^{3} I_i^1, \tag{2.12}$$

where $I_i^1, i = 1,2,3$, are given by

$$I_1^1 = \int_{\mathbb{T}} \left((\rho n w - \rho u n - \rho u u_x)u_t\right)(x,t)dx,$$

$$I_2^1 = \int_{\mathbb{T}} (\rho^\gamma u_{xt})(x,t)dx,$$

$$I_3^1 = \frac{1}{2}\int_{\mathbb{T}} \left((\mu(\rho))_t |u_x|^2\right)(x,t)dx.$$

We deal with the terms I_i^1 $(i = 1,2,3)$ as follows. One deduces by (2.4e) and (2.5c) that

$$I_1^1 \leq \rho_+^{\frac{1}{2}} \|(\sqrt{\rho}u_t)(t)\|_{L^2(\mathbb{T})} \Big(\|nw(t)\|_{L^2(\mathbb{T})} + \|u(t)\|_{L^\infty(\mathbb{T})} \|n(t)\|_{L^2(\mathbb{T})}$$

$$+ \|u_x(t)\|_{L^2(\mathbb{T})} \|u(t)\|_{L^\infty(\mathbb{T})} \Big)$$

$$\leq \frac{1}{8} \|(\sqrt{\rho}u_t)(t)\|_{L^2(\mathbb{T})}^2 + C_T \|u(t)\|_{L^\infty(\mathbb{T})}^2 \left(1 + \|u_x(t)\|_{L^2(\mathbb{T})}^2\right) + C_T.$$

Since it holds

$$-\int_{\mathbb{T}} \left((\rho^\gamma)_t u_x\right)(x,t)dx = \gamma \int_{\mathbb{T}} \left(\left(\rho^\gamma - \mu(\rho)\int_1^\rho \frac{s^{\gamma-1}}{\mu(s)}ds\right)|u_x|^2\right)(x,t)dx$$

$$-\gamma\int_{\mathbb{T}}\left(u\int_1^\rho\frac{s^{\gamma-1}}{\mu(s)}ds(\rho u_t+\rho uu_x-\rho nw+\rho nu)\right)(x,t)dx$$

$$-\gamma\int_{\mathbb{T}}\left(u_x\left(\int_1^\rho\frac{s^{2\gamma-1}}{\mu(s)}ds-\rho^\gamma\int_1^\rho\frac{s^{\gamma-1}}{\mu(s)}ds\right)\right)(x,t)dx$$

derived from the Eqs. (1.1a)-(1.1b), we obtain by (2.4) and (2.5b) that

$$I_2^1=\frac{d}{dt}\int_{\mathbb{T}}(\rho^\gamma u_x)(x,t)dx-\int_{\mathbb{T}}((\rho^\gamma)_t u_x)(x,t)dx$$

$$\le\frac{d}{dt}\int_{\mathbb{T}}(\rho^\gamma u_x)(x,t)dx+\frac{1}{8}\|(\sqrt{\rho}u_t)(t)\|_{L^2(\mathbb{T})}^2$$

$$+C\|u_x(t)\|_{L^2(\mathbb{T})}^2+C_T\|u(t)\|_{L^\infty(\mathbb{T})}^2+C_T.$$

It follows from the Gagliardo-Nirenberg inequality, (1.1b), (2.4e), and (2.5c) that

$$\|u_x(t)\|_{L^\infty(\mathbb{T})}\le\|(\mu(\rho)u_x-\rho^\gamma)(t)\|_{L^\infty(\mathbb{T})}+\|\rho(t)\|_{L^\infty(\mathbb{T})}^\gamma \tag{2.13}$$

$$\le C_T\|(\sqrt{\rho}u_t)(t)\|_{L^2(\mathbb{T})}+C_T\|u(t)\|_{L^\infty(\mathbb{T})}\left(\|u_x(t)\|_{L^2(\mathbb{T})}+1\right)+C_T.$$

Similarly to the estimate of I_2^1, one deduces by (1.1a)-(1.1b), (2.4)-(2.5), and (2.13) that

$$I_3^1=-\frac{\beta}{2}\int_{\mathbb{T}}(\rho^\beta|u_x|^2u_x)(x,t)dx-\frac{\beta}{2}\int_{\mathbb{T}}(u\rho^{\beta-1}\rho_x|u_x|^2)(x,t)dx$$

$$\le\frac{1}{8}\|(\sqrt{\rho}u_t)(t)\|_{L^2(\mathbb{T})}^2+C_T\left(\|u_x(t)\|_{L^2(\mathbb{T})}^2+\|u(t)\|_{L^\infty(\mathbb{T})}^2+1\right)\|u_x(t)\|_{L^2(\mathbb{T})}^2+C_T.$$

Substituting the above estimates of I_i^1 ($i=1,2,3$) into (2.12), multiplying the resulting inequality by t, and then integrating it over $[0,t]$, we obtain

$$\frac{1}{2}t\|u_x(t)\|_{L^2(\mathbb{T})}^2+\frac{1}{2}\int_0^t\tau\|(\sqrt{\rho}u_t)(\tau)\|_{L^2(\mathbb{T})}^2d\tau$$

$$\le\frac{1}{2}\int_0^t\tau\|u_x(\tau)\|_{L^2(\mathbb{T})}^2d\tau+t\int_{\mathbb{T}}(\rho^\gamma u_x)(x,t)dx-\int_0^t\tau\int_{\mathbb{T}}(\rho^\gamma u_x)(x,\tau)dxd\tau$$

$$+C_T\int_0^t\left(1+\|u(\tau)\|_{L^\infty(\mathbb{T})}^2+\|u_x(\tau)\|_{L^2(\mathbb{T})}^2\right)\tau\|u_x(\tau)\|_{L^2(\mathbb{T})}^2d\tau+C_T,$$

which together with the Grönwall inequality, (2.4), and (2.13) gives rise to

$$\sup_{t\in[0,T]}t\|u_x(t)\|_{L^2(\mathbb{T})}^2+\int_0^t\tau\left(\|(\sqrt{\rho}u_t)(\tau)\|_{L^2(\mathbb{T})}^2+\|u_x(\tau)\|_{L^\infty(\mathbb{T})}^2\right)d\tau\le C_T. \tag{2.14}$$

By (1.1b), (2.4e), (2.5), (2.14), and the Gagliardo-Nirenberg inequality, we have

$$
\int_0^T \|(\rho^\gamma - \mu(\rho)u_x)(t)\|_{L^\infty(\mathbb{T})}^p dt
$$

$$
\leq C \int_0^T \Big(\|(\rho^\gamma - \mu(\rho)u_x)(t)\|_{L^2(\mathbb{T})}^{\frac{p}{2}} \|(\rho u_t + \rho u u_x - \rho n w + \rho u n)(t)\|_{L^2(\mathbb{T})}^{\frac{p}{2}}
$$

$$
+ \|(\rho^\gamma - \mu(\rho)u_x)(t)\|_{L^1(\mathbb{T})}^p \Big) dt
$$

$$
\leq C_T \sup_{t \in [0,T]} \Big(t^{\frac{1}{2}} \|u_x(t)\|_{L^2(\mathbb{T})} \Big)^{\frac{p}{2}} \Big(\int_0^T \Big(t \|(\sqrt{\rho} u_t)(t)\|_{L^2(\mathbb{T})}^2 + t \|u_x(t)\|_{L^\infty(\mathbb{T})}^2 \Big) dt \Big)^{\frac{p}{4}}
$$

$$
\times \Big(\int_0^T t^{-\frac{2p}{4-p}} dt \Big)^{\frac{4-p}{4}} + C_T \Big(\int_0^T t \|(\sqrt{\rho} u_t)(t)\|_{L^2(\mathbb{T})}^2 dt \Big)^{\frac{p}{4}} \Big(\int_0^T t^{-\frac{p}{4-p}} dt \Big)^{\frac{4-p}{4}} + C_T
$$

$$
\leq C_T
$$

provided $p \in [1, \frac{4}{3})$. Due to (2.4) and (2.14), it also holds

$$
\sup_{t \in [0,T]} t \|u(t)\|_{H^1(\mathbb{T})}^2 \leq C_T.
$$

The proof of Lemma 2.4 is complete. □

2.2　Higher-order estimates

We are ready to establish the higher-order estimates of the solution (ρ, u, f).

Lemma 2.5. *Let $T > 0$, and (ρ, u, f) be any regular solution to the IVP* (1.1)-(1.4) *for $t \in (0,T]$. Then under the assumptions of Theorem* 1.2, *it holds*

$$
\begin{cases}
\rho(x,t) \geq \rho_T > 0, \quad (x,t) \in \mathbb{T} \times [0,T], & \text{(2.15a)} \\[2mm]
\sup_{t \in [0,T]} \|(\rho, u)(t)\|_{H^1(\mathbb{T})} + \|u\|_{L^2(0,T;H^2(\mathbb{T}))} + \|u_t\|_{L^2(0,T;L^2(\mathbb{T}))} \leq C_T, & \text{(2.15b)} \\[2mm]
\sup_{t \in [0,T]} \|f(t)\|_{L^2_{k_0}(\mathbb{T} \times \mathbb{R})} + \|f_v\|_{L^2(0,T;L^2_{k_0}(\mathbb{T} \times \mathbb{R}))} \leq C_T, & \text{(2.15c)} \\[2mm]
\sup_{t \in [0,T]} t^{\frac{1}{2}} \|(u_{xx}, u_t)(t)\|_{L^2(\mathbb{T})} \leq C_T, & \text{(2.15d)}
\end{cases}
$$

where $\rho_T > 0$ and $C_T > 0$ are two constants.

Proof. Repeating the same arguments as used in [28], we can show (2.15a)-(2.15b), the details are omitted here. Then we multiply (1.1c) by $\langle v \rangle^{2k_0} f$ and integrate the resulting equation by parts over $\mathbb{T} \times \mathbb{R}$ to get

$$
\frac{d}{dt} \|f(t)\|^2_{L^2_{k_0}(\mathbb{T} \times \mathbb{R})} + \|(\sqrt{\rho} f_v)(t)\|^2_{L^2_{k_0}(\mathbb{T} \times \mathbb{R})}
$$
$$
\leq C \|\rho(t)\|_{L^\infty(\mathbb{T})} \big(1 + \|u(t)\|_{L^\infty(\mathbb{T})}\big) \|f(t)\|^2_{L^2_{k_0}(\mathbb{T} \times \mathbb{R})},
$$

which together with (2.4), (2.15a), and the Grönwall inequality leads to (2.15c).

In addition, due to (2.15c) and the fact $k_0 - 2 > \frac{1}{2}$, it also holds

$$
\sup_{t \in [0,T]} \left\| \int_{\mathbb{R}} |v|^2 f(\cdot, v, t) dv \right\|_{L^2(\mathbb{T})}
$$
$$
\leq \left(\int_{\mathbb{R}} \frac{1}{\langle v \rangle^{2(k_0 - 2)}} dv \right)^{\frac{1}{2}} \sup_{t \in [0,T]} \|f(t)\|_{L^2_{k_0}(\mathbb{T} \times \mathbb{R})} \leq C_T. \tag{2.16}
$$

Differentiating the Eq. (1.1b) with respect to t, we obtain

$$
\rho(u_t + u u_{xt}) - (\mu(\rho) u_{xt})_x + \rho n u_t
$$
$$
= -\rho_t(u_t + u u_x) - \rho u_t u_x - (\rho^\gamma)_{xt} + \big((\mu(\rho))_t u_x\big)_x
$$
$$
+ \rho_t(nw - nu) - \rho u n_t + \rho(nw)_t. \tag{2.17}
$$

One deduces after a direct computation that

$$
\frac{1}{2} \frac{d}{dt} \Big(t \|(\sqrt{\rho} u_t)(t)\|^2_{L^2(\mathbb{T})} \Big) + t \left\| \Big(\sqrt{\mu(\rho)} u_{xt} \Big)(t) \right\|^2_{L^2(\mathbb{T})} + t \|(\sqrt{\rho n} u_t)(t)\|^2_{L^2(\mathbb{T})}
$$
$$
= \frac{1}{2} \|(\sqrt{\rho} u_t)(t)\|^2_{L^2(\mathbb{T})} + \sum_{i=1}^{3} I_i^2, \tag{2.18}
$$

where $I_i^2, i = 1, 2, 3$, are given by

$$
I_1^2 = t \int_{\mathbb{T}} \Big(\big(-\rho_t(u_t + u u_x) - \rho u_t u_x - (\rho^\gamma)_{xt} + \big((\mu(\rho))_t u_x\big)_x + \rho_t(nw - nu)\big) u_t \Big)(x,t) dx,
$$
$$
I_2^2 = t \int_{\mathbb{T}} (\rho u n_t u_t)(x,t) dx,
$$
$$
I_3^2 = -t \int_{\mathbb{T}} (\rho(nw)_t u_t)(x,t) dx.
$$

It holds by (2.4) and (2.15b) that

$$
\begin{aligned}
I_1^2 = t \int_{\mathbb{T}} &\Big(-\rho_t(u_t+uu_x)u_t - \rho u_x|u_t|^2 + (\rho^\gamma)_t u_{tx} \\
&- (\mu(\rho))_t u_x u_{xt} + \rho_t(nw-nu)u_t \Big)(x,t)dx \\
\leq \frac{t}{4} &\left\| \left(\sqrt{\mu(\rho)} u_{xt} \right)(t) \right\|_{L^2(\mathbb{T})}^2 + C_T \left(1 + \|u_x(t)\|_{L^\infty(\mathbb{T})} \right) t\|u_t(t)\|_{L^2(\mathbb{T})}^2 + C_T.
\end{aligned}
$$

By virtue of (2.13), (2.15b), and the fact $n_t = -(nw)_x$, we have

$$
\begin{aligned}
I_2^2 = t \int_{\mathbb{T}} &(\rho(u_x u_t + uu_{xt})nw + \rho_x uu_t)(x,t)dx \\
\leq t\Big(&\|u_x(t)\|_{L^\infty(\mathbb{T})}\|u_t(t)\|_{L^2(\mathbb{T})} + \|u(t)\|_{L^\infty(\mathbb{T})}\|u_{xt}(t)\|_{L^2(\mathbb{T})} \\
&+ \|\rho_x(t)\|_{L^2(\mathbb{T})}\|u(t)\|_{L^\infty(\mathbb{T})}\|u_t(t)\|_{L^2(\mathbb{T})} \Big) \|nw(t)\|_{L^2(\mathbb{T})} \\
\leq \frac{t}{4} &\left\| \left(\sqrt{\mu(\rho)} u_{xt} \right)(t) \right\|_{L^2(\mathbb{T})}^2 + C_T \left(1 + \|u_x(t)\|_{L^\infty(\mathbb{T})} \right) t\|u_t(t)\|_{L^2(\mathbb{T})}^2 + C_T.
\end{aligned}
$$

Similarly, it is easy to verify

$$
(nw)_t = -\left(\int_{\mathbb{R}} |v|^2 f dv \right)_x + \rho un - \rho nw,
$$

which together with (2.4), (2.15b), and (2.16) yields

$$
\begin{aligned}
I_3^2 = t \int_{\mathbb{T}} &\left(\int_{\mathbb{R}} |v|^2 f(x,v,t)dv \right)(\rho u_t)_x(x,t)dx - t \int_{\mathbb{T}} ((\rho un - \rho nw)\rho u_t)(x,t)dx \\
\leq \frac{t}{4} &\left\| \left(\sqrt{\mu(\rho)} u_{xt} \right)(t) \right\|_{L^2(\mathbb{T})}^2 + C_T t\|u_t(t)\|_{L^2(\mathbb{T})}^2 + C_T.
\end{aligned}
$$

Substituting the above estimates of I_i^2 $(i=1,2,3)$ into (2.18) and applying the Grönwall inequality, we obtain

$$
\sup_{t \in [0,T]} t\|u_t(t)\|_{L^2(\mathbb{T})}^2 \leq C_T. \tag{2.19}
$$

By (1.1b), (2.4), (2.15b), and (2.19), it also holds

$$
\begin{aligned}
&\sup_{t \in [0,T]} t^{\frac{1}{2}} \|u_{xx}(t)\|_{L^2(\mathbb{T})} \\
&\leq \sup_{t \in [0,T]} t^{\frac{1}{2}} \|(\rho u_t + \rho uu_x + (\rho^\gamma)_x - \rho nw + \rho nu)(t)\|_{L^2(\mathbb{T})} \leq C_T. \tag{2.20}
\end{aligned}
$$

The combination of (2.19)-(2.20) gives rise to (2.15d). The proof of Lemma 2.5 is complete. $\qquad\square$

Inspired by [12, 21, 38], we have the following hypoellipticity estimates of the distribution function f for the Vlasov-Fokker-Planck equation (1.1c):

Lemma 2.6. *Let $T > 0$, and (ρ, u, f) be any regular solution to the IVP (1.1)-(1.4) for $t \in (0, T]$. Then under the assumptions of Theorem 1.2, it holds*

$$\sup_{t\in[0,T]} \Big(t\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + t^2\|f_{vv}(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}$$

$$+ t^3\|(f_x, f_{vvv})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \Big) \leq C_T, \tag{2.21}$$

where $C_T > 0$ is a constant.

Proof. First, we differentiate the Vlasov-Fokker-Planck equation (1.1c) with respect to v to obtain

$$(f_v)_t + v(f_v)_x + \big(\rho(u-v)f_v - \rho(f_v)_v\big)_v = \rho f_v + f_x. \tag{2.22}$$

Multiplying (2.22) by $\langle v\rangle^{2(k_0-2)} f_v$, integrating the resulting equation by parts over $\mathbb{T}\times\mathbb{R}$, and then making use of (2.4e) and (2.15), we have

$$\frac{1}{2}\frac{d}{dt}\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \|(\sqrt{\rho}f_{vv})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}$$

$$\leq C_T\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}, \tag{2.23}$$

from which we infer

$$\frac{1}{2}\frac{d}{dt}\Big(t\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \Big) + t\|(\sqrt{\rho}f_{vv})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}$$

$$\leq C_T\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + t\|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}. \tag{2.24}$$

Similarly, due to the equation

$$(f_x)_t + v(f_x)_x + \big(\rho(u-v)f_x - \rho(f_x)_v\big)_v = -\big(\rho_x(u-v)f + \rho u_x f - \rho_x f_v\big)_v, \tag{2.25}$$

one has

$$\frac{1}{2}\frac{d}{dt}\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \|(\sqrt{\rho}f_{vx})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}$$

$$\leq C_T\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + C_T\|(\rho_x, u_x)(t)\|_{L^2(\mathbb{T})}$$

$$\times \sup_{x\in\mathbb{T}}\|(\langle v\rangle^{k_0-1}f, \langle v\rangle^{k_0-2}f_v)(x,t)\|_{L^2(\mathbb{R})}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}. \tag{2.26}$$

To overcome the difficulties caused by the loss of one velocity weight v and the low regularity of ρ_x on the right-hand side of (2.26), we need

$$
\sup_{x\in\mathbb{T}}\int_{\mathbb{R}}\langle v\rangle^{2(k_0-1)}f^2(x,v,t)dv
$$

$$
\leq\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2(k_0-1)}f^2(x,v,t)dvdx+2\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2(k_0-1)}(f|f_x|)(x,v,t)dvdx
$$

$$
\leq2\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2k_0}f^2(x,v,t)dvdx+\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2(k_0-2)}|f_x|^2(x,v,t)dvdx, \qquad (2.27)
$$

where one has used the fact

$$
\sup_{x\in\mathbb{T}}|g(x)|\leq\|g\|_{L^1(\mathbb{T})}+\|g_x\|_{L^1(\mathbb{T})}\quad\text{for any}\quad g\in W^{1,1}(\mathbb{T}).
$$

Similarly, we get for any $\eta\in(0,1)$ that

$$
\sup_{x\in\mathbb{T}}\int_{\mathbb{R}}\langle v\rangle^{2(k_0-2)}|f_v|^2(x,v,t)dv \qquad (2.28)
$$

$$
\leq\left(1+\frac{1}{\eta}\right)\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2(k_0-2)}|f_v|^2(x,v,t)dvdx+\eta\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2(k_0-2)}|f_{vx}|^2(x,v,t)dvdx.
$$

We combine (2.26)-(2.28) together to have

$$
\frac{1}{2}\frac{d}{dt}\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}+\|(\sqrt{\rho}f_{vx})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}
$$

$$
\leq C_T\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}+C_T\left(\|f(t)\|_{L^2_{k_0}(\mathbb{T}\times\mathbb{R})}+\left(1+\frac{1}{\eta^{\frac{1}{2}}}\right)\|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right.
$$

$$
\left.+\|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}+\eta^{\frac{1}{2}}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right)\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})},
$$

from which we derive

$$
\frac{1}{2}\frac{d}{dt}\left(t^3\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right)+t^3\|(\sqrt{\rho}f_{vx})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \qquad (2.29)
$$

$$
\leq C_Tt^2\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}+C_T\left(\|f(t)\|_{L^2_{k_0}(\mathbb{T}\times\mathbb{R})}+\left(1+\frac{1}{\eta^{\frac{1}{2}}}\right)\|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right.
$$

$$
\left.+t\|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}+\eta^{\frac{1}{2}}t^{\frac{3}{2}}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right)t^{\frac{3}{2}}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}.
$$

To obtain the dissipation $t^2\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}$, it follows by (2.22) and (2.25) that

$$
\begin{aligned}
&\frac{d}{dt}(f_v f_x) + v(f_v f_x)_x + |f_x|^2 \\
&= -f_x\big(\rho(u-v)f - \rho f_v\big)_{vv} - f_v\big(\rho(u-v)f - \rho f_v\big)_{vx}.
\end{aligned}
\tag{2.30}
$$

Multiplying (2.30) by $t^2\langle v\rangle^{2(k_0-2)}$ and integrating the resulting equation by parts over $\mathbb{T}\times\mathbb{R}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(t^2\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2(k_0-2)}(f_v f_x)(x,v,t)\,dvdx\right) + t^2\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \\
&\leq C_T\Big(\|f(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \|f_v(t)\|_{L^2_{k_0-1}(\mathbb{T}\times\mathbb{R})} + t^{\frac{1}{2}}\|f_{vv}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\Big) \\
&\quad\times\Big(t\|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + t^{\frac{3}{2}}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\Big).
\end{aligned}
\tag{2.31}
$$

Introducing

$$
\begin{cases}
L^\eta(t) := \dfrac{1}{2}t\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \dfrac{1}{2}\eta^3 t^3\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \\
\qquad\qquad + \eta^2 t^2\displaystyle\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{2(k_0-2)}(f_v f_x)(x,v,t)\,dvdx, \\
D^\eta(t) := \|(\sqrt{\rho}f_{vx})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \eta^3 t^3\|(\sqrt{\rho}f_{vx})(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \\
\qquad\qquad + \eta^2 t^2\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})},
\end{cases}
$$

for the constant $\eta\in(0,1)$ to be determined, we obtain from (2.24), (2.29), and (2.31) that

$$
\begin{aligned}
&\frac{d}{dt}L^\eta(t) + D^\eta(t) \\
&\leq C_T\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + t\|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \\
&\quad + C_T\eta^3 t^2\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + C_T\Big(\|f(t)\|_{L^2_{k_0}(\mathbb{T}\times\mathbb{R})} + \|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \\
&\quad + \eta^{\frac{3}{2}}t\|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \eta^2 t^{\frac{3}{2}}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\Big)\eta^{\frac{3}{2}}t^{\frac{3}{2}}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \\
&\quad + C_T\Big(\|f(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \|f_v(t)\|_{L^2_{k_0-1}(\mathbb{T}\times\mathbb{R})} + t^{\frac{1}{2}}\|f_{vv}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\Big) \\
&\quad \times\eta^{\frac{1}{2}}\Big(\eta t\|f_x(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \eta^{\frac{3}{2}}t^{\frac{3}{2}}\|f_{vx}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\Big).
\end{aligned}
\tag{2.32}
$$

It is easy to verify

$$
\begin{cases}
L^{\eta}(t) \geq \left(1 - \dfrac{1}{2}\eta^{\frac{1}{2}}\right)\left(t\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \eta^3 t^3 \|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right), \\
D^{\eta}(t) \geq \min\{1,\rho_T\}\left(t\|f_{vv}(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \eta^3 t^3 \|f_{vx}(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right. \\
\qquad\qquad\left. + \eta^2 t^2 \|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right),
\end{cases}
\tag{2.33}
$$

where the constant $\rho_T > 0$ is given by (2.15a). Then for any $\eta \in (0,1)$, one has by (2.32)-(2.33) that

$$
\frac{d}{dt}L^{\eta}(t) + D^{\eta}(t)
$$

$$
\leq C_T \|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \frac{C}{\eta}\|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\sqrt{D^{\eta}(t)} + C_T \eta D^{\eta}(t)
$$

$$
+ C_T\left(\|f(t)\|_{L^2_{k_0}(\mathbb{T}\times\mathbb{R})} + \|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \eta^{\frac{1}{2}}\sqrt{D^{\eta}(t)}\right)\sqrt{D^{\eta}(t)}
$$

$$
+ C_T\left(\|f(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + \|f_v(t)\|_{L^2_{k_0-1}(\mathbb{T}\times\mathbb{R})} + \sqrt{D^{\eta}(t)}\right)\eta^{\frac{1}{2}}\sqrt{D^{\eta}(t)}
$$

$$
\leq C_T \eta^{\frac{1}{2}} D^{\eta}(t) + C_T\left(1 + \frac{1}{\eta^{\frac{5}{2}}}\right)\|f_v(t)\|^2_{L^2_{k_0-1}(\mathbb{T}\times\mathbb{R})} + C_T\|f(t)\|^2_{L^2_{k_0}(\mathbb{T}\times\mathbb{R})}.
\tag{2.34}
$$

Since it follows $L^{\eta}(0) = 0$, we choose $\eta = \min\{1, 1/4C_T^2\}$ and make use of (2.15) and (2.32)-(2.34) to get

$$
\sup_{t\in[0,T]}\left(t\|f_v(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + t^3\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right)
\tag{2.35}
$$

$$
+ \int_0^T \left(t\|f_{vv}(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + t^3\|f_{vx}(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} + t^2\|f_x(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}\right)dt \leq C_T.
$$

Finally, differentiating (2.22) thrice with respect to v and multiplying the resulting equation by t^3, we have

$$
\left(t^3 f_{vvv}\right)_t + v\left(t^3 f_{vvv}\right)_x + \left(\rho(u-v)t^3 f_{vvv} - \rho\left(t^3 f_{vvv}\right)_v\right)_v
$$

$$
= \left(3t^2 f_{vv} + 3t^3\rho f_{vv} + 3t^3 f_{xv}\right)_v.
$$

Similarly to (2.22)-(2.26), one can show

$$
\sup_{t\in[0,T]} t^3\|f_{vvv}(t)\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \leq C_T \int_0^T \left\|\left(3t^2 f_{vv} + 3t^3\rho f_{vv} + 3t^3 f_{xv}\right)(t)\right\|^2_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}dt
$$

$$\leq C_T \int_0^T \left(t^2 \|f_{vv}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}^2 + t^3 \|f_{xv}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}^2 \right) dt \leq C_T. \qquad (2.36)$$

By the Gagliardo-Nirenberg inequality, we also have

$$\sup_{t\in[0,T]} t\|f_{vv}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})}$$

$$\leq C \sup_{t\in[0,T]} \left(t^{\frac{1}{2}}\|f_v(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \right)^{\frac{1}{2}} \left(t^{\frac{3}{2}}\|f_{vvv}(t)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \right)^{\frac{1}{2}} \leq C_T. \qquad (2.37)$$

The combination of (2.35)-(2.37) leads to (2.21). The proof of Lemma 2.6 is complete. $\qquad\square$

3 Uniqueness

Proposition 3.1. *Let $T > 0$, and (ρ_i, u_i, f_i), $i = 1, 2$, be two weak solutions to the IVP (1.1)-(1.4) on $[0,T]$ in the sense of Definition 1.1 satisfying (1.5),(1.9), and (1.17). Then it holds $(\rho_1, u_1, f_1) = (\rho_2, u_2, f_2)$ a.e. in $(x, v, t) \in \mathbb{T} \times \mathbb{R} \times [0, T]$.*

Proof. We first estimate $(\rho_1 - \rho_2)$. It is easy to verify

$$(\rho_1 - \rho_2)_t + \left((\rho_1 - \rho_2) u_1 \right)_x = -\rho_{2x}(u_1 - u_2) - \rho_2(u_1 - u_2)_x. \qquad (3.1)$$

Multiplying (3.1) by $(\rho_1 - \rho_2)$ and integrating the resulting equation by parts over $\mathbb{T}$, we have

$$\frac{1}{2}\frac{d}{dt}\|(\rho_1 - \rho_2)(t)\|_{L^2(\mathbb{T})}^2$$

$$\leq C_T \left(\|(u_1 - u_2)_x(t)\|_{L^2(\mathbb{T})} + \|(\sqrt{\rho_1}(u_1 - u_2))(t)\|_{L^2(\mathbb{T})} \right)$$

$$\times \|(\rho_1 - \rho_2)(t)\|_{L^2(\mathbb{T})} + C_T \|u_{2x}(t)\|_{H^1(\mathbb{T})} \|(\rho_1 - \rho_2)(t)\|_{L^2(\mathbb{T})}^2, \qquad (3.2)$$

where one has used (1.17) and the fact

$$\|(u_1 - u_2)(t)\|_{L^\infty(\mathbb{T})}$$

$$\leq \|(u_1 - u_2)_x(t)\|_{L^2(\mathbb{T})} + \frac{\|(\sqrt{\rho_1}(u_1 - u_2))(t)\|_{L^2(\mathbb{T})}}{\|\rho_0\|_{L^1(\mathbb{T})}^{\frac{1}{2}}}, \quad \forall t \in [0, T]. \qquad (3.3)$$

By (3.2), we obtain

$$\frac{d}{dt}\|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})}$$

$$\leq C_T\Big(\|(u_1-u_2)_x(t)\|_{L^2(\mathbb{T})}+\|\big(\sqrt{\rho_1}(u_1-u_2)\big)(t)\|_{L^2(\mathbb{T})}\Big)$$

$$+\|u_2(t)\|_{H^2(\mathbb{T})}\|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})},$$

which together with the Grönwall inequality leads to

$$\|(\rho_1-\rho_2)(t)\|_{L^2(\mathbb{T})}$$

$$\leq C_T t^{\frac{1}{2}}\Big(\sup_{\tau\in[0,t]}\|\big(\sqrt{\rho_1}(u_1-u_2)\big)(\tau)\|_{L^2(\mathbb{T})}+\|(u_1-u_2)_x\|_{L^2(0,t;L^2(\mathbb{T}))}\Big). \qquad (3.4)$$

To estimate (u_1-u_2), noticing that it holds

$$\rho_1(u_1-u_2)_t+\rho_1 u_1(u_1-u_2)_x-\big[\mu(\rho_1)(u_1-u_2)_x\big]_x+\rho_1\int_{\mathbb{R}}f_1 dv(u_1-u_2)$$

$$=-\big(\rho_1^\gamma-\rho_2^\gamma\big)_x-(\rho_1-\rho_2)(u_{2t}+u_2 u_{2x})-\rho_1(u_1-u_2)u_{2x}+\big[(\mu(\rho_1)-\mu(\rho_2))u_{2x}\big]_x$$

$$+\int_{\mathbb{R}}(\rho_1-\rho_2)(v-u_2)f_1 dv+\int_{\mathbb{R}}\rho_2(v-u_2)(f_1-f_2)dv, \qquad (3.5)$$

we multiply (3.5) by (u_1-u_2) and integrate the resulting equation by parts over $\mathbb{T}\times[0,t]$ to derive

$$\frac{1}{2}\|\big(\sqrt{\rho_1}(u_1-u_2)\big)(t)\|^2_{L^2(\mathbb{T})}+\int_0^t\left\|\Big(\sqrt{\mu(\rho_1)}(u_1-u_2)_x\Big)(\tau)\right\|^2_{L^2(\mathbb{T})}d\tau$$

$$+\int_0^t\|\big(\sqrt{\rho_1 f_1}(u_1-u_2)\big)(\tau)\|^2_{L^2(\mathbb{T}\times\mathbb{R})}d\tau=\sum_{i=1}^4 I_i^3, \qquad (3.6)$$

where $I_i^3, i=1,\dots,4$, are given by

$$I_1^3=\int_0^t\int_{\mathbb{T}}\big((\rho_1^\gamma-\rho_2^\gamma)(u_1-u_2)_x\big)(x,\tau)dxd\tau,$$

$$I_2^3=-\int_0^t\int_{\mathbb{T}}\big((\rho_1-\rho_2)(u_{2t}+u_2 u_{2x})(u_1-u_2)+\rho_1 u_{2x}|u_1-u_2|^2\big)(x,\tau)dxd\tau,$$

$$I_3^3=\int_0^t\int_{\mathbb{T}}\big(-(\mu(\rho_1)-\mu(\rho_2))u_{2x}(u_1-u_2)_x+(\rho_1-\rho_2)(u_1-u_2)(v-u_2)f_1\big)(x,\tau)dxd\tau,$$

$$I_4^3=\int_0^t\int_{\mathbb{T}\times\mathbb{R}}\big(\rho_2(v-u_2)(f_1-f_2)(u_1-u_2)\big)(x,v,\tau)dxdvd\tau.$$

By (1.9) and the Young inequality, one has

$$I_1^3 \le C \int_0^t \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})}^2 d\tau + \frac{1}{100} \int_0^t \|(u_1-u_2)_x(\tau)\|_{L^2(\mathbb{T})}^2 d\tau.$$

For the term I_2^3, it follows from (1.17) and (3.3) that

$$\begin{aligned}
I_2^3 \le C_T \int_0^t &\left(1+\|u_{2t}(\tau)\|_{L^2(\mathbb{T})}^2 + \|u_2(\tau)\|_{H^2(\mathbb{T})}\right) \\
&\times \left(\|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})}^2 + \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})}^2\right) d\tau \\
&+ \frac{1}{100} \int_0^t \|(u_1-u_2)_x(\tau)\|_{L^2(\mathbb{T})}^2 d\tau.
\end{aligned}$$

Due to (1.9), (1.17), and (3.3), it also holds

$$\begin{aligned}
I_3^3 \le C_T \int_0^t &\left(1+\|u_2(\tau)\|_{H^2(\mathbb{T})}\right) \left(\|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})}^2 + \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})}^2\right) d\tau \\
&+ \frac{1}{100} \int_0^t \|(u_1-u_2)_x(\tau)\|_{L^2(\mathbb{T})}^2 d\tau.
\end{aligned}$$

Finally, the term I_4^3 can be estimated as follows:

$$\begin{aligned}
I_4^3 &\le C \int_0^t \left(1+\|u_2(\tau)\|_{L^\infty(\mathbb{T})}\right) \|(u_1-u_2)(\tau)\|_{L^\infty(\mathbb{T})} \|(f_1-f_2)(\tau)\|_{L_1^1(\mathbb{T}\times\mathbb{R})} d\tau \\
&\le C_T \int_0^t \left(\|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})}^2 d\tau + \|(f_1-f_2)(\tau)\|_{L_1^1(\mathbb{T}\times\mathbb{R})}^2\right) d\tau \\
&+ \frac{1}{100} \int_0^t \|(u_1-u_2)_x(\tau)\|_{L^2(\mathbb{T})}^2 d\tau.
\end{aligned}$$

Substituting the above estimates of I_i^3 $(i=1,2,3,4)$ into (3.6), we get

$$\begin{aligned}
\sup_{\tau \in [0,t]} &\|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})}^2 + \int_0^t \|(u_1-u_2)_x(\tau)\|_{L^2(\mathbb{T})}^2 d\tau \\
&\le \int_0^t \left(1+\|u_{2t}(\tau)\|_{L^2(\mathbb{T})}^2 + \|u_2(\tau)\|_{H^2(\mathbb{T})}\right) \\
&\qquad \times \left(\|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})}^2 + \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})}^2\right) d\tau \\
&\qquad + \int_0^t \|(f_1-f_2)(\tau)\|_{L_1^1(\mathbb{T}\times\mathbb{R})}^2 d\tau.
\end{aligned} \tag{3.7}$$

We are ready to estimate (f_1-f_2). It can be verified by (1.1c) that

$$(f_1-f_2)_t+v(f_1-f_2)_x+\big[\rho_1(u_1-v)(f_1-f_2)-\rho_1(f_1-f_2)_v\big]_v$$
$$=-\rho_1(u_1-u_2)f_{2v}-(\rho_1-\rho_2)(u_2-v)f_{2v}+(\rho_1-\rho_2)f_2+(\rho_1-\rho_2)f_{2vv}. \qquad (3.8)$$

Applying Lemma 3.1 below to the Eq. (3.8) for $b(s)=b^\delta(s)=(s^2+\delta^2)^{\frac{1}{2}}$ satisfying

$$\big|(b^\delta(s))'\big|=\Big|s(s^2+\delta^2)^{-\frac{1}{2}}\Big|\le 1, \quad (b^\delta(s))''=\delta^2(s^2+\delta^2)^{-\frac{3}{2}}\ge 0$$

for any $\delta>0$, we obtain

$$\big(b^\delta(f_1-f_2)\big)_t+v\big(b^\delta(f_1-f_2)\big)_x+\big[\rho_1(u_1-v)b_\delta(f_1-f_2)\big]_v-\rho\big[b^\delta(f_1-f_2)\big]_{vv}$$
$$\le\big|-\rho_1(u_1-u_2)f_{2v}-(\rho_1-\rho_2)(u_2-v)f_{2v}+(\rho_1-\rho_2)f_2+(\rho_1-\rho_2)f_{2vv}\big|. \qquad (3.9)$$

Since it follows $b^\delta(s)\to|s|$ as $\delta\to 0$, we choose the text function $\psi^\delta(x,v,t)\in\mathcal{D}(\mathbb{T}\times\mathbb{R}\times(0,T))$ in (3.9) satisfying $\psi^\delta(x,v,t)\to\langle v\rangle$ as $\delta\to 0$ and employ the dominated convergence theorem to have

$$\|(f_1-f_2)(t)\|_{L_1^1(\mathbb{T}\times\mathbb{R})}\le\sum_{i=1}^{3}I_i^4, \qquad (3.10)$$

where I_i^4, $i=1,2,3$, are given by

$$I_1^4=\int_0^t\|(\rho_1(u_1-v)(f_1-f_2))(\tau)\|_{L^1(\mathbb{T}\times\mathbb{R})}d\tau,$$

$$I_2^4=\int_0^t\|(\rho_1(u_1-u_2)f_{2v}+(\rho_1-\rho_2)(u_2-v)f_{2v}-(\rho_1-\rho_2)f_2)(\tau)\|_{L_1^1(\mathbb{T}\times\mathbb{R})}d\tau,$$

$$I_3^4=\int_0^t\|((\rho_1-\rho_2)f_{2vv})(\tau)\|_{L_1^1(\mathbb{T}\times\mathbb{R})}d\tau.$$

It follows from (1.9a) that

$$I_1^4\le\rho_++\int_0^t\big(\|u_1(\tau)\|_{L^\infty(\mathbb{T})}+1\big)\|(f_1-f_2)(\tau)\|_{L_1^1(\mathbb{T}\times\mathbb{R})}d\tau.$$

Due to (1.17), (3.4) and the fact $k_0-2>\frac{1}{2}$, we have

$$I_2^4\le\rho_+^{\frac{1}{2}}\int_0^t\|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})}\left(\int_{\mathbb{R}}\frac{1}{\langle v\rangle^{2(k_0-1)}}dv\right)^{\frac{1}{2}}\|f_{2v}(\tau)\|_{L_{k_0}^2(\mathbb{T}\times\mathbb{R})}d\tau$$

$$+ \int_0^t \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})} \big(1+\|u_2(\tau)\|_{L^\infty(\mathbb{T})}\big)$$

$$\times \left(\int_{\mathbb{R}} \frac{1}{\langle v \rangle^{2(k_0-2)}} dv \right)^{\frac{1}{2}} \|f_{2v}(\tau)\|_{L^2_{k_0}(\mathbb{T}\times\mathbb{R})} d\tau$$

$$+ \int_0^t \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})} \left(\int_{\mathbb{R}} \frac{1}{\langle v \rangle^{2(k_0-1)}} dv \right)^{\frac{1}{2}} \|f_2(\tau)\|_{L^2_{k_0}(\mathbb{T}\times\mathbb{R})} d\tau$$

$$\leq C_T \sup_{\tau\in[0,t]} \|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})} + C_T \|(u_1-u_2)_x\|_{L^2(0,t;L^2(\mathbb{T}))}.$$

For the key term I_3^4, we make use of (1.17d), (3.4) and the fact $k_0-3 > \frac{1}{2}$ to obtain

$$I_3^4 \leq \int_0^t \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})} \left(\int_{\mathbb{R}} \frac{1}{\langle v \rangle^{2(k_0-3)}} dv \right)^{\frac{1}{2}} \|f_{2vv}(\tau)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} d\tau$$

$$\leq \left(\sup_{\tau\in[0,t]} \frac{1}{\tau^{\frac{1}{2}}} \|(\rho_1-\rho_2)(\tau)\|_{L^2(\mathbb{T})} \right) \left(\sup_{\tau\in[0,t]} \tau \|f_{2vv}(\tau)\|_{L^2_{k_0-2}(\mathbb{T}\times\mathbb{R})} \right) \int_0^t \frac{1}{\tau^{\frac{1}{2}}} d\tau$$

$$\leq C_T \sup_{\tau\in[0,t]} \|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})} + C_T \|(u_1-u_2)_x\|_{L^2(0,t;L^2(\mathbb{T}))}.$$

Substituting the above estimates of I_i^4 $(i=1,2,3)$ into (3.10) and applying the Grönwall inequality, we have

$$\sup_{\tau\in[0,t]} \|(f_1-f_2)(\tau)\|_{L^1_1(\mathbb{T}\times\mathbb{R})}$$

$$\leq C_T \sup_{\tau\in[0,t]} \|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})} + C_T \|(u_1-u_2)_x\|_{L^2(0,t;L^2(\mathbb{T}))}, \tag{3.11}$$

which together with (3.4)-(3.5) leads to

$$\sup_{\tau\in[0,t]} \|(\sqrt{\rho_1}(u_1-u_2))(\tau)\|_{L^2(\mathbb{T})}^2 + \|(u_1-u_2)_x\|_{L^2(0,t;L^2(\mathbb{T}))}^2$$

$$\leq C_T \int_0^t \left(1+\|u_{2t}(\tau)\|_{L^2(\mathbb{T})}^2 + \|u_2(\tau)\|_{H^2(\mathbb{T})} \right)$$

$$\times \left(\sup_{\omega\in[0,\tau]} \|(\sqrt{\rho_1}(u_1-u_2))(\omega)\|_{L^2(\mathbb{T})}^2 + \|(u_1-u_2)_x\|_{L^2(0,\tau;L^2(\mathbb{T}))}^2 \right) d\tau. \tag{3.12}$$

The combination of (1.17), (3.4), (3.11)-(3.12), and the Grönwall inequality gives rise to $(\rho_1,u_1,f_1) = (\rho_2,u_2,f_2)$ a.e. in $\mathbb{T}\times\mathbb{R}\times[0,T]$. The proof of Proposition 3.1 is complete. $\qquad\square$

Similarly to [7], we can show the following result about the renormalized solutions for Fokker-Planck equations with variable coefficients:

Lemma 3.1. *For any $T > 0$, assume*

$$\begin{cases} f \in L^\infty(0,T;L^1_2 \cap L^\infty(\mathbb{T} \times \mathbb{R})), & f_v \in L^2(0,T;L^2(\mathbb{T} \times \mathbb{R})), \\ \rho \in L^\infty(0,T;L^\infty(\mathbb{T})), & u \in L^2(0,T;L^\infty(\mathbb{T})), \quad G \in L^1(0,T;L^1(\mathbb{T} \times \mathbb{R})). \end{cases}$$

If f solves the equation

$$f_t + v f_x + \big(\rho(u-v)f\big)_v - \rho f_{vv} = G \quad in \quad \mathcal{D}'(\mathbb{T} \times \mathbb{R} \times (0,T)),$$

then f is a renormalized solution, i.e., for any $b(s) \in C^2(\mathbb{R})$, it holds

$$\big(b(f)\big)_t + v\big(b(f)\big)_x + \big(\rho(u-v)b(f)\big)_v - \big(\rho b(f)\big)_{vv} + \rho\big(b(f) - b'(f)f\big)$$
$$+ b''(f)|f_v|^2 = b'(f)G \quad in \quad \mathcal{D}'(\mathbb{T} \times \mathbb{R} \times (0,T)).$$

4 Long time behavior

In this section, we study the large time behavior of global solutions to the IVP (1.1)-(1.4).

Proposition 4.1. *Let (ρ,u,f) be any global weak solution to the IVP (1.1)-(1.4) in the sense of Definition 1.1 satisfying (1.9)-(1.10) for any $T > 0$. Then under the assumptions of Theorem 1.1, it holds*

$$\lim_{t \to \infty} \Big(\|(\rho - \overline{\rho_0})(t)\|_{L^p(\mathbb{T})} + \|(\sqrt{\rho}(u-\overline{u}))(t)\|_{L^2(\mathbb{T})} \Big) = 0, \quad p \in [1,\infty), \qquad (4.1)$$

where $\overline{\rho_0}$ and $\overline{u}(t)$ are given by (1.13).

The proof of (4.1) is based on relative entropy estimates for the compressible Navier-Stokes equation (1.1a)-(1.1b), please refer to [27,28] for details.

Proposition 4.2. *Let (ρ,u,f) be any global weak solution to the IVP (1.1)-(1.4) in the sense of Definition 1.1 satisfying (1.9)-(1.10) for any $T > 0$. Then under the assumptions of Theorem 1.2, it holds*

$$\begin{cases} \rho(x,t) \geq \rho_-, & (x,t) \in \mathbb{T} \times [0,T], \\ \sup_{t \in [0,T]} \|\rho_x(t)\|_{L^2(\mathbb{T})} \leq C, \end{cases} \qquad (4.2)$$

where $\rho_- > 0$ and $C > 0$ are two constant independent of the time $T > 0$. Moreover, we have

$$\lim_{t\to\infty} \left(\|(\rho - \overline{\rho_0})(t)\|_{L^\infty(\mathbb{T})} + \|(u - u_c)(t)\|_{L^2(\mathbb{T})} + |(m_i - u_c)(t)| \right) = 0, \quad i = 1,2, \quad (4.3)$$

where the constants $\overline{\rho_0}$ and u_c are given by (1.12) and (1.13) respectively, and $m_i(t)$, $i = 1,2$, are defined by

$$m_1(t) := \int_{\mathbb{T}} \rho u(x,t) dx \left(\int_{\mathbb{T}} \rho_0(x) dx \right)^{-1}, \quad (4.4a)$$

$$m_2(t) := \int_{\mathbb{T}} nw(x,t) dx \left(\int_{\mathbb{T}} n_0(x) dx \right)^{-1}. \quad (4.4b)$$

Proof. Repeating the arguments as in [28] with few modifications, we can show the uniform estimates (4.2). To get (4.3), we first prove $|(m_1 - m_2)(t)|^2 \in L^1(\mathbb{R}_+)$. Indeed, a straightforward computation yields

$$\left\| \left[\sqrt{\rho}\left((u-v)\sqrt{f} - 2(\sqrt{f})_v\right) \right](t) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2$$

$$\geq \rho_- \left\| \left[(u-m_1)\sqrt{f} + (m_1-m_2)\sqrt{f} + (m_2-v)\sqrt{f} - 2(\sqrt{f})_v \right](t) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2$$

$$= \rho_- \left\| (\sqrt{f}(u-m_1))(t) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2 + \rho_- |(m_1-m_2)(t)|^2 \|f(t)\|_{L^1(\mathbb{T}\times\mathbb{R})}^{\frac{1}{2}}$$

$$+ \rho_- \left\| ((m_2-v)\sqrt{f} - 2(\sqrt{f})_v)(t) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2$$

$$+ 2\rho_-(m_1-m_2)(t) \int_{\mathbb{T}\times\mathbb{R}} (f(u-m_1))(x,v,t) dv dx$$

$$+ 2\rho_- \int_{\mathbb{T}\times\mathbb{R}} \left[\sqrt{f}(u-m_1)((m_2-v)\sqrt{f} - (\sqrt{f})_v) \right](x,v,t) dv dx,$$

which together with (2.1) and the fact $\|(u-m_1)(t)\|_{L^\infty(\mathbb{T})} \leq \|u_x(t)\|_{L^2(\mathbb{T})}$ gives rise to

$$\left\| \left[\sqrt{\rho}((u-v)\sqrt{f} - 2(\sqrt{f})_v) \right](t) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2$$

$$\geq -3\rho_- \|n_0\|_{L^1(\mathbb{T})} \|u_x(t)\|_{L^2(\mathbb{T})}^2 + \frac{\rho_- \|n_0\|_{L^1(\mathbb{T})}}{2} |(m_1-m_2)(t)|^2$$

$$+ \frac{\rho_-}{2} \left\| ((m_2-v)\sqrt{f} - 2(\sqrt{f})_v)(t) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2.$$

Therefore, we have

$$\int_0^\infty |(m_1-m_2)(t)|^2 dt \leq \frac{2(3\rho_- \|n_0\|_{L^1(\mathbb{T})} + 1)}{\|n_0\|_{L^1(\mathbb{T})} \rho_-}$$

$$\times \int_0^\infty \left(\|u_x(t)\|_{L^2(\mathbb{T})}^2 + \left\| \left[\sqrt{\rho}((u-v)\sqrt{f} - 2(\sqrt{f})_v) \right](t) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2 \right) dt < \infty. \quad (4.5)$$

Next, it can be verified by (1.1b)-(1.1c) that

$$\frac{1}{2}\frac{d}{dt}|(m_2 - m_1)(t)|^2$$
$$= C_1(m_2 - m_1)(t) \int_{\mathbb{T}\times\mathbb{R}} \left(\rho\sqrt{f}((u-v)\sqrt{f} - 2(\sqrt{f})_v) \right)(x,v,t) dv dx, \quad (4.6)$$

where the constant $C_1 > 0$ is given by

$$C_1 := \frac{\|\rho_0\|_{L^1(\mathbb{T})} + \|n_0\|_{L^1(\mathbb{T})}}{\|\rho_0\|_{L^1(\mathbb{T})}\|n_0\|_{L^1(\mathbb{T})}}.$$

For any $0 \leq s \leq t < \infty$, we integrate (4.6) over $[s,t]$ to derive

$$\frac{1}{2}|(m_1 - m_2)(t)|^2$$
$$\leq \frac{1}{2}|(m_1 - m_2)(s)|^2 + \rho_+^{\frac{1}{2}}\|n_0\|_{L^1(\mathbb{T})}^{\frac{1}{2}} C_1 \int_s^t \left(|(m_1 - m_2)(\tau)|^2 \right.$$
$$\left. + \left\| (\sqrt{\rho}((u-v)\sqrt{f} - 2(\sqrt{f})_v))(\tau) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2 \right) d\tau, \quad (4.7)$$

and thence integrate (4.7) with respect to s over $[t-1,t]$ to have

$$\frac{1}{2}|(m_1 - m_2)(t)|^2$$
$$\leq \frac{1}{2}\int_{t-1}^t |(m_1 - m_2)(s)|^2 ds + \rho_+^{\frac{1}{2}}\|n_0\|_{L^1(\mathbb{T})}^{\frac{1}{2}} C_1 \int_{t-1}^t \left(|(m_1 - m_2)(\tau)|^2 \right.$$
$$\left. + \left\| (\sqrt{\rho}((u-v)\sqrt{f} - 2(\sqrt{f})_v))(\tau) \right\|_{L^2(\mathbb{T}\times\mathbb{R})}^2 \right) d\tau. \quad (4.8)$$

Due to (1.6) and (4.5), the right-hand side of (4.8) tends to 0 as $t \to \infty$. Thus, it follows

$$\lim_{t\to\infty} |(m_1 - m_2)(t)|^2 = 0. \quad (4.9)$$

Finally, it holds by the conservation laws (1.10) that

$$m_1(t) \int_{\mathbb{T}} \rho_0(x) dx + m_2(t) \int_{\mathbb{T}} n_0(x) dx = \int_{\mathbb{T}} (m_0 + j_0)(x) dx,$$

which leads to

$$|(m_1 - m_2)(t)| = \frac{\int_{\mathbb{T}} (\rho_0 + n_0)(x) dx}{\int_{\mathbb{T}} \rho_0(x) dx} |(m_2 - u_c)(t)|, \quad (4.10)$$

where the constant u_c is given by (1.13). One deduces by (4.9)-(4.10) that

$$\lim_{t\to\infty} \left(|(m_1 - u_c)| + |(m_2 - u_c)(t)| \right)$$
$$\leq \lim_{t\to\infty} \left(|(m_1 - m_2)(t)| + 2|(m_2 - u_c)(t)| \right) = 0. \tag{4.11}$$

Combining (4.1)-(4.2) and (4.11) together, we have

$$\lim_{t\to\infty} \|(u - u_c)(t)\|_{L^2(\mathbb{T})}$$
$$\leq \lim_{t\to\infty} \left(\frac{1}{\rho_-^{\frac{1}{2}}} \|(\sqrt{\rho}(u - m_1))(t)\|_{L^2(\mathbb{T})} + |(m_1 - u_c)(t)| \right) = 0.$$

It follows from (1.9a), (4.1), and the Gagliardo-Nirenberg inequality that

$$\lim_{t\to\infty} \|(\rho - \overline{\rho_0})(t)\|_{L^\infty(\mathbb{T})} \leq \sqrt{2} \lim_{t\to\infty} \|\rho_x(t)\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \|(\rho - \overline{\rho_0})(t)\|_{L^2(\mathbb{T})}^{\frac{1}{2}} = 0.$$

The proof of Proposition 4.2 is complete. $\qquad\square$

Furthermore, the time convergence of the distribution function f to the global Maxwellian $M_{\overline{n_0},u_c}(v)$ is proved based on the ideas inspired by [4] and the compactness tool developed in [2].

Proposition 4.3. *Let (ρ,u,f) be any global weak solution to the IVP (1.1)-(1.4) in the sense of Definition 1.1 satisfying (1.9)-(1.10) for any $T>0$. Then under the assumptions of Theorem 1.2, it holds*

$$\begin{cases} \lim_{t\to\infty} \|(f - M_{\overline{n_0},u_c})(t)\|_{L^1(\mathbb{T}\times\mathbb{R})} = 0, & \text{(4.12a)} \\ \lim_{t\to\infty} \left(\|(n - \overline{n_0})(t)\|_{L^1(\mathbb{T})} + \|(n(w - u_c))(t)\|_{L^1(\mathbb{T})} \right) = 0, & \text{(4.12b)} \end{cases}$$

where the global Maxwellian $M_{\overline{n_0},u_c}(v)$ is denoted through (1.14), and the constants u_c and $\overline{n_0}$ are given by (1.13) and (1.15) respectively.

Proof. For any $(x,v,t) \in \mathbb{T} \times \mathbb{R} \times (0,1)$ and $s \geq 0$, define

$$\rho^s(t,x) := \rho(x,t+s), \quad u^s(x,t) := u(x,t+s), \quad f^s(x,v,t) := f(x,v,t+s),$$

so that it satisfies

$$(f^s)_t + v(f^s)_x + \rho^s \left[(u^s - v)f^s - (f^s)_v \right]_v = 0. \tag{4.13}$$

By (1.6), (1.9a), (1.10), and (4.2), we have

$$
\begin{cases}
\|\rho^s(t)\|_{L^1(\mathbb{T})} = \|\rho_0\|_{L^1(\mathbb{T})}, \quad \|f^s(t)\|_{L^1(\mathbb{T}\times\mathbb{R})} = \|n_0\|_{L^1(\mathbb{T})}, \\
0 < \rho_- \le \rho^s(x,t) \le \rho_+ < \infty, \\
\sup_{s\ge 0}\Big(\|f^s\|_{L^\infty(0,1;L_2^1(\mathbb{T}\times\mathbb{R}))} + \|f^s\log f^s\|_{L^\infty(0,1;L^1(\mathbb{T}\times\mathbb{R}))} \\
\quad + \|u^s\|_{L^2(0,1;H^1(\mathbb{T}))} + \|(u^s-v)\sqrt{f^s} - 2(\sqrt{f^s})_v\|_{L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))} \Big) < \infty.
\end{cases}
\tag{4.14}
$$

Thence, one deduces from (2.4d), (4.1), (4.3), and (4.14) as $s\to\infty$ that

$$
\begin{cases}
\rho^s \overset{*}{\rightharpoonup} \overline{\rho_0} & \text{in } L^\infty(0,1;L^\infty(\mathbb{T})), \\
u^s \to u_c & \text{in } L^2(0,1;H^1(\mathbb{T})) \hookrightarrow L^2(0,1;L^\infty(\mathbb{T})).
\end{cases}
\tag{4.15}
$$

By (4.14) and the fact

$$
-4\int_0^1\int_{\mathbb{T}\times\mathbb{R}}\big((u^s-v)\sqrt{f^s}(\sqrt{f^s})_v\big)(x,v,t)\,dvdxdt = -2\|n_0\|_{L^1(\mathbb{T})},
$$

we have

$$
\begin{aligned}
&\big\|(u^s-v)\sqrt{f^s} - 2(\sqrt{f^s})_v\big\|_{L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))}^2 \\
&= \big\|(u^s-v)\sqrt{f^s}\big\|_{L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))}^2 + 4\big\|(\sqrt{f^s})_v\big\|_{L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))}^2 - 2\|n_0\|_{L^1(\mathbb{T})},
\end{aligned}
$$

which implies

$$
\sup_{s\ge 0}\big\|(\sqrt{f^s})_v\big\|_{L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))} < \infty.
\tag{4.16}
$$

We are ready to establish the strong $L^1(\mathbb{T}\times\mathbb{R})$-compactness of $f^s(x,v,t)$ for a.e. $t\in(0,1)$. It follows from (4.13) and Lemma 3.1 for any $\delta\in(0,1)$ and $s\in(0,\infty)$ that

$$
\begin{aligned}
&(\sqrt{f^s+\delta})_t + v(\sqrt{f^s+\delta})_x + [\rho^s(u^s-v)(\sqrt{f^s+\delta})]_v - [\rho^s(\sqrt{f^s+\delta})_v]_v \\
&\quad + \frac{\rho^s(f^s+2\delta)}{2\sqrt{f^s+\delta}} - \frac{\rho^s|(f^s)_v|^2}{4(f^s+\delta)^{\frac{3}{2}}} = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^3).
\end{aligned}
\tag{4.17}
$$

In order to employ Lemma 4.1 below, we prolong (ρ^s,u^s,f^s) to be zero outside $(0,1)\times\mathbb{R}\times(0,1)$:

$$
\begin{aligned}
&\big(\widetilde{\rho^s}(x,t), \widetilde{u^s}(x,t), \widetilde{f^s}(x,v,t)\big) \\
&:= \begin{cases}
(\rho^s(x,t), u^s(x,t), f^s(x,v,t)), & (x,v,t)\in(0,1)\times\mathbb{R}\times(0,1), \\
0, & (x,v,t)\notin(0,1)\times\mathbb{R}\times(0,1)
\end{cases}
\end{aligned}
$$

for any $(x,v,t) \in \mathbb{R}^3$. According to the uniform estimates (4.14) and Dunford-Pettis criterion, there is a limit f^∞ satisfying $f^\infty \in L^\infty(0,1;L^1_2(\mathbb{T} \times \mathbb{R}))$ and $(\sqrt{f^\infty})_v \in L^2(0,1;L^2(\mathbb{T} \times \mathbb{R}))$ such that it holds as $s \to \infty$,

$$\widetilde{f^s} \rightharpoonup \widetilde{f^\infty} \qquad \text{in } L^1(\mathbb{R}^2) \quad \text{a.e.} \quad t \in \mathbb{R}, \qquad (4.18a)$$

$$\left(\sqrt{\widetilde{f^s}}\right)_v \rightharpoonup \left(\sqrt{\widetilde{f^\infty}}\right)_v \quad \text{in } L^2(\mathbb{R}^3), \qquad (4.18b)$$

where $\widetilde{f^\infty}$ is given by

$$\widetilde{f^\infty}(x,v,t) := \begin{cases} f^\infty(x,v,t), & (x,v,t) \in (0,1) \times \mathbb{R} \times (0,1), \\ 0, & (x,v,t) \notin (0,1) \times \mathbb{R} \times (0,1). \end{cases}$$

Denote

$$\vartheta^{s,\delta,r}(t,x,v) := \mathbf{1}_{|v| \le r} \sqrt{\widetilde{f^s}(x,v,t) + \delta}, \quad s > 0, \quad \delta \in (0,1), \quad r > 0, \qquad (4.19)$$

where $\mathbf{1}_{|v| \le r} \in \mathcal{D}(\mathbb{R})$ stands for the cut-off function satisfying $\mathbf{1}_{|v| \le r} = 1$ for $|v| \le r$ and $\mathbf{1}_{|v| \le r} = 0$ for $|v| \ge 2r$. Obviously, $\vartheta^{s,\delta,r}$ has compact support in $(-2r,2r) \times (0,1) \times (0,1)$, and we obtain by (4.17) that

$$(\vartheta^{s,\delta,r})_t + v(\vartheta^{s,\delta,r})_v = (I - \partial_{tt}^2 - \partial_{xx}^2)^{\frac{9}{22}}(I - \partial_{vv}^2)^{\frac{31}{22}}(g_1^{s,\delta} + g_2^{s,\delta}) \quad \text{in } \mathcal{D}'(\mathbb{R}^3) \qquad (4.20)$$

with

$$\begin{cases} g_1^{s,\delta} = (I - \partial_{tt}^2 - \partial_{xx}^2)^{-\frac{9}{22}}(I - \partial_{vv}^2)^{-\frac{31}{22}}\partial_{vv}^2(\widetilde{\rho^s}\vartheta^{s,\delta,r}), \\ g_2^{s,\delta} = (I - \partial_{tt}^2 - \partial_{xx}^2)^{-\frac{9}{22}}(I - \partial_{vv}^2)^{-\frac{31}{22}} \\ \qquad \times \left(-(\mathbf{1}_{|v| \le r})_v \widetilde{\rho^s}(\widetilde{u^s} - v)\sqrt{\widetilde{f^s} + \delta} - (\mathbf{1}_{|v| \le r})_{vv}\widetilde{\rho^s}\sqrt{\widetilde{f^s} + \delta} \right. \\ \qquad \left. -2(\mathbf{1}_{|v| \le r})_v \widetilde{\rho^s}\left(\sqrt{\widetilde{f^s} + \delta}\right)_v - \frac{\mathbf{1}_{|v| \le r}\widetilde{\rho^s}(\widetilde{f^s} + 2\delta)}{2\sqrt{\widetilde{f^s} + \delta}} + \frac{\mathbf{1}_{|v| \le r}\widetilde{\rho^s}|(\widetilde{f^s})_v|^2}{4(\widetilde{f^s} + \delta)^{\frac{3}{2}}} \right). \end{cases}$$

Note that the operator $(I - \partial_{tt}^2 - \partial_{xx}^2 - \partial_{vv}^2)^{-\frac{s}{2}}$ maps $L^1(\mathbb{R}^3)$ to $L^{p,\infty}(\mathbb{R}^3)$ (i.e., the weak $L^p(\mathbb{R}^3)$ space) for $p \in (1,\infty)$ and $s \in (0,3)$ satisfying $1 - \frac{1}{p} = \frac{s}{3}$, and it holds $\|g\|_{L^q(\mathbb{R}^3)} \le C\|g\|_{L^{p,\infty}(\mathbb{R}^3)}$ for $1 \le q < p$ provided that g has compact support. Thus,

one deduces by (4.14) that

$$
\begin{aligned}
\big\|g_1^{s,\delta}\big\|_{L^{\frac{5}{4}}(\mathbb{R}^3)} &\leq C_r \big\|(I-\partial_{tt}^2-\partial_{xx}^2)^{-\frac{9}{22}}(I-\partial_{vv}^2)^{-\frac{9}{22}}(\widetilde{\rho^s}\vartheta^{s,\delta,r})\big\|_{L^{\frac{5}{4}}(\mathbb{R}^3)} \\
&\leq C_r \big\|(I-\partial_{tt}^2-\partial_{xx}^2-\partial_{vv}^2)^{-\frac{9}{22}}(\widetilde{\rho^s}\vartheta^{s,\delta,r})\big\|_{L^{\frac{11}{8},\infty}(\mathbb{R}^3)} \\
&\leq C_r \big\|\widetilde{\rho^s}\vartheta^{s,\delta,r}\big\|_{L^1(\mathbb{R}^3)} \\
&\leq C_r \|\rho^s\|_{L^\infty(0,1;L^\infty(\mathbb{T}))}\|f^s\|_{L^\infty(0,1;L^1(\mathbb{T}\times\mathbb{R}))}^{\frac{1}{2}} \leq C_r, \qquad (4.21)
\end{aligned}
$$

where $C_r > 0$ denotes a sufficiently large constant only dependent of $r > 0$. Similarly, it follows by (4.14) and (4.16) that

$$
\begin{aligned}
\big\|g_2^{s,\delta}\big\|_{L^{\frac{5}{4}}(\mathbb{R}^3)} &\leq C_r \|\rho^s\|_{L^\infty(0,1;L^\infty(\mathbb{T}))}\big(\|u^s\|_{L^2(0,1;L^\infty(\mathbb{T}))}+1\big) \\
&\quad \times \Big(\|f^s\|_{L^\infty(0,1;L^1(\mathbb{T}\times\mathbb{R}))}^{\frac{1}{2}}+\|(\sqrt{f^s})_v\|_{L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))} \\
&\qquad +\delta^{-\frac{1}{2}}\|(\sqrt{f^s})_v\|_{L^2(0,1;L^2(\mathbb{T}\times\mathbb{R}))}^2\Big) \leq C_r. \qquad (4.22)
\end{aligned}
$$

Due to the uniform estimate (4.16), the sequence $\vartheta^{s,\delta,r}$ given by (4.20) is locally strongly compact with respect to v in $L^{\frac{5}{4}}(\mathbb{R}^3)$. Thus, we make use of Lemma 4.1 below, (4.14), (4.16), (4.18a), and (4.21)-(4.22) to derive

$$
\vartheta^{s,\delta,r} \to \mathbf{1}_{|v|\leq r}\sqrt{\widetilde{f^\infty}+\delta} \quad \text{in} \quad L^{\frac{5}{4}}_{loc}(\mathbb{R}^3), \quad \delta\in(0,1), \quad r>0, \quad \text{as} \quad s\to\infty,
$$

which implies for any $\delta\in(0,1)$ and $r>0$ that

$$
\mathbf{1}_{|v|\leq r}\sqrt{f^s+\delta} \to \mathbf{1}_{|v|\leq r}\sqrt{f^\infty+\delta} \quad \text{in} \quad L^{\frac{5}{4}}(\mathbb{T}\times\mathbb{R}\times(0,1)), \quad \text{as} \quad s\to\infty. \qquad (4.23)
$$

In addition, we have

$$
\begin{aligned}
&\big\|\sqrt{f^s}-\sqrt{f^\infty}\big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))} \\
&\leq \big\|\mathbf{1}_{|v|\leq r}(\sqrt{f^s+\delta}-\sqrt{f^\infty+\delta})\big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))} + \big\|\sqrt{f^s}-\mathbf{1}_{|v|\leq r}\sqrt{f^s+\delta}\big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))} \\
&\quad +\big\|\mathbf{1}_{|v|\leq r}\sqrt{f^\infty+\delta}-\sqrt{f^\infty}\big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))}. \qquad (4.24)
\end{aligned}
$$

To control the right-hand side of (4.24), we deduce by (4.14) and (4.18)-(4.19) that

$$
\begin{aligned}
&\big\|\sqrt{f^s}-\mathbf{1}_{|v|\leq r}\sqrt{f^s+\delta}\big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))} \\
&= \big\|(1-\mathbf{1}_{|v|\leq r})\sqrt{f^s}+\mathbf{1}_{|v|\leq r}(\sqrt{f^s}-\sqrt{f^s+\delta})\big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))}
\end{aligned}
$$

$$\leq \frac{1}{r} \big\| \langle v \rangle \sqrt{f^s} \big\|_{L^1(0,1;L^1(\mathbb{T}\times\mathbb{R}))} + \Big\| \mathbf{1}_{|v|\leq r} \frac{\delta}{\sqrt{f^s}+\sqrt{f^s+\delta}} \Big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))}$$

$$\leq \frac{1}{r} \left(\int_{\mathbb{R}} \frac{1}{\langle v \rangle^2} dv \right)^{\frac{1}{2}} \|f^s\|_{L^\infty(0,1;L^1_2(\mathbb{T}\times\mathbb{R}))}^{\frac{1}{2}} + C_r\sqrt{\delta} \leq C_r\sqrt{\delta}+\frac{C_2}{r}, \tag{4.25}$$

where $C_2>0$ stands for a constant uniformly in s,δ, and r, and $C_r>0$ is a constant only dependent of $r>0$. Similarly, it holds

$$\big\| \sqrt{f^\infty} - \mathbf{1}_{|v|\leq r}\sqrt{f^\infty+\delta} \big\|_{L^1(\mathbb{T}\times\mathbb{R}\times(0,1))} \leq C_r\sqrt{\delta}+\frac{C_2}{r}. \tag{4.26}$$

Substituting (4.25)-(4.26) into (4.24) and using (4.23), we obtain

$$\lim_{s\to\infty} \Big\| \sqrt{\widetilde{f^s}} - \sqrt{\widetilde{f^\infty}} \Big\|_{L^1(\mathbb{R}^3)}$$

$$\leq \lim_{r\to\infty}\lim_{\delta\to0}\lim_{s\to\infty} \left(C_2 r^{\frac{1}{5}} \big\| \mathbf{1}_{|v|\leq r}(\sqrt{f^s+\delta}-\sqrt{f^\infty+\delta}) \big\|_{L^{\frac{5}{4}}(\mathbb{R}^3)} + C_r\sqrt{\delta}+\frac{C_2}{r} \right) = 0,$$

which gives rise to

$$f^s \to f^\infty \quad \text{a.e. in } \mathbb{T}\times\mathbb{R}\times(0,1), \quad \text{as} \quad s\to\infty. \tag{4.27}$$

By virtue of (4.18), (4.27), and [6, p. 468], we have

$$f^s(x,v,t) \to f^\infty(x,v,t) \quad \text{in } L^1(\mathbb{T}\times\mathbb{R}), \quad \text{a.e.} \quad t\in(0,1), \quad \text{as} \quad s\to\infty. \tag{4.28}$$

We are going to show $f^\infty(x,v,t)=M_{\overline{n_0},u_c}(v)$. By (1.6), (1.9a), and (1.10b), we have

$$\lim_{t\to\infty} \|\rho^s(u^s-v)f^s-\rho^s(f^s)_v\|_{L^1(0,1;L^1(\mathbb{T}\times\mathbb{R}))}$$

$$= \lim_{t\to\infty} \|\rho(u-v)f-\rho f_v\|_{L^1(s,s+1;L^1(\mathbb{T}\times\mathbb{R}))}$$

$$\leq \rho_+^{\frac{1}{2}} \|f_0\|_{L^1(\mathbb{T}\times\mathbb{R})}^{\frac{1}{2}} \lim_{t\to\infty} \big\| \sqrt{\rho}((u-v)\sqrt{f}-(\sqrt{f})_v) \big\|_{L^2(s,s+1;L^2(\mathbb{T}\times\mathbb{R}))} = 0. \tag{4.29}$$

Due to (4.13), (4.15), (4.18), and (4.29), the limit f^∞ satisfies

$$(f^\infty)_v - (u_c-v)f^\infty = e^{-\frac{|v-u_c|^2}{2}} \left(e^{\frac{|v-u_c|^2}{2}} f^\infty \right)_v = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}\times\mathbb{R}\times(0,1)), \tag{4.30a}$$

$$(f^\infty)_t + v(f^\infty)_x = 0 \qquad\qquad\qquad\qquad\quad \text{in } \mathcal{D}'(\mathbb{T}\times\mathbb{R}\times(0,1)). \tag{4.30b}$$

With the help of (4.30a), we deduce that exists a function

$$g^\infty = g^\infty(x,t) \in L^\infty\big(0,1;L^1(\mathbb{T})\big)$$

satisfying

$$g^\infty(x,t) := e^{\frac{|v-u_c|^2}{2}} f^\infty(x,v,t). \tag{4.31}$$

And by virtue of (4.30b) and (4.31), we have for any $\phi \in \mathcal{D}(\mathbb{T} \times (0,1))$ and $\chi \in \mathcal{D}(\mathbb{R})$ that

$$\int_0^1 \int_{\mathbb{T} \times \mathbb{R}} g^\infty(x,t)\big(\phi_t(x,t) + v\phi_x(x,t)\big)\chi(v)dvdxdt = 0. \tag{4.32}$$

By a density argument, (4.32) indeed holds for any $\chi \in \mathcal{S}(\mathbb{R})$. Thus, we choose $\chi(v) = e^{-|v|^2}$ and $\chi(v) = ve^{-|v|^2}$ in (4.32) to derive

$$g_t^\infty(x,t) = g_x^\infty(x,t) = 0 \quad \text{in } \mathcal{D}\big(\mathbb{T} \times (0,1)\big),$$

which together with (1.10b) shows $g^\infty = \frac{\overline{n_0}}{\sqrt{2\pi}}$. Thus, we conclude that the unique formula of f^∞ is $M_{\overline{n_0},u_c}(v)$.

Finally, we claim (4.12a). Indeed, if $f^s(t)$ does not converge to $M_{\overline{n_0},u_c}$ in $L^1(\mathbb{T} \times \mathbb{R})$ for any $t \in (0,1)$ as $s \to \infty$, then there are a constant $\delta > 0$, a time $t_0 \in (0,1)$, and a sufficiently large subsequence s_j such that we have

$$\big\|(f^{s_j} - M_{\overline{n_0},u_c})(t_0)\big\|_{L^1(\mathbb{T} \times \mathbb{R})} > \delta,$$

which contradicts (4.28). In addition, it holds

$$\lim_{t \to \infty} \big\|(n - \overline{n_0})(t)\big\|_{L^1(\mathbb{T})} \leq \lim_{t \to \infty} \big\|(f - M_{\overline{n_0},u_c})(t)\big\|_{L^1(\mathbb{T} \times \mathbb{R})} = 0. \tag{4.33}$$

One concludes from (1.7)-(1.6) and (4.12a) that

$$\lim_{t \to \infty} \big\|(nw - \overline{n_0}u_c)(t)\big\|_{L^1(\mathbb{T})}$$
$$\leq \lim_{t \to \infty} \big\||v|^2(f + M_{\overline{n_0},u_c})(t)\big\|_{L^1(\mathbb{T} \times \mathbb{R})}^{\frac{1}{2}} \big\|(f - M_{\overline{n_0},u_c})(t)\big\|_{L^1(\mathbb{T} \times \mathbb{R})}^{\frac{1}{2}} = 0. \tag{4.34}$$

The combination of (4.12a) and (4.33)-(4.34) gives rise to

$$\lim_{t \to \infty} \big\|(n|w - u_c|)(t)\big\|_{L^1(\mathbb{T})}$$
$$\leq \lim_{t \to \infty} \Big(\big\|(nw - \overline{n_0}u_c)(t)\big\|_{L^1(\mathbb{T})} + |u_c|\big\|(n - \overline{n_0})(t)\big\|_{L^1(\mathbb{T})}\Big) = 0.$$

The proof of Proposition 4.3 is complete.   □

We need the following lemma, which implies the strong compactness in all variables of the distribution function for the Vlasov-Fokker-Planck equation (1.1c) provided that this sequence is strongly compact with respect to velocity variable.

Lemma 4.1 ([2])**.** *Let $d \geq 1, 1 < p < \infty$, $\alpha_1 \geq 0$, $0 \leq \alpha_2 < 1$, and the nonnegative sequence $f^n \in L^p(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R})$ uniformly in $n \geq 0$, be locally strongly compact with respect to v in $L^p(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R})$, and satisfy*

$$(f^n)_t + v \cdot \nabla_x f^n = (I - \Delta_{t,x})^{\frac{\alpha_2}{2}} (I - \Delta_v)^{\frac{\alpha_1}{2}} g^n \quad in \ \ \mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R})$$

for $g^n \in L^p(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R})$ uniformly in $n \geq 0$. Then, the sequence f^n is locally strongly compact in $L^p(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R})$ (in all variables).

5   Proofs of main results

Proof of Theorem 1.1. Step 1: Construction of approximate sequence.
We regularize the initial data as follows:

$$\begin{aligned}
&\left(\rho_0^\varepsilon(x), u_0^\varepsilon(x), f_0^\varepsilon(x,v)\right) \\
&:= \left(J_1^\varepsilon * \rho_0(x) + \varepsilon, \frac{J_1^\varepsilon * (\frac{m_0}{\sqrt{\rho_0}})(x)}{\sqrt{J_1^\varepsilon * \rho_0(x) + \varepsilon}}, J_1^\varepsilon * J_2^\varepsilon * (f_0 \mathbf{1}_{|v| \leq \varepsilon^{-1}})(x,v)\right), \quad \varepsilon \in (0,1), \quad (5.1)
\end{aligned}$$

where $J_1^\varepsilon(x)$ and $J_2^\varepsilon(v)$ are the Friedrichs mollifier with respect to the variables x and v, and $\mathbf{1}_{|v| \leq \varepsilon^{-1}} \in \mathcal{D}(\mathbb{R})$ is the cut-off function satisfying $\mathbf{1}_{|v| \leq \varepsilon^{-1}} = 1$ for $|v| \leq r$ and $\mathbf{1}_{|v| \leq \varepsilon^{-1}} = 0$ for $|v| \geq 2\varepsilon^{-1}$. It is easy to verify that $(\rho_0^\varepsilon, \rho_0^\varepsilon u_0^\varepsilon, f_0^\varepsilon)$ satisfies the assumptions (1.8) of Theorem 1.1 uniformly in $\varepsilon \in (0,1)$.

We are able to obtain a local regular solution $(\rho^\varepsilon, u^\varepsilon, f^\varepsilon)$ for any $\varepsilon \in (0,1)$ to the IVP (1.1)-(1.4) for the initial data (5.1) in a standard way based on linearization techniques [28], the details are omitted. Then, by Remark 1.3, we can extend the local regular solution $(\rho^\varepsilon, u^\varepsilon, f^\varepsilon)$ to a global one.

Step 2: Compactness and convergence.
Let $T > 0$ be any given time. It follows from the a-priori estimates established in Lemmas 2.2-2.3 and standard arguments as in [34] that there exist a limit (ρ, u, f) such that up to a subsequence (still labeled by $(\rho^\varepsilon, u^\varepsilon, f^\varepsilon)$ here and in what

follows), it holds as $\varepsilon \to 0$ that

$$
\begin{cases}
(\rho^\varepsilon, f^\varepsilon) \xrightarrow{*} (\rho, f) & \text{in } L^\infty(0,T;L^\infty(\mathbb{T})) \times L^\infty(0,T;L^\infty(\mathbb{T} \times \mathbb{R})), & \text{(5.2a)} \\
u^\varepsilon \rightharpoonup u & \text{in } L^2(0,T;H^1(\mathbb{T})), & \text{(5.2b)} \\
\rho^\varepsilon \to \rho & \text{in } C([0,T];L^p_{weak}(\mathbb{T})) \cap C([0,T];H^{-1}(\mathbb{T})), \quad p \in (1,\infty), & \text{(5.2c)} \\
\rho^\varepsilon u^\varepsilon \to \rho u & \text{in } C([0,T];L^2_{weak}(\mathbb{T})) \cap C([0,T];H^{-1}(\mathbb{T})), & \text{(5.2d)} \\
\rho^\varepsilon |u^\varepsilon|^2 \to \rho |u|^2 & \text{in } \mathcal{D}(\mathbb{T} \times (0,T)), & \text{(5.2e)}
\end{cases}
$$

where $C([0,T];X_{weak})$ is the space of weak topology on X defined by

$$
C([0,T];X_{weak}) := \left\{ g : [0,T] \to X \mid <g,\phi>_{X,X^*} \in C([0,T]) \text{ for any } \phi \in X^* \right\}.
$$

In addition, it follows from Lemmas 2.2-2.3, (1.1c), and the average compactness lemma (for instance, refer to [26]) for any $\chi(v) \in \mathcal{D}(\mathbb{R})$ satisfying $|\chi(v)| \leq C(1+|v|)$ that

$$
\int_{\mathbb{R}} f^\varepsilon \chi(v) dv \to \int_{\mathbb{R}} f \chi(v) dv \quad \text{in } L^1(0,T;L^1(\mathbb{T})), \quad \text{as} \quad \varepsilon \to 0, \tag{5.3}
$$

which together with (2.5c) implies as $\varepsilon \to 0$ that

$$
\begin{cases}
n^\varepsilon := \displaystyle\int_{\mathbb{R}} f^\varepsilon dv \to n = \int_{\mathbb{R}} f dv & \text{in } L^q(0,T;L^p(\mathbb{T})), \\
& \qquad q \in [1,\infty), \quad p \in [1,4), \\
n^\varepsilon w^\varepsilon := \displaystyle\int_{\mathbb{R}} f^\varepsilon v dv \to nw = \int_{\mathbb{R}} fv dv & \text{in } L^q(0,T;L^p(\mathbb{T})), \\
& \qquad q \in [1,\infty), \quad p \in [1,2).
\end{cases} \tag{5.4}
$$

Then, denote by $\mathcal{G}^\varepsilon$ the effect viscous flux as $\mathcal{G}^\varepsilon := (\rho^\varepsilon)^\gamma - \mu(\rho^\varepsilon)(u^\varepsilon)_x$. By Lemma 2.4, there is a limit $\mathcal{G} \in L^p(0,T;L^\infty(\mathbb{T}))$ for $p \in (1,\frac{4}{3})$ satisfying as $\varepsilon \to 0$ that

$$
\mathcal{G}^\varepsilon \xrightarrow{*} \mathcal{G} \quad \text{in } L^p(0,T;L^\infty(\mathbb{T})).
$$

Similarly to the arguments used in [23], one can prove the following inequality holds for a.e. $t \in [0,T]$:

$$
\int_{\mathbb{T}} |\rho^\varepsilon - \rho|^2 (x,t) dx \leq C_T \int_0^T \left(1 + \|\mathcal{G}(t)\|_{L^\infty(\mathbb{T})}\right) \int_{\mathbb{T}} |\rho^\varepsilon - \rho|^2 (x,t) dx dt, \tag{5.5}
$$

the details are omitted here. The combination of the Grönwall inequality, (2.4e), and (5.5) gives rise to

$$
\rho^\varepsilon \to \rho \quad \text{in } L^p(0,T;L^p(\mathbb{T})), \quad \text{as} \quad \varepsilon \to 0, \quad p \in [1,\infty). \tag{5.6}
$$

Owing to Lemma 2.3, (5.2a), (5.6), and integration by parts, we obtain

$$\sqrt{\rho^\varepsilon}(f^\varepsilon)_v \rightharpoonup \sqrt{\rho}f_v \quad \text{in } L^2\big(0,T;L^2(\mathbb{T}\times\mathbb{R})\big), \quad \text{as} \quad \varepsilon\to 0. \tag{5.7}$$

By virtue of (5.2)-(5.6) and (5.7), one can show that the limit (ρ,u,f) indeed satisfies the Eqs. (1.1) in the sense of distributions.

Step 3: The properties (1.5)-(1.6) and (1.9)-(1.11).

For any nonnegative function $\phi\in\mathcal{D}(0,T)$ and constant $R>0$, one deduces by (2.4b) and (5.2c) that

$$\int_0^T \phi(t)\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^3 1_{|v|\leq R}f(x,v,t)dvdxdt$$

$$\leq \limsup_{\varepsilon\to 0}\int_0^T \phi(t)\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^3 f^\varepsilon(x,v,t)dvdxdt.$$

We take the limit as $R\to\infty$ and apply the monotone convergence theorem to get

$$f\in L^\infty\big(0,T;L^1_3(\mathbb{T}\times\mathbb{R})\big). \tag{5.8}$$

Therefore, it follows by Lemmas 2.2-2.3, (5.2), and (5.8) that (ρ,u,f) satisfies (1.9). By (1.1c) and (2.4), it holds for any $p\in(1,3]$ that

$$\sup_{\varepsilon>0}\int_0^T \left\|\langle v\rangle^{\frac{3}{p}-1}(f^\varepsilon)_t(t)\right\|^2_{W^{-2,p}(\mathbb{T}\times\mathbb{R})}dt$$

$$=\sup_{\varepsilon>0}\int_0^T \left|\sup_{\|\varphi\|_{W^{2,\frac{p}{p-1}}(\mathbb{T}\times\mathbb{R})}\leq 1}\int_{\mathbb{T}\times\mathbb{R}}\left[\langle v\rangle^{\frac{3}{p}-1}vf^\varepsilon\varphi_x+\rho^\varepsilon(u^\varepsilon-v)f^\varepsilon\left(\langle v\rangle^{\frac{3}{p}-1}\varphi\right)_v\right.\right.$$

$$\left.\left.+\rho^\varepsilon f^\varepsilon\left(\langle v\rangle^{\frac{3}{p}-1}\varphi\right)_{vv}\right](x,v,t)dvdx\right|^2 dt$$

$$\leq C\sup_{\varepsilon>0}\left(\big(1+\|u^\varepsilon\|_{L^2(0,T;L^\infty(\mathbb{T}))}\big)\|f^\varepsilon\|^{\frac{1}{p}}_{L^\infty(0,T;L^1_3(\mathbb{T}\times\mathbb{R}))}\|f^\varepsilon\|^{1-\frac{1}{p}}_{L^\infty(0,T;L^\infty(\mathbb{T}\times\mathbb{R}))}\right)^2<\infty,$$

which together with [33, Lemma C.1] yields for any $p\in(1,3)$ and $q\in[0,\frac{3}{p}-1]$ that

$$\langle v\rangle^q f^\varepsilon\to\langle v\rangle^q f \quad \text{in } C\big([0,T];L^p_{weak}(\mathbb{T}\times\mathbb{R})\big), \quad \text{as} \quad \varepsilon\to 0. \tag{5.9}$$

Since we have $1\in L^2(\mathbb{T})$ and $\langle v\rangle^{-\frac{2}{5}}\in L^5(\mathbb{T}\times\mathbb{R})$, one deduces by (5.2c) and (5.9) for $(p,q)=(\frac{5}{4},\frac{2}{5})$ that

$$\begin{cases} \displaystyle\lim_{\varepsilon\to 0}\int_{\mathbb{T}}\rho^\varepsilon(x,t)dx=\int_{\mathbb{T}}\rho(x,t)dx, & \forall t\in[0,T], \\[2mm] \displaystyle\lim_{\varepsilon\to 0}\int_{\mathbb{T}\times\mathbb{R}}\langle v\rangle^{-\frac{2}{5}}\langle v\rangle^{\frac{2}{5}}f^\varepsilon(x,v,t)dvdx=\int_{\mathbb{T}\times\mathbb{R}}f(x,v,t)dvdx, & \forall t\in[0,T]. \end{cases} \tag{5.10}$$

Similarly, it follows from $v\langle v\rangle^{-\frac{7}{5}}\in L^5(\mathbb{T}\times\mathbb{R})$, (5.2d), and (5.9) for $(p,q)=(\frac{5}{4},\frac{7}{5})$ that

$$
\begin{cases}
\lim\limits_{\varepsilon\to 0}\displaystyle\int_{\mathbb{T}}\rho^\varepsilon u^\varepsilon(x,t)dx=\int_{\mathbb{T}}\rho u(x,t)dx, & \forall t\in[0,T], \\
\lim\limits_{\varepsilon\to 0}\displaystyle\int_{\mathbb{T}\times\mathbb{R}}v\langle v\rangle^{-\frac{7}{5}}\langle v\rangle^{\frac{7}{5}}f^\varepsilon(x,v,t)dvdx=\int_{\mathbb{T}\times\mathbb{R}}vf(x,v,t)dvdx, & \forall t\in[0,T].
\end{cases}
\tag{5.11}
$$

According to (2.1)-(2.2) and (5.10)-(5.11), the conservation laws (1.10) hold for any $t\in[0,T]$. We conclude from (5.2)-(5.7), (5.9), and the lower semi-continuity of the limit (ρ,u,f) that the entropy inequality (1.6) holds for a.e. $t\in[0,T]$. By Proposition 4.1, one has the long time behavior (1.11). The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Let $(\rho_0,\frac{m_0}{\rho_0},f_0)$ satisfy (1.16), and $(\rho_0^\varepsilon,u_0^\varepsilon,f_0^\varepsilon)$ be given through (5.1) for $\varepsilon\in(0,1)$. Similarly, we can obtain an approximate sequence $(\rho^\varepsilon,u^\varepsilon,f^\varepsilon)$ and show its convergence to a global weak solution (ρ,u,f) to the IVP (1.1)-(1.4) in the sense of Definition 1.1 as $\varepsilon\to 0$. Thus, it follows from the a-priori estimates established in Lemmas 2.5-2.6 that (ρ,u,f) satisfies the further properties (1.17). In addition, this weak solution (ρ,u,f) is unique due to Proposition 3.1, and the time convergence (1.18) can be derived by Propositions 4.2-4.3. The proof of Theorem 1.2 is complete. □

Acknowledgments

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