Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **1**, No. 1, pp. 107-124 (2009)
February 2009

Posteriori Error Estimation for an Interior Penalty Discontinuous Galerkin Method for Maxwell's Equations in Cold Plasma

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Received 17 November 2008; Accepted (in revised version) 2 December 2008

Available online 16 February 2009

Abstract. In this paper, we develop a residual-based a posteriori error estimator for the time-dependent Maxwell's equations in the cold plasma. Here we consider a semi-discrete interior penalty discontinuous Galerkin (DG) method for solving the governing equations. We provide both the upper bound and lower bound analysis for the error estimator. This is the first posteriori error analysis carried out for the Maxwell's equations in dispersive media.

AMS subject classifications: 65N30, 35L15, 78-08

Key words: Posteriori error estimator, Maxwell's equations, cold plasma, discontinuous Galerkin method.

1 Introduction

Dispersive electromagnetic media are those materials with wavelength dependent physical parameters (such as permittivity). Examples of dispersive media include human tissue, soil, snow, ice, plasma, optical fibers and radar-absorbing materials. Hence, the study of wave interaction with dispersive media is very important to our daily life.

In recent years, there is a growing interest in the finite element modeling and analysis of Maxwell's equations (see books [10,17,30] and references cited therein). However, most work is restricted to the discussion of simple medium such as air in the free

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space. Work on dispersive media is still very limited. In 2001, Jiao and Jin [22] initiated the application of time-domain finite element method (TDFEM) for the dispersive media. Then in 2004, Lu et al. [28] developed a discontinuous Galerkin (DG) method for solving Maxwell's equations in dispersive media. In 2005, Banks and Browning [5] considered a Debye medium problem solved by finite element method. Unfortunately, no any error analysis has been carried out for Maxwell's equations in dispersive media. Since 2006, we carried out some a priori error analysis of TDFEM for dispersive media [21,23–27]. In this paper, we initiate our effort on developing a posteriori error estimation for Maxwell's equations in dispersive media. For simplicity, we only consider the cold plasma model in this paper. Analysis of other dispersive media can be carried out similarly.

A posteriori error estimation plays an important role in adaptive finite element methods (FEMs), and the literature on this is vast (see books [1,3,4,34], reviews [11,13] and references cited therein). However, to our best knowledge, there are only dozens of papers devoted to the study of posteriori error estimation for Maxwell's equations [6–9,16,19,20,29,32,35]. No any paper has discussed the posteriori error estimation for dispersive media yet. Here we want to fill the gap by carrying out the first posteriori error analysis for the Maxwell's equations in dispersive media.

The governing equations that describe electromagnetic wave propagation in isotropic nonmagnetized cold electron plasma are [25]

$$\epsilon_0 \boldsymbol{E}_{tt} + \nabla \times (\mu_0^{-1} \nabla \times \boldsymbol{E}) + \epsilon_0 \omega_p^2 \boldsymbol{E} - \nu \boldsymbol{J}(\boldsymbol{E}) = 0, \tag{1.1}$$

where $\boldsymbol{E}$ is the electric field, ϵ_0 is the permittivity of free space, μ_0 is the permeability of free space, ω_p is the plasma frequency, $\nu \geq 0$ is the electron-neutral collision frequency, and the polarization current density $\boldsymbol{J}$ is represented as

$$\boldsymbol{J}(\boldsymbol{x}, t; \boldsymbol{E}) \equiv \boldsymbol{J}(\boldsymbol{E}) = \epsilon_0 \omega_p^2 \int_0^t e^{-\nu(t-s)} \boldsymbol{E}(\boldsymbol{x}, s) ds. \tag{1.2}$$

Moreover, we assume that the boundary of Ω is a perfect conductor so that

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{1.3}$$

where $\boldsymbol{n}$ denotes the unit outward normal of $\partial\Omega$. Furthermore, we assume that the initial conditions for (1.1) are given as

$$\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{E}_t(\boldsymbol{x}, 0) = \boldsymbol{E}_1(\boldsymbol{x}), \tag{1.4}$$

where $\boldsymbol{E}_0(\boldsymbol{x})$ and $\boldsymbol{E}_1(\boldsymbol{x})$ are some given functions.

The rest of the paper is organized as follows. In Section 2, we describe the semi-discrete DG formulation for the plasma model. In Section 3, we construct our a posteriori error estimator and present the main result. Detailed proof of the error estimator is given in Section 3.1. Here we adopted many ideas and techniques from [20] originally developed for Maxwell's equations in the simple medium. Then in Section 4, we prove some lower bounds for the local error estimators. We conclude the paper in Section 5. In this paper, C (sometimes with subindex) denotes a generic constant which is independent of both the time step τ and the finite element mesh size h.

2 Semi-discrete DG formulation

We consider a shape-regular mesh T_h that partitions the domain Ω into disjoint tetrahedral elements $\{K\}$, such that $\overline{\Omega} = \bigcup_{K\in T_h} K$. We denote the diameter of K by h_K, and the mesh size h by $h = \max_{K\in T_h} h_K$. Furthermore, we denote the set of all interior faces by F_h^I, the set of all boundary faces by F_h^B, and the set of all faces by $F_h = F_h^I \cup F_h^B$.

We assume that the finite element space is given by

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in L^2(\Omega)^3 : \ \boldsymbol{v}|_K \in (P_l(K))^3, \ K \in T_h\}, \ \ l \geq 1, \tag{2.1}$$

where $P_l(K)$ denotes the space of polynomials of total degree at most l on K. We want to remark that all results below hold true for a mesh of affine hexahedral elements, in which case on each element K, $\boldsymbol{v}|_K$ is a polynomial of degree at most l in each variable.

To simplify the presentation, we rewrite the governing equation (1.1) as:

$$\boldsymbol{E}_{tt} + \nabla\times\nabla\times\boldsymbol{E} + \boldsymbol{E} - \boldsymbol{J}(\boldsymbol{E}) = \boldsymbol{j}, \tag{2.2}$$

where we assumed that there exists a given external source field $\boldsymbol{j} \in L^2(\Omega)^3$, and all physical parameters $\epsilon_0 = \mu_0 = \nu = \omega_p = 1$, in which case the polarization current density $\boldsymbol{J}$ of (1.2) becomes as

$$\boldsymbol{J}(\boldsymbol{x}, t; \boldsymbol{E}) \equiv \boldsymbol{J}(\boldsymbol{E}) = \int_0^t e^{-(t-s)}\boldsymbol{E}(\boldsymbol{x}, s)ds. \tag{2.3}$$

We can form a semi-discrete DG scheme for (2.2): For any $t \in (0, T)$, find $\boldsymbol{E}^h(\cdot, t) \in \boldsymbol{V}_h$ such that

$$(\boldsymbol{E}^h_{tt}, \boldsymbol{\phi}) + a_h(\boldsymbol{E}^h, \boldsymbol{\phi}) - (\boldsymbol{J}(\boldsymbol{E}^h), \boldsymbol{\phi}) = (\boldsymbol{j}, \boldsymbol{\phi}), \ \ \forall\, \boldsymbol{\phi} \in \boldsymbol{V}_h, \tag{2.4}$$

subject to the initial conditions

$$\boldsymbol{E}^h|_{t=0} = \Pi_2\boldsymbol{E}_0, \quad \boldsymbol{E}^h_t|_{t=0} = \Pi_2\boldsymbol{E}_1, \tag{2.5}$$

where Π_2 denotes the standard L_2- projection onto $\boldsymbol{V}_h$. Moreover, the bilinear form a_h is defined on $\boldsymbol{V}_h \times \boldsymbol{V}_h$ as

$$\begin{aligned} a_h(\boldsymbol{u}, \boldsymbol{v}) = &\sum_{K\in T_h}\int_K (\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{v})d\boldsymbol{x} - \sum_{F\in F_h}\int_F [[\boldsymbol{u}]]_T\cdot\{\{\nabla\times\boldsymbol{v}\}\}dA \\ &- \sum_{F\in F_h}\int_F [[\boldsymbol{v}]]_T\cdot\{\{\nabla\times\boldsymbol{u}\}\}dA + \sum_{F\in F_h}\int_F a[[\boldsymbol{u}]]_T\cdot[[\boldsymbol{v}]]_T dA. \end{aligned}$$

Here $[[\boldsymbol{v}]]$ and $\{\{\boldsymbol{v}\}\}$ are the standard notation for the tangential jumps and averages of $\boldsymbol{v}$ across an interior face $F = \partial K^+ \cap \partial K^-$ between two neighboring elements K^+ and K^-:

$$[[\boldsymbol{v}]]_T = \boldsymbol{n}^+\times\boldsymbol{v}^+ + \boldsymbol{n}^-\times\boldsymbol{v}^-, \quad \{\{\boldsymbol{v}\}\} = (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2, \tag{2.6}$$

where $\boldsymbol{v}^\pm$ denote the traces of $\boldsymbol{v}$ from within $K^\pm$, and $\boldsymbol{n}^\pm$ denote the unit outward normal vectors on the boundaries $\partial K^\pm$, respectively.

While on a boundary face $F = \partial K \cap \partial\Omega$, we define $[[\boldsymbol{v}]]_T = \boldsymbol{n} \times \boldsymbol{v}$ and $\{\{\boldsymbol{v}\}\} = \boldsymbol{v}$. Finally, a is a penalty function, which is defined on each face $F \in F_h$ as:

$$a|_F = \gamma \hbar^{-1},$$

where $\hbar|_F = \min\{h_{K^+}, h_{K^-}\}$ for an interior face $F = \partial K^+ \cap \partial K^-$, and $\hbar|_F = h_K$ for a boundary face $F = \partial K \cap \partial\Omega$. The penalty parameter γ is a positive constant.

Furthermore, we denote the space $\boldsymbol{V}(h) = H_0(\text{curl};\Omega) + \boldsymbol{V}_h$ and define the DG energy norm by

$$||\boldsymbol{v}||_h^2 = ||\boldsymbol{v}||_{0,\Omega}^2 + \sum_{K \in F_h} ||\nabla \times \boldsymbol{v}||_{0,K}^2 + \sum_{F \in F_h} ||a^{1/2}[[\boldsymbol{v}]]_T||_{0,F}^2.$$

In order to carry out the posteriori analysis, we introduce an auxiliary bilinear form $\tilde{a}_h$ on $\boldsymbol{V}(h) \times \boldsymbol{V}(h)$ defined as

$$\begin{aligned}\tilde{a}_h(\boldsymbol{u}, \boldsymbol{v}) = &\sum_{K \in T_h} \int_K (\nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{v}) d\boldsymbol{x} - \sum_{K \in T_h} \int_K \mathcal{L}(\boldsymbol{u}) \cdot (\nabla \times \boldsymbol{v}) dx \\ &- \sum_{K \in T_h} \int_K \mathcal{L}(\boldsymbol{v}) \cdot (\nabla \times \boldsymbol{u}) dx + \sum_{F \in F_h} \int_F a[[\boldsymbol{u}]]_T \cdot [[\boldsymbol{v}]]_T dA,\end{aligned}$$

where the lifting operator $\mathcal{L}(\boldsymbol{v}) \in \boldsymbol{V}_h$ for any $\boldsymbol{v} \in \boldsymbol{V}_h$ is defined by

$$\int_\Omega \mathcal{L}(\boldsymbol{v}) \cdot \boldsymbol{w} dx = \sum_{F \in F_h} \int_F [[\boldsymbol{v}]]_T \cdot \{\{\boldsymbol{w}\}\} dA \quad \forall\, \boldsymbol{w} \in \boldsymbol{V}_h. \tag{2.7}$$

Moreover, the lifting operator $\mathcal{L}(\boldsymbol{v})$ can be bounded as follows [20]

$$||\mathcal{L}(\boldsymbol{v})||_{0,\Omega}^2 \leq \alpha^{-1} C_{lift} \sum_{F \in F_h} ||a^{\frac{1}{2}}[[\boldsymbol{v}]]_T||_{0,F}^2. \tag{2.8}$$

Note that $\tilde{a}_h = a_h$ on $\boldsymbol{V}_h \times \boldsymbol{V}_h$ and $\tilde{a}_h = a$ on $H_0(\text{curl};\Omega) \times H_0(\text{curl};\Omega)$. Furthermore, we have

$$\tilde{a}_h(\boldsymbol{v}, \boldsymbol{v}) = ||\nabla \times \boldsymbol{v}||_0^2 + ||\boldsymbol{v}||_0^2 = ||\boldsymbol{v}||_h^2 \quad \forall\, \boldsymbol{v} \in H_0(\text{curl};\Omega). \tag{2.9}$$

Moreover, it is easy to prove [2, 20] that the bilinear form $\tilde{a}_h$ is both coercive and continuous on $\boldsymbol{V}(h)$, i.e.,

Lemma 2.1. *There exists some positive constant C_{cont} such that*

$$|\tilde{a}_h(\boldsymbol{u}, \boldsymbol{v})| \leq C_{cont} ||\boldsymbol{u}||_h ||\boldsymbol{v}||_h, \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h).$$

Furthermore, provided that $\gamma \geq \gamma_{min} > 0$, where γ_{min} depends only on the shape-regularity of the mesh and the approximation order l of $\boldsymbol{V}(h)$, we have

$$\tilde{a}_h(\boldsymbol{v}, \boldsymbol{v}) \geq C_{coer} ||\boldsymbol{v}||_h^2, \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}(h).$$

3 A posteriori error estimator

One of the main tools in posteriori error estimate is to find a conforming finite element function close to the discontinuous one. For this purpose, we define the conforming finite element space
$$\boldsymbol{V}_h^c = \boldsymbol{V}_h \cap H_0(\mathrm{curl};\Omega), \tag{3.1}$$
i.e., $\boldsymbol{V}_h^c$ is the second family of Nédélec element [31]. Moreover, we have the following approximation property [19,20].

Lemma 3.1. *For any $\boldsymbol{v}^h \in \boldsymbol{V}_h$, there exists a conforming approximation $\boldsymbol{v}_c^h \in \boldsymbol{V}_h^c$ such that*

$$\begin{aligned}
&\sum_{K\in T_h} ||\nabla\times(\boldsymbol{v}^h-\boldsymbol{v}_c^h)||_{0,K}^2 \le C_{app}\sum_{F\in F_h} h_F^{-1}||[[\boldsymbol{v}^h]]_T||_{0,F}^2,\\
&||\boldsymbol{v}^h-\boldsymbol{v}_c^h||_{0,\Omega}^2 \le C_{app}\sum_{F\in F_h} h_F||[[\boldsymbol{v}^h]]_T||_{0,F}^2,\\
&||\boldsymbol{v}^h-\boldsymbol{v}_c^h||_h^2 \le (2\alpha^{-1}C_{app}+1)\sum_{F\in F_h}||a^{\frac12}[[\boldsymbol{v}^h]]_T||_{0,F}^2,
\end{aligned}\tag{3.2}$$

where the constant $C_{app}>0$ depends only on the shape regularity of the mesh and the approximation order l.

To obtain the posterior error estimator, we need the following result.

Lemma 3.2. (i) [18, Lemma 2.4] *For any $\boldsymbol{w} \in H_0(curl;\Omega)$, there exists the regular decomposition*
$$\boldsymbol{w} = \boldsymbol{w}^0 + \nabla w^1, \tag{3.3}$$
where $\boldsymbol{w}^0 \in H_0(curl;\Omega)\cap H^1(\Omega)^3$ and $w^1 \in H_0^1(\Omega)$. Moreover, there is a constant $C_{hip}>0$ depending only on Ω such that

$$||\boldsymbol{w}^0||_1 \le C_{hip}||\boldsymbol{w}||_{curl}, \quad ||w^1||_1 \le C_{hip}||\boldsymbol{w}||_{curl}, \tag{3.4}$$

where and in the following we define the norm

$$||\boldsymbol{w}||_{curl} = \Big(||\boldsymbol{w}||_0^2 + ||\nabla\times\boldsymbol{w}||_0^2\Big)^{1/2}.$$

(ii) [7] *For any $\boldsymbol{w}^0 \in H_0(curl;\Omega)\cap H^1(\Omega)^3$, there exists the quasi-interpolation $\boldsymbol{w}_h^0 \in \boldsymbol{V}_h^c$ such that*

$$\begin{aligned}
&\sum_{K\in T_h}\Big(||\nabla\times(\boldsymbol{w}^0-\boldsymbol{w}_h^0)||_{0,K}^2 + h_K^{-2}||\boldsymbol{w}^0-\boldsymbol{w}_h^0||_{0,K}^2 + h_K^{-1}||\boldsymbol{w}^0-\boldsymbol{w}_h^0||_{0,\partial K}^2\Big)\\
&\le C_{bec}^2||\boldsymbol{w}^0||_1^2,
\end{aligned}\tag{3.5}$$

where the constant $C_{bec}>0$ depends only on the shape regularity of the mesh.

(iii) [14, Sec.I.A.3] *For any* $w^1 \in H_0^1(\Omega)$, *there exists a piecewise linear approximation* $w_h^1 \in H_0^1(\Omega)$ *such that*

$$\begin{aligned}&\sum_{K\in T_h}\Big(||\nabla(w^1-w_h^1)||_{0,K}^2+h_K^{-2}||w^1-w_h^1||_{0,K}^2+h_K^{-1}||w^1-w_h^1||_{0,\partial K}^2\Big)\\ &\le C_{cle}^2||w^1||_1^2,\end{aligned} \tag{3.6}$$

where the constant $C_{cle}>0$ *depending only on the shape regularity of the mesh.*

Before we state the main theorem, we first introduce some local error indicators. Let

$$\eta_{R_K}^2=h_K^2||\boldsymbol{j}_h-\boldsymbol{E}_{tt}^h-\nabla\times\nabla\times\boldsymbol{E}^h-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)||_{0,K}^2,$$

which measures the residual of the governing Maxwell's equations (2.2). We denote

$$\eta_{T_K}^2=\frac{1}{2}\sum_{F\in\partial K\backslash\Gamma}h_F||[[\nabla\times\boldsymbol{E}^h]]_T||_{0,F}^2$$

for the face residual about the jump of $\nabla\times\boldsymbol{E}^h$. To measure the tangential jumps of the approximate solution $\boldsymbol{E}^h$, we denote

$$\eta_{J_K}^2=\frac{1}{2}\sum_{F\in\partial K\backslash\Gamma}||a^{\frac{1}{2}}[[\boldsymbol{E}^h]]_T||_{0,F}^2.$$

Noting that $\nabla\cdot\nabla\times(\nabla\times\boldsymbol{E}^h)=0$, hence

$$\eta_{D_K}^2=h_K^2||\nabla\cdot(\boldsymbol{j}_h-\boldsymbol{E}_{tt}^h-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h))||_{0,K}^2$$

measures the error in the divergence of the governing Maxwell's equations (2.2). Furthermore, we denote

$$\eta_{N_K}^2=\frac{1}{2}\sum_{F\in\partial K\backslash\Gamma}h_K||[[\boldsymbol{j}_h-\boldsymbol{E}_{tt}^h-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)]]_N||_{0,F}^2$$

for measuring the normal jump of $\boldsymbol{j}_h-\boldsymbol{E}_{tt}^h-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)$ over the interior faces.

Similarly, we can define the following local estimators:

$$\begin{aligned}\eta_{R_K^t}^2&=h_K^2||(\boldsymbol{j}_h-\boldsymbol{E}_{tt}^h-\nabla\times\nabla\times\boldsymbol{E}^h-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h))_t||_{0,K}^2,\\ \eta_{T_K^t}^2&=\frac{1}{2}\sum_{F\in\partial K\backslash\Gamma}h_F||[[\nabla\times\boldsymbol{E}_t^h]]_T||_{0,F}^2,\\ \eta_{J_K^t}^2&=\frac{1}{2}\sum_{F\in\partial K\backslash\Gamma}||a^{\frac{1}{2}}[[\boldsymbol{E}_t^h]]_T||_{0,F}^2,\\ \eta_{D_K^t}^2&=h_K^2||\nabla\cdot(\boldsymbol{j}_h-\boldsymbol{E}_{tt}^h-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h))_t||_{0,K}^2,\\ \eta_{N_K^t}^2&=\frac{1}{2}\sum_{F\in\partial K\backslash\Gamma}h_K||[[(\boldsymbol{j}_h-\boldsymbol{E}_{tt}^h-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h))_t]]_N||_{0,F}^2.\end{aligned}$$

In the rest of this section, we denote $\boldsymbol{w} = \boldsymbol{E} - \boldsymbol{E}_c^h \in H_0(\text{curl};\Omega)$, where $\boldsymbol{E}_c^h$ is the conforming approximation of $\boldsymbol{E}^h$. Before we can prove the main result, we need prepare some estimates.

Lemma 3.3. *Let*

$$Err_7 = \Big(\boldsymbol{j} - \boldsymbol{E}_{tt}^h + \boldsymbol{J}(\boldsymbol{E}^h), \boldsymbol{w}^0 - \boldsymbol{w}_h^0\Big)(t) - \tilde{a}_h(\boldsymbol{E}^h, \boldsymbol{w}^0 - \boldsymbol{w}_h^0)(t),$$

where $\boldsymbol{w}^0$ is the first part of the regular decomposition (3.3)*, and $\boldsymbol{w}_h^0$ is the corresponding quasi-interpolation* (*Lemma 3.2*)*. Then we have*

$$Err_7 \leq C\Big(\sum_{K\in T_h}(h_K^2||\boldsymbol{j} - \boldsymbol{j}_h||_{0,K}^2 + \eta_{R_K}^2 + \eta_{T_K}^2 + \eta_{J_K}^2\Big)^{\frac{1}{2}}||\boldsymbol{w}(t)||_{curl}. \tag{3.7}$$

Proof: Using the definition of $\tilde{a}_h$ and the fact that $[[\boldsymbol{w}^0 - \boldsymbol{w}_h^0]]_T = 0$, we have

$$\begin{aligned} Err_7 = {} & (\boldsymbol{j} - \boldsymbol{E}_{tt}^h + \boldsymbol{J}(\boldsymbol{E}^h) - \boldsymbol{E}^h, \boldsymbol{w}^0 - \boldsymbol{w}_h^0) - \sum_{K\in T_h}\int_K (\nabla\times\boldsymbol{E}^h)\cdot(\nabla\times(\boldsymbol{w}^0 \\ & -\boldsymbol{w}_h^0))dx + \sum_{K\in T_h}\int_K \mathcal{L}(\boldsymbol{E}^h)\cdot(\nabla\times(\boldsymbol{w}^0 - \boldsymbol{w}_h^0))dx. \end{aligned}$$

Using integration by parts and the conformity of $\boldsymbol{w}^0 - \boldsymbol{w}_h^0$, we obtain

$$\begin{aligned} & -\sum_{K\in T_h}\int_K (\nabla\times\boldsymbol{E}^h)\cdot(\nabla\times(\boldsymbol{w}^0 - \boldsymbol{w}_h^0))dx \\ = & -\sum_{K\in T_h}\int_K (\nabla\times\nabla\times\boldsymbol{E}^h)\cdot(\boldsymbol{w}^0 - \boldsymbol{w}_h^0)dx \\ & + \sum_{K\in T_h}\int_{\partial K} \boldsymbol{n}_K\times(\nabla\times\boldsymbol{E}^h)\cdot(\boldsymbol{w}^0 - \boldsymbol{w}_h^0)dA \\ = & -\sum_{K\in T_h}\int_K (\nabla\times\nabla\times\boldsymbol{E}^h)\cdot(\boldsymbol{w}^0 - \boldsymbol{w}_h^0)dx \\ & + \sum_{K\in T_h}\sum_{F\in\partial K\backslash\Gamma}\frac{1}{2}\int_F [[\nabla\times\boldsymbol{E}^h]]_T\cdot(\boldsymbol{w}^0 - \boldsymbol{w}_h^0)dA. \end{aligned}$$

Therefore, we have

$$\begin{aligned} Err_7 = {} & \Big(\boldsymbol{j}_h - \boldsymbol{E}_{tt}^h - \nabla\times\nabla\times\boldsymbol{E}^h - \boldsymbol{E}^h + \boldsymbol{J}(\boldsymbol{E}^h), \boldsymbol{w}^0 - \boldsymbol{w}_h^0\Big) \\ & + \sum_{K\in T_h}\sum_{F\in\partial K\backslash\Gamma}\frac{1}{2}\int_F [[\nabla\times\boldsymbol{E}^h]]_T\cdot(\boldsymbol{w}^0 - \boldsymbol{w}_h^0)dA \\ & + \sum_{K\in T_h}\int_K \mathcal{L}(\boldsymbol{E}^h)\cdot\Big(\nabla\times(\boldsymbol{w}^0 - \boldsymbol{w}_h^0))dx + (\boldsymbol{j} - \boldsymbol{j}_h, \boldsymbol{w}^0 - \boldsymbol{w}_h^0\Big) \\ = {} & \sum_{i=1}^4 Err_{7i}. \end{aligned} \tag{3.8}$$

By Cauchy-Schwarz inequality and Lemma 3.2, we have

$$\begin{aligned} Err_{71} &\leq \sum_{K\in T_h} \eta_{R_K} \cdot h_K^{-1}||\boldsymbol{w}^0 - \boldsymbol{w}_h^0||_{0,K} \\ &\leq \sum_{K\in T_h} \eta_{R_K} \cdot C||\boldsymbol{w}^0||_{1,K} \leq \Big(\sum_{K\in T_h} \eta_{R_K}^2\Big)^{\frac{1}{2}} \cdot C||\boldsymbol{w}||_{curl}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} Err_{72} &\leq \sum_{K\in T_h} \Big(\sum_{F\in\partial K\setminus\Gamma} \frac{1}{2} h_K||[[\nabla\times\boldsymbol{E}^h]]_T||_{0,F}^2\Big)^{\frac{1}{2}} \Big(\sum_{F\in\partial K} \frac{1}{2} h_K^{-1}||\boldsymbol{w}^0 - \boldsymbol{w}_h^0||_{0,F}^2\Big)^{\frac{1}{2}} \\ &\leq \sum_{K\in T_h} \eta_{T_K} h_K^{-\frac{1}{2}}||\boldsymbol{w}^0 - \boldsymbol{w}_h^0||_{0,\partial K} \leq \Big(\sum_{K\in T_h} \eta_{T_K}^2\Big)^{\frac{1}{2}} \cdot C||\boldsymbol{w}||_{curl}. \end{aligned}$$

By Cauchy-Schwarz inequality, Lemma 3.2 and (2.8), we have

$$\begin{aligned} Err_{73} &\leq \Big(\sum_{K\in T_h} ||\mathcal{L}(\boldsymbol{E}^h)||_{0,K}^2\Big)^{\frac{1}{2}} \Big(\sum_{K\in T_h} ||\nabla\times(\boldsymbol{w}^0 - \boldsymbol{w}_h^0)||_{0,K}^2\Big)^{\frac{1}{2}} \\ &\leq \alpha^{-\frac{1}{2}} C_{lift}^{\frac{1}{2}} \Big(\sum_{F\in F_h} ||a^{\frac{1}{2}}[[\boldsymbol{E}^h]]_T||_{0,F}^2\Big)^{\frac{1}{2}} \Big(\sum_{K\in T_h} ||\nabla\times(\boldsymbol{w}^0 - \boldsymbol{w}_h^0)||_{0,K}^2\Big)^{\frac{1}{2}} \\ &\leq \alpha^{-\frac{1}{2}} C_{lift}^{\frac{1}{2}} \Big(\sum_{K\in T_h} \eta_{J_K}^2\Big)^{\frac{1}{2}} \cdot C||\boldsymbol{w}||_{curl}. \end{aligned}$$

Similarly, we have

$$Err_{74} \leq \sum_{K\in T_h} h_K||\boldsymbol{j} - \boldsymbol{j}_h||_{0,K} \cdot h_K^{-1}||\boldsymbol{w}^0 - \boldsymbol{w}_h^0||_{0,K} \leq \Big(\sum_{K\in T_h} h_K^2||\boldsymbol{j} - \boldsymbol{j}_h||_{0,K}^2\Big)^{\frac{1}{2}} \cdot C||\boldsymbol{w}||_{curl}.$$

The proof is completed by substituting the above estimates into (3.8). □

Lemma 3.4. *Let*

$$Err_8 = \Big(\boldsymbol{j} - \boldsymbol{E}_{tt}^h + \boldsymbol{J}(\boldsymbol{E}^h), \nabla(w^1 - w_h^1)\Big)(t) - \tilde{a}_h\Big(\boldsymbol{E}^h, \nabla(w^1 - w_h^1)\Big)(t),$$

where $\boldsymbol{w}^1$ *is the second part of the regular decomposition* (3.3)*, and* $\boldsymbol{w}_h^1$ *is the corresponding piecewise linear approximation* (*Lemma* 3.2)*. Then we have*

$$Err_8 \leq C\Big(||\boldsymbol{j} - \boldsymbol{j}_h||_{0,\Omega} + \sum_{K\in T_h} (\eta_{D_K}^2 + \eta_{N_K}^2)^{\frac{1}{2}}\Big)||\boldsymbol{w}(t)||_{curl}. \tag{3.9}$$

Proof: By the definition of $\tilde{a}_h$ and integration by parts, we obtain

$$\begin{aligned} Err_8 &= \Big(\boldsymbol{j} - \boldsymbol{E}_{tt}^h + \boldsymbol{J}(\boldsymbol{E}^h) - \boldsymbol{E}^h, \nabla(w^1 - w_h^1)\Big) \\ &= \Big(\boldsymbol{j} - \boldsymbol{j}_h, \nabla(w^1 - w_h^1)\Big) + \Big(\boldsymbol{j}_h - \boldsymbol{E}_{tt}^h + \boldsymbol{J}(\boldsymbol{E}^h) - \boldsymbol{E}^h, \nabla(w^1 - w_h^1)\Big) \\ &= \Big(\boldsymbol{j} - \boldsymbol{j}_h, \nabla(w^1 - w_h^1)\Big) - \sum_{K\in T_h} \int_K \nabla\cdot\Big(\boldsymbol{j}_h - \boldsymbol{E}_{tt}^h - \boldsymbol{E}^h + \boldsymbol{J}(\boldsymbol{E}^h)\Big)(w^1 - w_h^1)dx \end{aligned}$$

$$+\sum_{K\in T_h}\sum_{F\in\partial K\backslash\Gamma}\frac{1}{2}\int_F[[\boldsymbol{j}_h-\boldsymbol{E}^h_{tt}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)]]_N(w^1-w^1_h)dA=\sum_{i=1}^3 Err_{8i}. \tag{3.10}$$

By the Cauchy-Schwarz inequality and Lemma 3.2, we easily have

$$Err_{81}\le ||\boldsymbol{j}-\boldsymbol{j}_h||_{0,\Omega}||\nabla(w^1-w^1_h)||_{0,\Omega}\le ||\boldsymbol{j}-\boldsymbol{j}_h||_{0,\Omega}\cdot C||\boldsymbol{w}||_{curl},$$

$$\begin{aligned}Err_{82}&\le \sum_{K\in T_h}h_K||\nabla\cdot\Big(\boldsymbol{j}_h-\boldsymbol{E}^h_{tt}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)\Big)||_{0,K}h_K^{-1}||w^1-w^1_h||_{0,K}\\ &\le C\Big(\sum_{K\in T_h}\eta^2_{D_K}\Big)^{\frac{1}{2}}||\boldsymbol{w}||_{curl},\\ Err_{83}&\le \sum_{K\in T_h}\Big(\sum_{F\in\partial K\backslash\Gamma}\frac{1}{2}h_K||[[\boldsymbol{j}_h-\boldsymbol{E}^h_{tt}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)]]_N||^2_{0,F}\Big)^{\frac{1}{2}}\\ &\quad\cdot\Big(\sum_{F\in\partial K\backslash\Gamma}\frac{1}{2}h_K^{-1}||w^1-w^1_h||^2_{0,F}\Big)^{\frac{1}{2}}\le C\Big(\sum_{K\in T_h}\eta^2_{N_K}\Big)^{\frac{1}{2}}||\boldsymbol{w}||_{curl},\end{aligned}$$

substituting which into (3.10) concludes the proof. □

With the above preparations, we can state our main result.

Theorem 3.1. *Let* $\boldsymbol{E}$ *be the solution of* (2.2) *and* $\boldsymbol{E}^h$ *be the DG solution of* (2.4) *with* $\gamma\ge\gamma_{min}$. *Then the following estimation holds:*

$$\begin{aligned}&||\boldsymbol{E}-\boldsymbol{E}^h||^2_h(t)+||(\boldsymbol{E}-\boldsymbol{E}^h)_t||^2_h(t)\\ \le\, & C\Big[||\boldsymbol{E}-\boldsymbol{E}^h||^2_h(0)+||(\boldsymbol{E}-\boldsymbol{E}^h)_t||^2_h(0)\Big]+C\int_0^t\sum_{F\in F_h}h_F\Big(||[[\boldsymbol{E}^h_{tt}]]_T||^2_{0,F}\\ &+||[[\boldsymbol{E}^h_t]]_T||^2_{0,F}+||[[\boldsymbol{E}^h]]_T||^2_{0,F}\Big)dt+C\sum_{F\in F_h}\Big[||a^{\frac{1}{2}}[[\boldsymbol{E}^h]]_T||^2_{0,F}(t)\\ &+||a^{\frac{1}{2}}[[\boldsymbol{E}^h_t]]_T||^2_{0,F}(t)+||a^{\frac{1}{2}}[[\boldsymbol{E}^h]]_T||^2_{0,F}(0)+||a^{\frac{1}{2}}[[\boldsymbol{E}^h_t]]_T||^2_{0,F}(0)\Big]\\ &+C\Big[||\boldsymbol{j}-\boldsymbol{j}_h||^2_{0,\Omega}(t)+\sum_{K\in T_h}(\eta^2_{R_K}+\eta^2_{T_K}+\eta^2_{J_K}+\eta^2_{D_K}+\eta^2_{N_K})(t)\Big]\\ &+C\Big[||\boldsymbol{j}-\boldsymbol{j}_h||^2_{0,\Omega}(0)+\sum_{K\in T_h}(\eta^2_{R_K}+\eta^2_{T_K}+\eta^2_{J_K}+\eta^2_{D_K}+\eta^2_{N_K})(0)\Big]\\ &+C\int_0^t\Big[\sum_{K\in T_h}(\eta^2_{R^t_K}+\eta^2_{T^t_K}+\eta^2_{J^t_K}+\eta^2_{D^t_K}+\eta^2_{N^t_K})+||(\boldsymbol{j}-\boldsymbol{j}_h)_t||^2_0\Big]dt.\end{aligned} \tag{3.11}$$

Proof: Recall that $\boldsymbol{w}=\boldsymbol{E}-\boldsymbol{E}^h_c$. Then for any $\phi\in H_0(\text{curl};\Omega)$, we have

$$\begin{aligned}&(\boldsymbol{w}_{tt},\phi)+\tilde{a}_h(\boldsymbol{w},\phi)\\ =\,&((\boldsymbol{E}-\boldsymbol{E}^h+\boldsymbol{E}^h-\boldsymbol{E}^h_c)_{tt},\phi)+\tilde{a}_h(\boldsymbol{E}-\boldsymbol{E}^h+\boldsymbol{E}^h-\boldsymbol{E}^h_c,\phi)\\ =\,&(\boldsymbol{E}_{tt}-\boldsymbol{E}^h_{tt},\phi)+\tilde{a}_h(\boldsymbol{E},\phi)-\tilde{a}_h(\boldsymbol{E}^h,\phi)+((\boldsymbol{E}^h-\boldsymbol{E}^h_c)_{tt},\phi)\\ &+\tilde{a}_h(\boldsymbol{E}^h-\boldsymbol{E}^h_c,\phi).\end{aligned} \tag{3.12}$$

Using the fact that $\tilde{a}_h(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{v})\,dx$ on $H_0(\mathrm{curl};\Omega)\times H_0(\mathrm{curl};\Omega)$, we can write the weak formulation of (2.2) as: Find $\boldsymbol{E}\in H_0(\mathrm{curl};\Omega)$ such that

$$(\boldsymbol{E}_{tt},\boldsymbol{\phi}) + \tilde{a}_h(\boldsymbol{E},\boldsymbol{\phi}) - (\boldsymbol{J}(\boldsymbol{E}),\boldsymbol{\phi}) = (\boldsymbol{j},\boldsymbol{\phi}), \quad \forall\, \boldsymbol{\phi}\in H_0(\mathrm{curl};\Omega). \tag{3.13}$$

Using the fact that $\tilde{a}_h = a_h$ on $\boldsymbol{V}_h\times\boldsymbol{V}_h$, we can rewrite the semi-discrete scheme (2.4) as

$$\left(\boldsymbol{E}^h_{tt},\boldsymbol{\phi}_h\right) + \tilde{a}_h\left(\boldsymbol{E}^h,\boldsymbol{\phi}_h\right) - \left(\boldsymbol{J}(\boldsymbol{E}^h),\boldsymbol{\phi}_h\right) = (\boldsymbol{j},\boldsymbol{\phi}_h), \ \forall\, \boldsymbol{\phi}_h\in\boldsymbol{V}_h. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\begin{aligned}
&\left(\boldsymbol{E}_{tt}-\boldsymbol{E}^h_{tt},\boldsymbol{\phi}\right) + \tilde{a}_h(\boldsymbol{E},\boldsymbol{\phi}) - \tilde{a}_h\left(\boldsymbol{E}^h,\boldsymbol{\phi}\right)\\
=&\left(\boldsymbol{j}+\boldsymbol{J}(\boldsymbol{E})-\boldsymbol{E}^h_{tt},\boldsymbol{\phi}\right) - \tilde{a}_h\left(\boldsymbol{E}^h,\boldsymbol{\phi}_h\right) - \tilde{a}_h\left(\boldsymbol{E}^h,\boldsymbol{\phi}-\boldsymbol{\phi}_h\right)\\
=&\left(\boldsymbol{j}+\boldsymbol{J}(\boldsymbol{E}^h)-\boldsymbol{E}^h_{tt},\boldsymbol{\phi}-\boldsymbol{\phi}_h\right) + \left(\boldsymbol{J}(\boldsymbol{E}-\boldsymbol{E}^h),\boldsymbol{\phi}\right) - \tilde{a}_h\left(\boldsymbol{E}^h,\boldsymbol{\phi}-\boldsymbol{\phi}_h\right),
\end{aligned}$$

substituting which into (3.12), we obtain

$$\begin{aligned}
&(\boldsymbol{w}_{tt},\boldsymbol{\phi}) + \tilde{a}_h(\boldsymbol{w},\boldsymbol{\phi})\\
=&\left(\boldsymbol{j}+\boldsymbol{J}(\boldsymbol{E}^h)-\boldsymbol{E}^h_{tt},\boldsymbol{\phi}-\boldsymbol{\phi}_h\right) + \left(\boldsymbol{J}(\boldsymbol{w}+\boldsymbol{E}^h_c-\boldsymbol{E}^h),\boldsymbol{\phi}\right)\\
&-\tilde{a}_h\left(\boldsymbol{E}^h,\boldsymbol{\phi}-\boldsymbol{\phi}_h\right) + \left((\boldsymbol{E}^h-\boldsymbol{E}^h_c)_{tt},\boldsymbol{\phi}\right) + \tilde{a}_h\left(\boldsymbol{E}^h-\boldsymbol{E}^h_c,\boldsymbol{\phi}\right).
\end{aligned} \tag{3.15}$$

Choosing $\boldsymbol{\phi} = \boldsymbol{w}_t$ in (3.15), then integrating both sides from 0 to t, multiplying both sides by 2, and using the property (2.9), we obtain

$$||\boldsymbol{w}(t)||_h^2 + ||\boldsymbol{w}_t(t)||_0^2 \le ||\boldsymbol{w}(0)||_h^2 + ||\boldsymbol{w}_t(0)||_0^2 + \sum_{i=1}^5 Err_i. \tag{3.16}$$

In the following, we just need to estimate all $Err_i, i=1,\cdots,5$. Let us first estimate Err_2. Using Cauchy-Schwarz inequality and the definition of $J(\boldsymbol{E})$, we have

$$\begin{aligned}
&2\int_0^t (\boldsymbol{J}(\boldsymbol{w}),\boldsymbol{w}_t)\,dt \le \int_0^t ||\boldsymbol{J}(\boldsymbol{w})||_0^2(t)dt + \int_0^t ||\boldsymbol{w}_t||_0^2(t)dt\\
\le& C\int_0^t\left(\int_0^t ||\boldsymbol{w}(s)||_0^2 ds\right)dt + \int_0^t ||\boldsymbol{w}_t||_0^2(t)dt\\
\le& Ct\int_0^t ||\boldsymbol{w}(t)||_0^2 dt + \int_0^t ||\boldsymbol{w}_t(t)||_0^2 dt.
\end{aligned}$$

Similarly, by Cauchy-Schwarz inequality and Lemma 3.1, we have

$$\begin{aligned}
&2\int_0^t \left(\boldsymbol{J}(\boldsymbol{E}^h-\boldsymbol{E}^h_c),\boldsymbol{w}_t\right)dt\\
\le& Ct\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}^h_c||_0^2(t)dt + \int_0^t ||\boldsymbol{w}_t||_0^2(t)dt\\
\le& Ct\int_0^t \sum_{F\in F_h} h_F||[[\boldsymbol{E}^h]]_T||_{0,F}^2 dt + \int_0^t ||\boldsymbol{w}_t(t)||_0^2 dt.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
Err_2 &= 2\int_0^t \Big(\boldsymbol{J}(\boldsymbol{w}+\boldsymbol{E}_c^h-\boldsymbol{E}^h),\boldsymbol{w}_t\Big)\,dt \\
&\le Ct\int_0^t ||\boldsymbol{w}(s)||_0^2 ds + 2\int_0^t ||\boldsymbol{w}_t(s)||_0^2 ds + Ct\int_0^t \sum_{F\in F_h} h_F||[[\boldsymbol{E}^h]]_T||_{0,F}^2(s)ds.
\end{aligned}
$$

Using Cauchy-Schwarz inequality and Lemma 3.1, we have

$$
\begin{aligned}
Err_4 &= 2\int_0^t \Big((\boldsymbol{E}^h-\boldsymbol{E}_c^h)_{tt},\boldsymbol{w}_t\Big)\,dt \le 2\int_0^t ||(\boldsymbol{E}^h-\boldsymbol{E}_c^h)_{tt}||_0||\boldsymbol{w}_t||_0 dt \\
&\le \int_0^t ||(\boldsymbol{E}^h-\boldsymbol{E}_c^h)_{tt}||_0^2(t)dt + \int_0^t ||\boldsymbol{w}_t(t)||_0^2 dt \\
&\le C_{app}\int_0^t \sum_{F\in F_h} h_F||[[\boldsymbol{E}_{tt}^h]]_T||_{0,F}^2(t)dt + \int_0^t ||\boldsymbol{w}_t(t)||_0^2 dt.
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
Err_5 &= 2\int_0^t \tilde{a}_h\Big(\boldsymbol{E}^h-\boldsymbol{E}_c^h,\boldsymbol{w}_t\Big)\,dt = 2\tilde{a}_h\Big((\boldsymbol{E}^h-\boldsymbol{E}_c^h)(t),\boldsymbol{w}(t)\Big) \\
&\quad -2\tilde{a}_h\Big((\boldsymbol{E}^h-\boldsymbol{E}_c^h)(0),\boldsymbol{w}(0)\Big) - 2\int_0^t \tilde{a}_h\Big((\boldsymbol{E}^h-\boldsymbol{E}_c^h)_t,\boldsymbol{w}\Big)\,dt \\
&\le 2C_{cont}||(\boldsymbol{E}^h-\boldsymbol{E}_c^h)(t)||_h||\boldsymbol{w}(t)||_h + 2C_{cont}||(\boldsymbol{E}^h-\boldsymbol{E}_c^h)(0)||_h||\boldsymbol{w}(0)||_h \\
&\quad +C_{cont}\Big[\int_0^t ||(\boldsymbol{E}^h-\boldsymbol{E}_c^h)_t||_h^2(t)dt + \int_0^t ||\boldsymbol{w}(t)||_h^2 dt\Big] \\
&\le C\Big[\frac{1}{\delta_1}\sum_{F\in F_h} h_F\Big(||[[\boldsymbol{E}^h]]_T||_{0,F}^2(t) + ||[[\boldsymbol{E}^h]]_T||_{0,F}^2(0)\Big) + ||\boldsymbol{w}(0)||_h^2\Big] + \delta_1||\boldsymbol{w}(t)||_h^2 \\
&\quad +C_{cont}\Big[\int_0^t \sum_{F\in F_h} h_F||[[\boldsymbol{E}_t^h]]_T||_{0,F}^2(t)dt + \int_0^t ||\boldsymbol{w}(t)||_h^2 dt\Big].
\end{aligned}
$$

Finally, note that

$$
\begin{aligned}
&\frac{1}{2}(Err_1+Err_3) \\
&= \int_0^t \Big[\Big(\boldsymbol{j}-\boldsymbol{E}_{tt}^h+\boldsymbol{J}(\boldsymbol{E}^h),(\boldsymbol{w}-\boldsymbol{w}_h)_t\Big) - \tilde{a}_h\Big(\boldsymbol{E}^h,(\boldsymbol{w}-\boldsymbol{w}_h)_t\Big)\Big]dt \\
&= \Big((\boldsymbol{j}-\boldsymbol{E}_{tt}^h+\boldsymbol{J}(\boldsymbol{E}^h))(t),(\boldsymbol{w}-\boldsymbol{w}_h)(t)\Big) - \tilde{a}_h\Big(\boldsymbol{E}^h(t),(\boldsymbol{w}-\boldsymbol{w}_h)(t)\Big) \\
&\quad -\Big((\boldsymbol{j}-\boldsymbol{E}_{tt}^h+\boldsymbol{J}(\boldsymbol{E}^h))(0),(\boldsymbol{w}-\boldsymbol{w}_h)(0)\Big) + \tilde{a}_h\Big(\boldsymbol{E}^h(0),(\boldsymbol{w}-\boldsymbol{w}_h)(0)\Big) \\
&\quad -\int_0^t \Big[\Big((\boldsymbol{j}-\boldsymbol{E}_{tt}^h+\boldsymbol{J}(\boldsymbol{E}^h))_t,\boldsymbol{w}-\boldsymbol{w}_h\Big) - \tilde{a}_h\Big(\boldsymbol{E}_t^h,\boldsymbol{w}-\boldsymbol{w}_h\Big)\Big]\,dt. \qquad (3.17)
\end{aligned}
$$

Let

$$
Err_6 = \Big(\boldsymbol{j}-\boldsymbol{E}_{tt}^h+\boldsymbol{J}(\boldsymbol{E}^h),\boldsymbol{w}-\boldsymbol{w}_h\Big)(t) - \tilde{a}_h\Big(\boldsymbol{E}^h,\boldsymbol{w}-\boldsymbol{w}_h\Big)(t).
$$

To bound Err_6, we need the regular decomposition (3.3), which splits Err_6 into a sum of Err_7 and Err_8, where

$$\begin{aligned}
Err_7 &= \left(\boldsymbol{j} - \boldsymbol{E}^h_{tt} + \boldsymbol{J}(\boldsymbol{E}^h), \boldsymbol{w}^0 - \boldsymbol{w}^0_h\right)(t) - \tilde{a}_h\left(\boldsymbol{E}^h, \boldsymbol{w}^0 - \boldsymbol{w}^0_h\right)(t),\\
Err_8 &= \left(\boldsymbol{j} - \boldsymbol{E}^h_{tt} + \boldsymbol{J}(\boldsymbol{E}^h), \nabla(w^1 - w^1_h)\right)(t) - \tilde{a}_h\left(\boldsymbol{E}^h, \nabla(w^1 - w^1_h)\right)(t).
\end{aligned}$$

Combining the estimates of Err_7 and Err_8 obtained by Lemmas 3.3 and 3.4, respectively, we have the estimate for Err_6:

$$Err_6 \leq C\Big(||\boldsymbol{j} - \boldsymbol{j}_h||_{0,\Omega} + \sum_{K\in T_h}(\eta^2_{R_K} + \eta^2_{T_K} + \eta^2_{J_K} + \eta^2_{D_K} + \eta^2_{N_K})^{\frac{1}{2}}\Big)||\boldsymbol{w}(t)||_{curl}. \tag{3.18}$$

Let

$$Err_9 = \left(\boldsymbol{j} - \boldsymbol{E}^h_{tt} + \boldsymbol{J}(\boldsymbol{E}^h), \boldsymbol{w} - \boldsymbol{w}_h\right)(0) - \tilde{a}_h\left(\boldsymbol{E}^h, \boldsymbol{w} - \boldsymbol{w}_h\right)(0).$$

Hence the estimate of Err_9 will be all the same as Err_6 except now all error terms are for $t = 0$. Denote

$$Err_{10} = -\int_0^t \Big[((\boldsymbol{j} - \boldsymbol{E}^h_{tt} + \boldsymbol{J}(\boldsymbol{E}^h))_t, \boldsymbol{w} - \boldsymbol{w}_h) - \tilde{a}_h(\boldsymbol{E}^h_t, \boldsymbol{w} - \boldsymbol{w}_h)\Big]dt.$$

By similar arguments as used for Err_6, we have

$$\begin{aligned}
Err_{10} \leq \int_0^t ||\boldsymbol{w}(t)||^2_h dt + C\int_0^t \Big[\sum_{K\in T_h}(\eta^2_{R^t_K} + \eta^2_{T^t_K}\\
+\eta^2_{J^t_K} + \eta^2_{D^t_K} + \eta^2_{N^t_K}) + ||(\boldsymbol{j} - \boldsymbol{j}_h)_t||^2_0\Big]dt.
\end{aligned}$$

Summation of Err_6, Err_9 and Err_{10} gives the estimate for $Err_1 + Err_3$. Substituting all above estimates into (3.16), we have

$$\begin{aligned}
&||\boldsymbol{w}(t)||^2_h + ||\boldsymbol{w}_t(t)||^2_h\\
\leq\ & C(||\boldsymbol{w}(0)||^2_h + ||\boldsymbol{w}_t(0)||^2_h) + C_1\int_0^t \left(||\boldsymbol{w}(t)||^2_h + ||\boldsymbol{w}_t(t)||^2_h\right)dt\\
&+C_2\int_0^t \sum_{F\in F_h} h_F\left(||[[\boldsymbol{E}^h_{tt}]]_T||^2_{0,F} + ||[[\boldsymbol{E}^h_t]]_T||^2_{0,F} + ||[[\boldsymbol{E}^h]]_T||^2_{0,F}\right)dt\\
&+\delta_1||\boldsymbol{w}(t)||^2_h + \frac{C_3}{\delta_1}\sum_{F\in F_h}||a^{\frac{1}{2}}[[\boldsymbol{E}^h]]_T||^2_{0,F} + C_4\sum_{F\in F_h}||a^{\frac{1}{2}}[[\boldsymbol{E}^h(0)]]_T||^2_{0,F}\\
&+\delta_2||\boldsymbol{w}(t)||^2_{curl} + \frac{C}{\delta_2}\Big[||\boldsymbol{j} - \boldsymbol{j}_h||^2_{0,\Omega}(t) + \sum_{K\in T_h}(\eta^2_{R_K} + \eta^2_{T_K} + \eta^2_{J_K} + \eta^2_{D_K} + \eta^2_{N_K})(t)\Big]\\
&+\delta_3||\boldsymbol{w}(0)||^2_{curl} + \frac{C}{\delta_3}\Big[||\boldsymbol{j} - \boldsymbol{j}_h||^2_{0,\Omega}(0) + \sum_{K\in T_h}(\eta^2_{R_K} + \eta^2_{T_K} + \eta^2_{J_K} + \eta^2_{D_K} + \eta^2_{N_K})(0)\Big]\\
&+C\int_0^t \Big[\sum_{K\in T_h}\left(\eta^2_{R^t_K} + \eta^2_{T^t_K} + \eta^2_{J^t_K} + \eta^2_{D^t_K} + \eta^2_{N^t_K}\right) + ||(\boldsymbol{j} - \boldsymbol{j}_h)_t||^2_0\Big]dt.
\end{aligned} \tag{3.19}$$

By the definition of $||\cdot||_h$ and Lemma 3.2, we easily have

$$||(\boldsymbol{E}^h-\boldsymbol{E}^h_c)(t)||_h^2 \le C\sum_{F\in F_h}||a^{\frac{1}{2}}[[\boldsymbol{E}^h]]_T||^2_{0,F},$$
$$||(\boldsymbol{E}^h-\boldsymbol{E}^h_c)_t(t)||_h^2 \le C\sum_{F\in F_h}||a^{\frac{1}{2}}[[\boldsymbol{E}^h_t]]_T||^2_{0,F},$$

which, along with (3.19), the triangle inequality, and the Gronwall inequality (choosing δ_1 and δ_2 small enough), concludes the proof. $\square$

4 Lower bounds of the local error estimators

To prove the lower bounds, we need to use the bubble function technique introduced by Verfurth [33]. We denote b_K for the standard polynomial bubble function on element K, and b_F for the standard polynomial bubble function on an interior element face F, shared by two elements K and K'. For simplicity, in the following we denote $UF=\{K,K'\}$ for the union of elements K and K'. With these notation, we have the following estimates.

Lemma 4.1. *For any polynomial function v on K, there exists a constant $C>0$ independent of v and h_K such that*

$$||b_Kv||_{0,K}\le C||v||_{0,K},\quad ||v||_{0,K}\le C||b_K^{\frac{1}{2}}v||_{0,K}, \tag{4.1}$$
$$||\nabla(b_Kv)||_{0,K}\le Ch_K^{-1}||v||_{0,K}. \tag{4.2}$$

On the other hand, for any polynomial function w on F, there exists a constant $C>0$ independent of w and h_F such that

$$||w||_{0,F}\le C||b_F^{\frac{1}{2}}w||_{0,F}, \tag{4.3}$$
$$||W_b||_{0,K}\le Ch_F^{\frac{1}{2}}||w||_{0,F}\quad \forall\, K\in UF, \tag{4.4}$$
$$||\nabla W_b||_{0,K}\le Ch_F^{-\frac{1}{2}}||w||_{0,F}\quad \forall\, K\in UF, \tag{4.5}$$

where $W_b\in H_0^1((\overline{K}\cup\overline{K}')^\circ)$ is an extension of b_Fw such that $W_b|_F=b_Fw$.

The same estimates as (4.1)-(4.5) *hold true for vector functions. Moreover, for a vector ploynomial function $\boldsymbol{v}$ on K, there exists a constant $C>0$ independent of $\boldsymbol{v}$ and h_K such that*

$$||\nabla\times(b_K\boldsymbol{v})||_{0,K}\le Ch_K^{-1}||\boldsymbol{v}||_{0,K}. \tag{4.6}$$

Similarly, for any vector polynomial function $\boldsymbol{w}$ on F, there exists a constant $C>0$ independent of $\boldsymbol{w}$ and h_F such that

$$||\nabla\times\boldsymbol{W}_b||_{0,K}\le Ch_F^{-\frac{1}{2}}||\boldsymbol{w}||_{0,F}\quad \forall\, K\in UF, \tag{4.7}$$

where $\boldsymbol{W}_b\in H_0^1((\overline{K}\cup\overline{K}')^\circ)^3$ is an extension of $b_F\boldsymbol{w}$ such that $\boldsymbol{W}_b|_F=b_F\boldsymbol{w}$.

Proof: The proof of (4.1), (4.3), and (4.4) can be found in [33, Lemma 4.1]. The proof of (4.2) and (4.5) can be obtained from Eqs. (2.35) and (2.39) of [1], respectively. The proof of (4.6) and (4.7) can be obtained by similar arguments as the proof of (4.2) and (4.5). $\square$

Theorem 4.1. *Let* $\boldsymbol{E}$ *be the solution of* (2.2) *and* $\boldsymbol{E}^h$ *be the DG solution of* (2.4) *with* $\gamma \geq \gamma_{min}$. *Then the following local bounds hold:*

$$\begin{aligned}
\text{(i)} \quad \eta_{R_K} \leq C\Big[& h_K||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + h_K||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} + h_K\int_0^t ||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}(s)ds \\
&+h_K||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} + ||\nabla\times(\boldsymbol{E}-\boldsymbol{E}^h)||_{0,K}\Big],\\
\text{(ii)} \quad \eta_{T_K} \leq C\sum_{K\in UF}\Big[& h_K||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + h_K||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} \\
&+h_K\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds + h_K||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} + ||\nabla\times(\boldsymbol{E}^h-\boldsymbol{E})||_{0,K}\Big],\\
\text{(iii)} \quad \eta_{D_K} \leq C\Big(& ||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} + ||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + ||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} \\
&+\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big),\\
\text{(iv)} \quad \eta_{N_K} \leq C\sum_{K\in UF}\Big(& ||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} + ||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + ||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} \\
&+\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big).
\end{aligned}$$

Proof: (i) Let $\boldsymbol{v}_h = \boldsymbol{j}_h - \boldsymbol{E}^h_{tt} - \nabla\times\nabla\times\boldsymbol{E}^h - \boldsymbol{E}^h + \boldsymbol{J}(\boldsymbol{E}^h)$, and $\boldsymbol{v}_b = b_K\boldsymbol{v}_h$. Using the governing equation (2.2), we have

$$\begin{aligned}
||b_K^{\frac{1}{2}}\boldsymbol{v}_h||^2_{0,K} &= \int_K\Big(\boldsymbol{j}_h - \boldsymbol{E}^h_{tt} - \nabla\times\nabla\times\boldsymbol{E}^h - \boldsymbol{E}^h + \boldsymbol{J}(\boldsymbol{E}^h)\Big)\cdot\boldsymbol{v}_b dx \\
&= \int_K\Big[(\boldsymbol{E}-\boldsymbol{E}^h)_{tt} + \nabla\times\nabla\times(\boldsymbol{E}-\boldsymbol{E}^h) + (\boldsymbol{E}-\boldsymbol{E}^h) \\
&\quad -\boldsymbol{J}(\boldsymbol{E}-\boldsymbol{E}^h)\Big]\cdot\boldsymbol{v}_b dx + \int_K(\boldsymbol{j}_h-\boldsymbol{j})\cdot\boldsymbol{v}_b dx \\
&= \int_K\Big[(\boldsymbol{E}-\boldsymbol{E}^h)_{tt} + (\boldsymbol{E}-\boldsymbol{E}^h) - \boldsymbol{J}(\boldsymbol{E}-\boldsymbol{E}^h) + (\boldsymbol{j}_h-\boldsymbol{j})\Big]\cdot\boldsymbol{v}_b dx \\
&\quad +\int_K\Big(\nabla\times(\boldsymbol{E}-\boldsymbol{E}^h)\Big)\cdot(\nabla\times\boldsymbol{v}_b)dx,
\end{aligned}$$

where in the last step we used integration by parts and the fact that $\boldsymbol{v}_b = 0$ on ∂K. Then by Lemma 4.1, we have

$$\begin{aligned}
||\boldsymbol{v}_h||_{0,K} \leq C\Big[& ||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + ||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} + \int_0^t ||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}(s)ds \\
&+||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} + h_K^{-1}||\nabla\times(\boldsymbol{E}-\boldsymbol{E}^h)||_{0,K}\Big],
\end{aligned}$$

which leads to

$$\eta_{R_K} \le C\Big[h_K||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + h_K||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} + h_K\int_0^t ||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}(s)ds$$
$$+h_K||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} + ||\nabla\times(\boldsymbol{E}-\boldsymbol{E}^h)||_{0,K}\Big].$$

(ii) Let $\boldsymbol{v}_h = [[\nabla\times\boldsymbol{E}^h]]_T$ and $\boldsymbol{v}_b = b_F\boldsymbol{v}_h$. Using the fact that $[[\nabla\times\boldsymbol{E}]]_T = 0$ on interior faces, we have

$$||b_F^{\frac{1}{2}}\boldsymbol{v}_h||_{0,F}^2 = \int_F [[\nabla\times\boldsymbol{E}^h]]_T\cdot\boldsymbol{v}_b ds = \int_F [[\nabla\times(\boldsymbol{E}^h-\boldsymbol{E})]]_T\cdot\boldsymbol{v}_b ds$$
$$= \sum_{K\in UF}\Big[\int_K \nabla\times\nabla\times(\boldsymbol{E}^h-\boldsymbol{E})\cdot\boldsymbol{V}_b dx - \int_K \nabla\times(\boldsymbol{E}^h-\boldsymbol{E})\cdot\nabla\times\boldsymbol{V}_b dx\Big], \quad (4.8)$$

where in the last we used integration by parts and the fact that $\boldsymbol{V}_b = 0$ on ∂K. Here $\boldsymbol{V}_b$ is an extension in $H_0^1((\overline{K}\cup\overline{K}')^\circ)^3$ of $\boldsymbol{v}_b$. Note that

$$\int_K \Big(\nabla\times\nabla\times\boldsymbol{E}^h - \nabla\times\nabla\times\boldsymbol{E}\Big)\cdot\boldsymbol{V}_b dx$$
$$= \int_K \Big[\nabla\times\nabla\times\boldsymbol{E}^h + \boldsymbol{E}_{tt} + \boldsymbol{E} - \boldsymbol{J}(\boldsymbol{E}) - \boldsymbol{j}\Big]\boldsymbol{V}_b dx$$
$$= \int_K \Big[\nabla\times\nabla\times\boldsymbol{E}^h + \boldsymbol{E}^h + \boldsymbol{E}^h_{tt} - \boldsymbol{J}(\boldsymbol{E}^h) - \boldsymbol{j}_h + (\boldsymbol{E}_{tt}-\boldsymbol{E}^h_{tt})$$
$$+(\boldsymbol{E}-\boldsymbol{E}^h) + \boldsymbol{J}(\boldsymbol{E}^h-\boldsymbol{E}) + (\boldsymbol{j}_h-\boldsymbol{j})\Big]\boldsymbol{V}_b dx$$
$$\le ||\boldsymbol{j}_h - \boldsymbol{E}^h_{tt} - \nabla\times\nabla\times\boldsymbol{E}^h - \boldsymbol{E}^h + \boldsymbol{J}(\boldsymbol{E}^h)||_{0,K}||\boldsymbol{V}_b||_{0,K} + (||\boldsymbol{E}_{tt}-\boldsymbol{E}^h_{tt}||_{0,K}$$
$$+||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} + ||\boldsymbol{J}(\boldsymbol{E}^h-\boldsymbol{E})||_{0,K} + ||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K})||\boldsymbol{V}_b||_{0,K}. \quad (4.9)$$

Substituting (4.9) into (4.8) and using Lemma 4.1, we obtain

$$||\boldsymbol{v}_h||_{0,F}^2 \le Ch_F^{\frac{1}{2}}\sum_{K\in UF}\Big(h_K^{-1}\eta_{R_K} + ||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + ||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} + ||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K}$$
$$+\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big)||\boldsymbol{v}_h||_{0,F} + C\sum_{K\in UF}||\nabla\times(\boldsymbol{E}^h-\boldsymbol{E})||_{0,K}\cdot h_F^{-\frac{1}{2}}||\boldsymbol{v}_h||_{0,F},$$

which along with the estimate (i), yields

$$\eta_{T_K} = h_F^{\frac{1}{2}}||[[\nabla\times\boldsymbol{E}^h]]_T||_{0,F} \le C\sum_{K\in UF}\Big[h_K||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} + h_K||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}$$
$$+h_K\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds + h_K||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} + ||\nabla\times(\boldsymbol{E}^h-\boldsymbol{E})||_{0,K}\Big],$$

from which the proof completes.

(iii) Let $\boldsymbol{v}_h = \nabla\cdot(\boldsymbol{j}_h - \boldsymbol{E}^h_{tt} - \boldsymbol{E}^h + \boldsymbol{J}(\boldsymbol{E}^h))$, and $\boldsymbol{v}_b = b_K\boldsymbol{v}_h$. Using the fact that $\nabla\cdot(\boldsymbol{j} - \boldsymbol{E}_{tt} - \boldsymbol{E} + \boldsymbol{J}(\boldsymbol{E})) = 0$ and Lemma 4.1, we have

$$||\boldsymbol{v}_h||_{0,K}^2 \le C||b_K^{\frac{1}{2}}\boldsymbol{v}_h||_{0,K}^2 = C\int_K \nabla\cdot\Big(\boldsymbol{j}_h - \boldsymbol{j} + \boldsymbol{E}_{tt} - \boldsymbol{E}^h_{tt} + \boldsymbol{E} - \boldsymbol{E}^h + \boldsymbol{J}(\boldsymbol{E}^h-\boldsymbol{E})\Big)\boldsymbol{v}_b dx$$

$$
\begin{aligned}
&= -C\int_K \Big[(\boldsymbol{j}_h-\boldsymbol{j})+(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}+\boldsymbol{E}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h-\boldsymbol{E})\Big]\cdot(\nabla\boldsymbol{v}_b)dx \\
&\le C\Big[||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K}+||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K}+||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K} \\
&\quad +\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big]\cdot h_K^{-1}||\boldsymbol{v}_h||_{0,K},
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\eta_{D_K} = h_K||\boldsymbol{v}_h||_{0,K} &\le C\Big(||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K}+||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} \\
&\quad +||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}+\int_0^t ||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big).
\end{aligned}
$$

(iv) Let $\boldsymbol{v}_h=[[\boldsymbol{j}_h-\boldsymbol{E}^h_{tt}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)]]_N$, and $\boldsymbol{v}_b=b_F\boldsymbol{v}_h$. Using the facts that $[[\boldsymbol{j}-\boldsymbol{E}_{tt}-\boldsymbol{E}+\boldsymbol{J}(\boldsymbol{E})]]_N=0$ on interior faces and $\nabla\cdot(\boldsymbol{j}-\boldsymbol{E}_{tt}-\boldsymbol{E}+\boldsymbol{J}(\boldsymbol{E}))=0$ in K, we have

$$
\begin{aligned}
&||b_F^{\frac{1}{2}}\boldsymbol{v}_h||^2_{0,F} = \int_F [[\boldsymbol{j}_h-\boldsymbol{E}^h_{tt}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h)]]_N\cdot\boldsymbol{v}_b ds \\
&= \int_F [[\boldsymbol{j}_h-\boldsymbol{j}+(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}+\boldsymbol{E}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h-\boldsymbol{E})]]_N\cdot\boldsymbol{v}_b ds \\
&= \sum_{K\in UF}\int_K \nabla\cdot(\boldsymbol{j}_h-\boldsymbol{E}^h_{tt}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h))\boldsymbol{v}_b dx+\sum_{K\in UF}\int_K\Big(\boldsymbol{j}_h-\boldsymbol{j} \\
&\quad +(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}+\boldsymbol{E}-\boldsymbol{E}^h+\boldsymbol{J}(\boldsymbol{E}^h-\boldsymbol{E})\Big)\cdot\nabla\boldsymbol{v}_b dx \\
&\le C\sum_{K\in UF} h_K^{-1}\eta_{D_K}||\boldsymbol{v}_b||_{0,K}+\sum_{K\in UF}\Big[||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K}+||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} \\
&\quad +||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}+\int_0^t||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big]||\nabla\boldsymbol{v}_b||_{0,K}.
\end{aligned}
$$

Using Lemma 4.1 and the estimate (iii), we have

$$
\begin{aligned}
&\eta_{N_K} = h_F^{\frac{1}{2}}||\boldsymbol{v}_h||_{0,F}\le C\sum_{K\in UF}\Big(\eta_{D_K}+||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K}+||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K} \\
&+||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}+\int_0^t||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big)\le C\sum_{K\in UF}\Big(||\boldsymbol{j}_h-\boldsymbol{j}||_{0,K} \\
&+||(\boldsymbol{E}-\boldsymbol{E}^h)_{tt}||_{0,K}+||\boldsymbol{E}-\boldsymbol{E}^h||_{0,K}+\int_0^t||\boldsymbol{E}^h-\boldsymbol{E}||_{0,K}(s)ds\Big),
\end{aligned}
$$

which concludes the proof of Theorem 4.1. $\Box$

5 Conclusions

In this paper, we initiated the study of a posteriori error estimator for a cold plasma model. We only obtain the result for the semi-discrete DG scheme. Since this is our first work in this area, many interesting issues worth exploring and will be explored in the future: for example, how to obtain a posteriori error estimator for the fully-discrete scheme; numerical tests of the proposed error estimator; extensions to more complicated dispersive media models.

Acknowledgments

The author would like to thank one anonymous referee for the helpful suggestions on improving the paper. This work is supported by National Science Foundation grant DMS-0810896.

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