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Optimal Decay Rates of Solutions to a Blood Flow Model

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Abstract. In this paper, we are concerned with the asymptotic behavior of solutions to Cauchy problem of a blood flow model. Under some smallness conditions on the initial perturbations, we prove that Cauchy problem of blood flow model admits a unique global smooth solution, and such solution converges time-asymptotically to corresponding equilibrium states. Furthermore, the optimal convergence rates are also obtained. The approach adopted in this paper is Green's function method together with time-weighted energy estimates.

AMS subject classifications: 85A25, 35L65, 35B40

Key words: Asymptotic behavior, blood flow model, Green's function method, time-weighted energy estimates.

1 Introduction

Cardiovascular disease is a common disease that seriously threatens the health of human beings. It is characterized by high prevalence rate, high disability rate and high mortality rate. Every year, there are approximately 15 million people die of cardiovascular disease, ranking first among all causes of death. So it is urgent for researchers to develop models and methods for prevention and treatment. To understand the fundamental mechanisms of this complex physiological system, numerous mathematical models were initiated in the 1950's. Among them, the hyperbolic PDE [10,18] has attracted considerable attentions in recent years

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$$\begin{cases} A_t + m_x = 0, \\ m_t + \left(\alpha \dfrac{m^2}{A}\right)_x + \dfrac{A}{\rho} p(A)_x = -\mu \dfrac{m}{A}, \end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Here, $A = A(x,t) \geq 0$ denotes cross-sectional area, $m = m(x,t)$ is the flow rate. $\rho > 0$ denotes fluid density, $\alpha \geq 1$ is the ratio of the averaged axial momentum and $\mu > 0$ is proportional to the viscosity of the fluid. It should be noted that the last three terms are assumed to be constant throughout this paper. Moreover, the pressure $p(A)$ is expressed by

$$p(A) = P_{ext} + G_0 \left(\left(\frac{A}{A_r}\right)^{\frac{\beta}{2}} - 1 \right) + \frac{\nu}{A_r} (\sqrt{A})_t, \tag{1.2}$$

where the constants $P_{ext} \geq 0$ denotes the constant external pressure, $A_r > 0$ is the reference cross-sectional area, $G_0 > 0$ describes the stiffness of the vessel wall, $\nu \geq 0$ denotes viscoelastic coefficient and $\beta > 0$ captures the linearity/nonlinearity of the stress-strain response.

As an important biological model, (1.1) was used to describe the complicated physiological phenomena with human vascular system, since its initiation, it has attracted considerable attentions and have been studied in a wide range of aspects. Here we only focus on the related work on the existence and stability of solutions. When $\nu = 0$, (1.1) reduces to a quasilinear hyperbolic system. Čanić and Kim [3] studied the existence of global-in-time regular solutions under some smallness conditions. Later Li and Čanić [11] illuminated the influence of the viscous damping μ on the solutions to the Cauchy problem of this model. They pointed out that for $\mu > 0$, the data classes that produce smooth solutions are richer than $\mu = 0$, and for the physiologically relevant data that give rise to shock formation, when $\mu > 0$, shock formation is delayed in time. Furthermore, Li and Zhao [12] showed the initial-boundary value problem of the blood flow model admits a unique global smooth solution for small initial data, and such solution converges to constant equilibrium states exponentially as time goes to infinity due to viscous damping and boundary effects. Subsequently, the authors studied the same type of asymptotic states of L^∞ entropy weak solution for large, rough initial data containing vacuum in [13]. In addition, the coupling of a quasilinear hyperbolic system with a Windkessel type boundary was considered in [4]. When considering $\nu > 0$, the viscous term makes the system of hyperbolic/parabolic nature. Even so, the hyperbolic nature of the system (1.1) is still dominant in the blood flow model. Recently, Maity [14] showed that the existence and uniqueness of maximal strong solutions of the system. For other results concerning numerical simulations of this model, we refer to the interesting works [1, 2, 5, 17, 22, 23].

However, to the best of our knowledge, there is very limited research on the stability of solutions to the Cauchy problem of this system.

In this paper, in order to study more conveniently, we rewrite system (1.1) into the following form:

$$\begin{cases} A_t+m_x=0, \\ m_t+\alpha\left(\dfrac{m^2}{A}\right)_x+P(A)_x+\dfrac{\nu A}{\rho A_r}(\sqrt{A})_{xt}=-\mu\dfrac{m}{A} \end{cases} \tag{1.3}$$

with initial data

$$(A,m)|_{t=0}=(A_0,m_0)(x) \;\to\; (A_\infty,m_\pm) \quad \text{as} \quad x\;\to\;\pm\infty. \tag{1.4}$$

Here

$$P(A)=\frac{\beta G_0}{(\beta+2)\rho A_r^{\frac{\beta}{2}}}A^{\frac{\beta}{2}+1}. \tag{1.5}$$

The main purpose of this paper is to study the global existence and large-time behavior of the solutions to (1.3)-(1.4). Note that (1.3) has stationary solutions $(A,m)(x,t)=(A_\infty,0)$, but there are some gaps between initial values $(A_0,m_0)(x)$ and $(A_\infty,0)$ (especially when $m_\pm\neq 0$), which will lead to boundary layer effect at the far field. However, by observing the structure of the system, we guess that as $t\to\infty$, the boundary layer will disappear and the solutions of this system will converge to the stationary solutions $(A_\infty,0)$. The main focus of this study is to prove this conjecture. In this paper, inspired by [7, 20], we artfully construct the correction functions to overcome the difficulties caused by the boundary layer effect at the far field, and then prove the conjecture by the energy method. Moreover, we will also give the optimal convergence rates.

We now point out the main difference between the study in this paper and the related works in the previous researches. Compared with works in [6, 21], the particularity of the structure of system (1.3) will cause some new mathematical difficulties, so some new ideas need to be used. Firstly, since the complexity of the expression for F_2-F_4 in (2.15), the energy estimates in this paper become more difficult and complex, because we have to deal with some additional troublesome terms, such as $e^{-ct}\|\omega(t)\|^2$ in (3.11) and (3.14), $e^{-ct}\|\omega_x(t)\|^2$ in (3.20), $\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}}\frac{\alpha m^2}{2A^2}\omega_{xx}^2\mathrm{d}x$ in (3.38) and so on. For the first two terms, although they cannot be absorbed by the corresponding good terms, for they all have the property of exponential decay, we finally deal with them by Gronwall's inequality successfully (see (3.44)-(3.45) for more details). And for the last term, we require a technical condition (2.16) for the sake of absorbing it by the good term (see (3.42)).

Secondly, one can see in the next section that after obtaining the global existence and decay rates of the solutions in the L^2-framework, we improve the decay rate of $\|\omega\|$ by analyzing the integral expression (3.127), and then by using time-weighted energy estimates, we can obtain (2.19)-(2.20) (see more details in Lemmas 3.9-3.10). Compared with the previous work in [6], we have weaker requirements for the regularity of initial values, and the calculation process is simpler and clearer.

Finally, it should be pointed out that although F_4 produces many time-space integrable good terms such as $\nu\int_0^t\|\omega_{xt}(\tau)\|^2\mathrm{d}\tau$ in (3.22), $\nu\int_0^t\|\omega_{xxt}(\tau)\|^2\mathrm{d}\tau$ in (3.40), $\nu\int_0^t\|\omega_{xxxt}(\tau)\|^2\mathrm{d}\tau$ in (3.68) and so on. In our analysis, we only utilize it to absorb the bad terms produced by itself. So Theorem 2.1 is always true whether $\nu=0$ or $\nu\neq 0$. When $\nu=0$, the calculation process will become more simpler.

Notations. Throughout this paper, we denote generic constant by C and c, which are independent of x,t and may change their value from line to line or even in the same line. $\|\cdot\|_{L^p}$ and $\|\cdot\|_l$ denote the $L^p(\mathbb{R})$-norm $(1\le p\le\infty)$ and $H^l(\mathbb{R})$-norm, respectively. For the sake of convenience, we denote $\|\cdot\|=\|\cdot\|_{L^2}$.

2 Reformulation of the problem and main theorem

Now we are ready to reformulate the system (1.3)-(1.4). First, as in [7,8,16,20], let us look into the behaviors of the solutions to (1.3)-(1.4) at the far fields $x=\pm\infty$, then we can know how big the gaps are between the solutions and the stationary solutions $(A_\infty,0)$ at the far fields. By taking the limits of (1.3) with respect to x, and noting that m_x, $\alpha(\frac{m^2}{A})_x$, $P(A)_x$ and $\frac{\nu A}{\rho A_r}(\sqrt{A})_{xt}$ will be vanished, then we find that $(A,m)(\pm\infty,t)$ satisfy formally the following ODEs:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}A(\pm\infty,t)=0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}m(\pm\infty,t)=-\mu\dfrac{m(\pm\infty,t)}{A(\pm\infty,t)}, \\ (A,m)(\pm\infty,0)=(A_0,m_0)(\pm\infty)=(A_\infty,m_\pm) \end{cases} \tag{2.1}$$

after some elementary computations, we have

$$\lim_{x\to\pm\infty}(A,m)(x,t)=(A,m)(\pm\infty,t)=\left(A_\infty,m_\pm e^{-\frac{\mu}{A_\infty}t}\right). \tag{2.2}$$

Next, in order to delete the gaps yield by the original solutions and the stationary solutions, the correction functions need to be introduced as in [7, 20]. We first suppose that

$$\int_{\mathbb{R}} \big(A_0(x) - A_\infty\big) \mathrm{d}x = \frac{A_\infty}{\mu}(m_+ - m_-). \tag{2.3}$$

Then integrating

$$(A - A_\infty)_t + m_x = 0$$

with respect to x over $\mathbb{R}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} (A - A_\infty)\mathrm{d}x = -(m_+ - m_-)e^{-\frac{\mu t}{A_\infty}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{A_\infty(m_+ - m_-)}{\mu}e^{-\frac{\mu t}{A_\infty}}, \tag{2.4}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left[A(x,t) - A_\infty - \frac{A_\infty(m_+ - m_-)}{\mu}e^{-\frac{\mu t}{A_\infty}}\varphi(x)\right]\mathrm{d}x = 0, \tag{2.5}$$

where $\varphi(x)$ satisfies

$$\varphi(x) \in C_0^\infty(\mathbb{R}), \quad \varphi(x) \geq 0, \quad \int_{\mathbb{R}} \varphi(x)\mathrm{d}x = 1. \tag{2.6}$$

Thus, thanks to (2.3), we reach the setting

$$\omega(x,t) = \int_{-\infty}^{x} \big[A(y,t) - A_\infty - \hat{A}(y,t)\big]\mathrm{d}y \tag{2.7}$$

with

$$\hat{A}(x,t) = \frac{A_\infty(m_+ - m_-)}{\mu}e^{-\frac{\mu t}{A_\infty}}\varphi(x). \tag{2.8}$$

Similar to [7], putting

$$\hat{m}(x,t) = e^{-\frac{\mu t}{A_\infty}}\left(m_- + (m_+ - m_-)\int_{-\infty}^{x}\varphi(y)\mathrm{d}y\right), \tag{2.9}$$

then it is easy to verify that

$$\begin{cases} \hat{A}_t + \hat{m}_x = 0, \\ \hat{m}_t = -\mu\dfrac{\hat{m}}{A_\infty}. \end{cases} \tag{2.10}$$

Now, we are going to make a perturbation around stationary solutions $(A_\infty,0)$, combining (1.3) and (2.10), we have

$$\begin{cases} \big(A - A_\infty - \hat{A}\big)_t + (m - \hat{m})_x = 0, \\ (m - \hat{m})_t + \big(P(A) - P(A_\infty)\big)_x + \alpha\left(\dfrac{m^2}{A}\right)_x \\ \quad + \dfrac{\nu A}{\rho A_r}(\sqrt{A})_{xt} + \mu\dfrac{m}{A} - \mu\dfrac{\hat{m}}{A_\infty} = 0. \end{cases} \tag{2.11}$$

Setting

$$z(x,t) = m(x,t) - \hat{m}(x,t), \tag{2.12}$$

together with (2.7), we can reformulate the system (2.11) as

$$\begin{cases} \omega_t + z = 0, \\ z_t + P'(A_\infty)\omega_{xx} + \mu \dfrac{z}{\omega_x + A_\infty + \hat{A}} = -F_1 - F_2 - F_3 - F_4, \\ (\omega, z)|_{t=0} := (\omega_0, \omega_1)(x) \to 0 \quad \text{as} \quad x \to \pm\infty, \end{cases} \tag{2.13}$$

or

$$\begin{cases} \omega_{tt} - P'(A_\infty)\omega_{xx} + \mu \dfrac{\omega_t}{\omega_x + A_\infty + \hat{A}} = F_1 + F_2 + F_3 + F_4, & \text{(2.14a)} \\ (\omega, \omega_t)|_{t=0} := (\omega_0, \omega_1)(x) \to 0 \quad \text{as} \quad x \to \pm\infty, & \text{(2.14b)} \end{cases}$$

where

$$\begin{cases} F_1 := \big(P(\omega_x + A_\infty + \hat{A}) - P(A_\infty) - P'(A_\infty)\omega_x\big)_x, \\ F_2 := -\mu \dfrac{\hat{m}(\omega_x + \hat{A})}{A_\infty(\omega_x + A_\infty + \hat{A})}, \\ F_3 := \alpha \left(\dfrac{(-\omega_t + \hat{m})^2}{\omega_x + A_\infty + \hat{A}} \right)_x, \\ F_4 := \dfrac{\nu}{\rho A_r} (\omega_x + A_\infty + \hat{A}) \left(\sqrt{\omega_x + A_\infty + \hat{A}} \right)_{xt}. \end{cases} \tag{2.15}$$

From now on, we will study the system (2.14)-(2.15). We list the main theorem as follows.

Theorem 2.1. *Suppose that* (2.3) *and*

$$\max\{|m_+|, |m_-|\} < \left(\frac{\beta G_0}{2\alpha\rho A_r^{\frac{\beta}{2}}} \right)^{\frac{1}{2}} A_\infty^{\frac{\beta}{4}+1} \tag{2.16}$$

holds, $\delta := |m_+ - m_-|$ *and* $\|\omega_0\|_3 + \|\omega_1\|_2$ *are sufficiently small, then the Cauchy problem of* (2.14) *admits a unique global solution* $\omega(x,t)$ *such that*

$$\begin{aligned} &\omega(x,t) \in C^k\big(0,\infty; H^{3-k}(\mathbb{R})\big), \quad k = 0,1,2,3, \\ &\omega_t(x,t) \in C^k\big(0,\infty; H^{2-k}(\mathbb{R})\big), \quad k = 0,1,2, \end{aligned}$$

and

$$\big\|\partial_x^k \partial_t^l \omega(t)\big\| \le C(1+t)^{-\frac{k}{2}-l}, \quad 0 \le k+l \le 3, \quad 0 \le l \le 2, \tag{2.17}$$

$$\big\|\partial_t^3 \omega(t)\big\| \le C(1+t)^{-\frac{5}{2}}. \tag{2.18}$$

Furthermore, under the additional assumption that $(\omega_0,\omega_1)(x)\in(L^1\times L^1)(\mathbb{R})$*, the solution* $\omega(x,t)$ *possesses improved decay estimates*

$$\left\|\partial_x^k\partial_t^l\omega(t)\right\|\le C(1+t)^{-\frac{1}{4}-\frac{k}{2}-l},\quad 0\le k+l\le 3,\quad 0\le l\le 2, \tag{2.19}$$

$$\left\|\partial_t^3\omega(t)\right\|\le C(1+t)^{-\frac{11}{4}}. \tag{2.20}$$

Noticing that $\omega_x=A-A_\infty-\hat{A}$, $z=m-\hat{m}$, and using Lemma 3.1 and Sobolev inequality, we immediately obtain the following estimates.

Corollary 2.1. *Under the conditions in Theorem* 2.1*, the system* (1.3)-(1.5) *possesses uniquely global solutions* $(A,m)(x,t)$ *satisfying*

$$\|(A-A_\infty)(t)\|_{L^\infty}\le C(1+t)^{-1}, \tag{2.21}$$

$$\|m(t)\|_{L^\infty}\le C(1+t)^{-\frac{3}{2}}. \tag{2.22}$$

3 Proof of the main results

This section is organized by two parts. In the first subsection, we establish the global existence by combing the local existence with a priori estimates under some smallness conditions, besides, we obtain the decay rates of solutions based on the standard L^2-energy method. In the second subsection, provided that the initial perturbation lies in $(H^3\times H^2)(\mathbb{R})\cap(L^1\times L^1)(\mathbb{R})$, we improve the decay estimates by Green's function and time-weighted energy method.

3.1 Proof of (2.17)-(2.18)

It is well known that the global existence can be obtained by the classical continuation method based on the local existence and a priori estimates. And the local existence of the reformulated Cauchy problem of (2.14) can be obtained by the standard iteration method (cf. [9,15,19]). Hence, to prove Theorem 2.1, we only need to focus on establishing the following a priori estimates.

Proposition 3.1 (a priori estimates)**.** *Suppose all the conditions in Theorem* 2.1 *hold. Let* $\omega(x,t)$ *be a smooth solution to the Cauchy problem* (2.14) *on* $0\le t\le T$ *for* $T>0$*. Then there exist positive constants* ε *and* C *such that if*

$$N(T):=\sup_{0\le t\le T}\left\{\sum_{0\le k+l\le 3,0\le l\le 2}(1+t)^{k+2l}\left\|\partial_x^k\partial_t^l\omega(t)\right\|^2+(1+t)^5\left\|\partial_t^3\omega(t)\right\|^2\right\}\le\varepsilon^2, \tag{3.1}$$

then it holds that

$$\begin{aligned}&\sum_{0\le k+l\le 3,0\le l\le 2}(1+t)^{k+2l}\left\|\partial_x^k\partial_t^l\omega(t)\right\|^2+(1+t)^5\left\|\partial_t^3\omega(t)\right\|^2\\&\quad+\int_0^t\left[\sum_{1\le k+l\le 3}(1+\tau)^{k+2l-1}\left\|\partial_x^k\partial_t^l\omega(\tau)\right\|^2\right]\mathrm{d}\tau\\&\le C\Big(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\Big).\end{aligned}\tag{3.2}$$

From (3.1) and the Sobolev inequality

$$\|f\|_{L^\infty}\le\sqrt{2}\|f\|^{\frac12}\|f_x\|^{\frac12}$$

one can immediately obtain

$$\left\|\partial_x^k\partial_t^l\omega(t)\right\|_{L^\infty}\le\sqrt{2}\varepsilon(1+t)^{-\frac14-\frac{k}{2}-l},\quad 0\le k+l\le 2.\tag{3.3}$$

Moreover, it is easy to deduce

$$0\le\frac12 A_\infty\le\omega_x+A_\infty+\hat{A}\le\frac32 A_\infty,\tag{3.4}$$

which will be used later. In order to prove Theorem 2.1, it is necessary to introduce some estimates of the correction functions $(\hat{A},\hat{m})(x,t)$ and heat kernel $G(x,t)$.

Lemma 3.1. *Let k,j be nonnegative integers and $p\in[1,\infty]$, it holds that*

$$\begin{aligned}&\left\|\partial_x^k\partial_t^j\hat{A}(t)\right\|_{L^p}\le C|m_+-m_-|e^{-ct},\quad k,j\ge 0,\\&\left\|\partial_x^k\partial_t^j\hat{m}(t)\right\|_{L^p}\le C|m_+-m_-|e^{-ct},\quad k\ge 1,\quad j\ge 0,\\&\|\hat{m}(t)\|_{L^\infty}\le\max\big\{|m_+|,|m_-|\big\}e^{-ct}.\end{aligned}\tag{3.5}$$

Proof. From (2.8) and (2.9), it is straightforward to check that $(\hat{A},\hat{m})(x,t)$ satisfies the above lemma. □

Lemma 3.2. *Let k,j be nonnegative integers and $p\in[1,\infty]$, it holds that*

$$\left\|\partial_x^k\partial_t^l G(t)\right\|_{L^p}\le Ct^{-\frac12(1-\frac1p)-\frac{k}{2}-l},\quad k,l\ge 0.\tag{3.6}$$

With the above preparation, we turn to prove Proposition 3.1, which will be finished by the following several lemmas.

Lemma 3.3. *Suppose that all the conditions in Proposition* 3.1 *hold. If ε and δ are sufficiently small, then for $0\le t\le T$, we have*

$$\begin{aligned}&\|\omega(t)\|_2^2+\|\omega_t(t)\|_1^2+\int_0^t\Big(\|\omega_x(\tau)\|_1^2+\|\omega_t(\tau)\|_1^2+\nu\|\omega_{xxt}(\tau)\|^2\Big)\mathrm{d}\tau\\ \le& C\Big(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\Big).\end{aligned}\tag{3.7}$$

Proof. Multiplying (2.14a) by ω and integrating the resulting equation with respect to x over $\mathbb{R}$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega^2}{2(\omega_x+A_\infty+\hat{A})}+\omega\omega_t\right)\mathrm{d}x+\int_{\mathbb{R}}P'(A_\infty)\omega_x^2\mathrm{d}x\\ =&\int_{\mathbb{R}}\omega_t^2\mathrm{d}x-\int_{\mathbb{R}}\frac{\mu\omega^2(\omega_{xt}+\hat{A}_t)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x+\sum_{i=1}^4\int_{\mathbb{R}}F_i\omega\mathrm{d}x.\end{aligned}\tag{3.8}$$

We now estimate the terms on the right-hand side of (3.8) as follows. First, by applying integration by parts, (3.3) and (3.4), together with Lemma 3.1 and Cauchy-Schwarz's inequality, one has

$$\begin{aligned}&-\int_{\mathbb{R}}\frac{\mu\omega^2(\omega_{xt}+\hat{A}_t)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x\\ =&-\int_{\mathbb{R}}\frac{\mu\omega^2\omega_{xt}}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x-\int_{\mathbb{R}}\frac{\mu\omega^2\hat{A}_t}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x\\ \le&\int_{\mathbb{R}}\frac{\mu\omega\omega_x\omega_t}{(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x-\int_{\mathbb{R}}\frac{\mu\omega^2\omega_t(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^3}\mathrm{d}x+C\int_{\mathbb{R}}\omega^2|\hat{A}_t|\mathrm{d}x\\ \le&C\int_{\mathbb{R}}\Big(|\omega\omega_x|+\omega^2\big(|\omega_{xx}|+|\hat{A}_x|\big)\Big)|\omega_t|\mathrm{d}x+C\int_{\mathbb{R}}|\hat{A}_t|\mathrm{d}x\\ \le&C\varepsilon\Big(\|\omega_x(t)\|^2+\|\omega_t(t)\|^2+\|\omega_{xx}(t)\|^2\Big)+C\delta e^{-ct}.\end{aligned}\tag{3.9}$$

Next, as shown in [20], we can estimate

$$\begin{aligned}\int_{\mathbb{R}}F_1\omega\mathrm{d}x=&-\int_{\mathbb{R}}\big[P(\omega_x+A_\infty+\hat{A})-P(A_\infty)-P'(A_\infty)\omega_x\big]\omega_x\mathrm{d}x\\ \le&\frac{P'(A_\infty)}{16}\|\omega_x(t)\|^2+C\delta e^{-ct}.\end{aligned}\tag{3.10}$$

Then by using (3.3), (3.4), Lemma 3.1 and Cauchy-Schwarz's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}}F_2\omega\mathrm{d}x\le& C\int_{\mathbb{R}}\big(|\omega_x|+|\hat{A}|\big)|\hat{m}||\omega|\mathrm{d}x\\ \le&\frac{P'(A_\infty)}{16}\|\omega_x(t)\|^2+Ce^{-ct}\|\omega(t)\|^2+C\delta e^{-ct}.\end{aligned}\tag{3.11}$$

Finally, due to the fact

$$F_3=\alpha\left(\frac{m^2}{A}\right)_x=\frac{2\alpha mm_x}{A}-\frac{\alpha m^2A_x}{A^2}, \tag{3.12}$$

$$F_4=\frac{\nu}{\rho A_r}A(\sqrt{A})_{xt}=-\frac{\nu}{4\rho A_r\sqrt{A}}A_xA_t+\frac{\nu\sqrt{A}}{2\rho A_r}A_{xt} \tag{3.13}$$

the last two terms on the right-hand side can be estimated as

$$\begin{aligned}\int_{\mathbb{R}}F_3\omega\mathrm{d}x\le& C\int_{\mathbb{R}}\left(|\omega_t|+|\hat{m}|\right)\left(|\omega_{xt}|+|\hat{m}_x|\right)|\omega|\mathrm{d}x\\&+C\int_{\mathbb{R}}\left(\omega_t^2+\hat{m}^2\right)\left(|\omega_{xx}|+|\hat{A}_x|\right)|\omega|\mathrm{d}x\\ \le& C\|\omega_t(t)\|_1^2+C\|\omega_{xx}(t)\|^2+Ce^{-ct}\|\omega(t)\|^2+C\delta e^{-ct},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\int_{\mathbb{R}}F_4\omega\mathrm{d}x\le&\frac{\nu}{2\rho A_r}\int_{\mathbb{R}}\sqrt{A}\omega_{xxt}\omega\mathrm{d}x+C\int_{\mathbb{R}}\left(|\hat{A}_{xt}|+|A_x||A_t|\right)|\omega|\mathrm{d}x\\ \le&-\frac{\nu}{4\rho A_r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\sqrt{A}\omega_x^2)\mathrm{d}x+C\int_{\mathbb{R}}|A_t|\omega_x^2\mathrm{d}x\\&+C\int_{\mathbb{R}}\Big(|\hat{A}_{xt}|+\left(|\omega_{xx}|+|\hat{A}_x|\right)\left(|\omega_{xt}|+|\hat{A}_t|\right)\Big)|\omega|\mathrm{d}x\\ \le&-\frac{\nu}{4\rho A_r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\sqrt{A}\omega_x^2)\mathrm{d}x+\frac{P'(A_\infty)}{16}\|\omega_x(t)\|^2\\&+C\|\omega_{xx}(t)\|^2+C\|\omega_{xt}(t)\|^2+C\delta e^{-ct}.\end{aligned} \tag{3.15}$$

Putting (3.9)-(3.11), (3.14)-(3.15) into (3.8), we can conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega^2}{2(\omega_x+A_\infty+\hat{A})}+\omega\omega_t+\frac{\nu\sqrt{A}}{4\rho A_r}\omega_x^2\right)\mathrm{d}x+\frac{3}{4}\int_{\mathbb{R}}P'(A_\infty)\omega_x^2\mathrm{d}x\\ \le& C\|\omega_t(t)\|_1^2+C\|\omega_{xx}(t)\|^2+Ce^{-ct}\|\omega(t)\|^2+C\delta e^{-ct}.\end{aligned} \tag{3.16}$$

On the other hand, multiplying (2.14a) by ω_t and integrating the resulting equation with respect to x over $\mathbb{R}$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_t^2+P'(A_\infty)\omega_x^2\right)\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu\omega_t^2}{\omega_x+A_\infty+\hat{A}}\mathrm{d}x=\sum_{i=1}^4\int_{\mathbb{R}}F_i\omega_t\mathrm{d}x. \tag{3.17}$$

The first term on the right-hand can be treated as follows:

$$\begin{aligned}\int_{\mathbb{R}}F_1\omega_t\mathrm{d}x\le&\int_{\mathbb{R}}\Big(|\omega_{xx}|\left(|\omega_x|+|\hat{A}|\right)+|\hat{A}_x|\Big)|\omega_t|\mathrm{d}x\\ \le&\frac{\mu}{12A_\infty}\|\omega_t(t)\|^2+C\varepsilon(1+t)^{-\frac{3}{2}}\|\omega_{xx}(t)\|^2+C\delta e^{-ct}.\end{aligned} \tag{3.18}$$

By using (3.3), (3.4), (3.12), (3.13), together with Lemma 3.1 and Cauchy-Schwarz's inequality, the other terms on the right-hand side of (3.17) can be bounded as

$$\begin{aligned}\int_{\mathbb{R}} F_2\omega_t \mathrm{d}x &\leq C\int_{\mathbb{R}} \left(|\omega_x|+|\hat{A}|\right)|\hat{m}||\omega_t|\mathrm{d}x\\ &\leq \frac{\mu}{12A_\infty}\|\omega_t(t)\|^2+Ce^{-ct}\|\omega_x(t)\|^2+C\delta e^{-ct},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\int_{\mathbb{R}} F_3\omega_t \mathrm{d}x &\leq C\int_{\mathbb{R}} \left(|\omega_t|+|\hat{m}|\right)\left(|\omega_{xt}|+|\hat{m}_x|\right)|\omega_t|\mathrm{d}x\\ &\quad +C\int_{\mathbb{R}}\left(\omega_t^2+\hat{m}^2\right)\left(|\omega_{xx}|+|\hat{A}_x|\right)|\omega_t|\mathrm{d}x\\ &\leq \frac{\mu}{12A_\infty}\|\omega_t(t)\|^2+Ce^{-ct}\left(\|\omega_{xt}(t)\|^2+\|\omega_{xx}(t)\|^2\right)+C\delta e^{-ct},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\int_{\mathbb{R}} F_4\omega_t \mathrm{d}x &\leq \int_{\mathbb{R}} \frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxt}\omega_t\mathrm{d}x+C\int_{\mathbb{R}}\left(|\hat{A}_{xt}|+|A_x||A_t|\right)|\omega_t|\mathrm{d}x\\ &\leq -\int_{\mathbb{R}} \frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xt}^2\mathrm{d}x+C\int_{\mathbb{R}}\Big(|\hat{A}_{xt}|+\left(|\omega_{xx}|+|\hat{A}_x|\right)\left(|\omega_{xt}|+|\hat{A}_t|\right)\Big)|\omega_t|\mathrm{d}x\\ &\leq -\int_{\mathbb{R}} \frac{c\nu\sqrt{A}}{4\rho A_r}\omega_{xt}^2\mathrm{d}x+\frac{\mu}{12A_\infty}\|\omega_t(t)\|^2+C\delta e^{-ct}.\end{aligned} \tag{3.21}$$

Putting (3.18)-(3.21) into (3.17), we can conclude that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_t^2+P'(A_\infty)\omega_x^2\right)\mathrm{d}x+c\int_{\mathbb{R}}\left(\omega_t^2+\nu\omega_{xt}^2\right)\mathrm{d}x\\ &\leq C\varepsilon(1+t)^{-\frac{3}{2}}\|\omega_{xx}(t)\|^2+Ce^{-ct}\left(\|\omega_{xt}(t)\|^2+\|\omega_x(t)\|_1^2\right)+C\delta e^{-ct}.\end{aligned} \tag{3.22}$$

We have from (3.16) $\times\lambda$ + (3.22) $(0<\lambda\ll 1)$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[\omega_t^2+\frac{\lambda\mu\omega^2}{\omega_x+A_\infty+\hat{A}}+2\lambda\omega\omega_t+\left(P'(A_\infty)+\frac{\nu\sqrt{A}}{2\rho A_r}\right)\omega_x^2\right]\mathrm{d}x\\ &\quad +c\int_{\mathbb{R}}\left(\omega_x^2+\omega_t^2+\nu\omega_{xt}^2\right)\mathrm{d}x\\ &\leq C\|\omega_{xt}(t)\|^2+C\|\omega_{xx}(t)\|^2+Ce^{-ct}\|\omega(t)\|_1^2+C\delta e^{-ct}.\end{aligned} \tag{3.23}$$

Integrating the above inequality over $[0,t]$, we have

$$\|\omega(t)\|_1^2+\|\omega_t(t)\|^2+\int_0^t\left(\|\omega_x(\tau)\|^2+\|\omega_t(\tau)\|^2+\nu\|\omega_{xt}(t)\|^2\right)\mathrm{d}\tau$$

$$\begin{aligned}&\leq C\left(\|\omega_0\|_1^2+\|\omega_1\|^2+\delta\right)+C\int_0^t\left(\|\omega_{xt}(t)\|^2+\|\omega_{xx}(\tau)\|^2\right)\mathrm{d}\tau\\&\quad+C\int_0^t e^{-\tau}\|\omega(\tau)\|_1^2\mathrm{d}\tau.\end{aligned}\tag{3.24}$$

Now we consider the higher order energy estimates. Multiplying (2.14a) by $-\omega_{xx}$ and integrating it with respect to x over $\mathbb{R}$, after some integrations by parts, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega_x^2}{2(\omega_x+A_\infty+\hat{A})}+\omega_x\omega_{xt}\right)\mathrm{d}x+\int_{\mathbb{R}}P'(A_\infty)\omega_{xx}^2\mathrm{d}x \qquad (3.25)\\=&\int_{\mathbb{R}}\omega_{xt}^2\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu\omega_x\omega_t(\omega_{xx}+\hat{A}_x)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x-\int_{\mathbb{R}}\frac{\mu\omega_x^2(\omega_{xt}+\hat{A}_t)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x-\sum_{i=1}^4\int_{\mathbb{R}}F_i\omega_{xx}\mathrm{d}x.\end{aligned}$$

By employing similar procedures from (3.18) to (3.21), the terms on the right-hand side of (3.25) can be treated as

$$\begin{aligned}&\int_{\mathbb{R}}\frac{\mu\omega_x\omega_t(\omega_{xx}+\hat{A}_x)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x-\int_{\mathbb{R}}\frac{\mu\omega_x^2(\omega_{xt}+\hat{A}_t)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x\\\leq&C\int_{\mathbb{R}}|\omega_x||\omega_t|\big(|\omega_{xx}|+|\hat{A}_x|\big)\mathrm{d}x+C\int_{\mathbb{R}}\omega_x^2\big(|\omega_{xt}|+|\hat{A}_t|\big)\mathrm{d}x\\\leq&\frac{P'(A_\infty)}{20}\|\omega_{xx}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_x(t)\|^2+C\delta e^{-ct},\end{aligned}\tag{3.26}$$

and

$$-\int_{\mathbb{R}}F_1\omega_{xx}\mathrm{d}x\leq C\int_{\mathbb{R}}\big(|\omega_{xx}|(|\omega_x|+|\hat{A}|)+|\hat{A}_x|\big)|\omega_{xx}|\mathrm{d}x\leq\frac{P'(A_\infty)}{20}\|\omega_{xx}(t)\|^2+C\delta e^{-ct},\tag{3.27}$$

$$-\int_{\mathbb{R}}F_2\omega_{xx}\mathrm{d}x\leq C\int_{\mathbb{R}}\big(|\omega_x|+|\hat{A}|\big)|\hat{m}||\omega_{xx}|\mathrm{d}x\leq\frac{P'(A_\infty)}{20}\|\omega_{xx}(t)\|^2+Ce^{-ct}\|\omega_x(t)\|^2+C\delta e^{-ct},\tag{3.28}$$

$$\begin{aligned}-\int_{\mathbb{R}}F_3\omega_{xx}\mathrm{d}x\leq&C\int_{\mathbb{R}}\big(|\omega_{xt}|+|\hat{m}_x|\big)\big(|\omega_t|+|\hat{m}|\big)|\omega_{xx}|\mathrm{d}x\\&+C\int_{\mathbb{R}}\big(\omega_t^2+\hat{m}^2\big)\big(|\omega_{xx}|+|\hat{A}_x|\big)|\omega_{xx}|\mathrm{d}x\\\leq&\frac{P'(A_\infty)}{20}\|\omega_{xx}(t)\|^2+C\|\omega_{xt}(t)\|^2+Ce^{-ct}\|\omega_{xx}(t)\|^2+C\delta e^{-ct},\end{aligned}\tag{3.29}$$

$$
\begin{aligned}
-\int_{\mathbb{R}} F_4\omega_{xx}\mathrm{d}x \leq & -\int_{\mathbb{R}} \frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxt}\omega_{xx}\mathrm{d}x + C\int_{\mathbb{R}} \left(|\hat{A}_{xt}|+|A_x||A_t|\right)|\omega_{xx}|\mathrm{d}x \\
\leq & -\frac{\nu}{4\rho A_r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left(\sqrt{A}\omega_{xx}^2\right)\mathrm{d}x \\
& + C\int_{\mathbb{R}} \Big(|\hat{A}_{xt}|+\left(|\omega_{xx}|+|\hat{A}_x|\right)\left(|\omega_{xt}|+|\hat{A}_t|\right)\Big)|\omega_{xx}|\mathrm{d}x \\
\leq & -\frac{\nu}{4\rho A_r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left(\sqrt{A}\omega_{xx}^2\right)\mathrm{d}x + \frac{P'(A_\infty)}{20}\|\omega_{xx}(t)\|^2 + C\delta e^{-ct}. \qquad (3.30)
\end{aligned}
$$

Putting (3.26)-(3.30) into (3.25), we can conclude that

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left(\frac{\omega_x^2}{2(\omega_x+A_\infty+\hat{A})} + \omega_x\omega_{xt} + \frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xx}^2\right)\mathrm{d}x + \frac{3}{4}\int_{\mathbb{R}} P'(A_\infty)\omega_{xx}^2\mathrm{d}x \\
\leq & C\|\omega_{xt}(t)\|^2 + C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_x(t)\|^2 + Ce^{-ct}\|\omega_x(t)\|_1^2 + C\delta e^{-ct}. \qquad (3.31)
\end{aligned}
$$

On the other hand, differentiating Eq. (2.14a) in x, we multiply the resulting equation by ω_{xt} and integrate the resulting equation in x over $\mathbb{R}$. Using integration by parts gives

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left(\omega_{xt}^2 + P'(A_\infty)\omega_{xx}^2\right)\mathrm{d}x + \int_{\mathbb{R}} \frac{\mu\omega_{xt}^2}{\omega_x+A_\infty+\hat{A}}\mathrm{d}x \\
= & \int_{\mathbb{R}} \frac{\mu\omega_t\omega_{xt}(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x + \sum_{i=1}^{4}\int_{\mathbb{R}} F_{ix}\omega_{xt}\mathrm{d}x. \qquad (3.32)
\end{aligned}
$$

We now estimate the righthand side of (3.32) term by term. First, we have from (3.3), (3.4), Lemma 3.1 and Cauchy-Schwarz's inequality that

$$
\begin{aligned}
& \int_{\mathbb{R}} \frac{\mu\omega_t\omega_{xt}(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})}\mathrm{d}x \\
\leq & C\int_{\mathbb{R}} |\omega_t||\omega_{xt}|\left(|\omega_{xx}|+|\hat{A}_x|\right)\mathrm{d}x \\
\leq & \frac{\mu}{12A_\infty}\|\omega_{xt}(t)\|^2 + C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{xx}(t)\|^2 + C\delta e^{-ct}. \qquad (3.33)
\end{aligned}
$$

Moreover, as shown in [20], we can estimate

$$
\begin{aligned}
\int_{\mathbb{R}} F_{1x}\omega_{xt}\mathrm{d}x \leq & -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right]\omega_{xx}^2\mathrm{d}x \\
& + C\int_{\mathbb{R}} \omega_{xx}^2\left(|\omega_{xt}|+|\hat{A}_t|\right)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{\mathbb{R}}\big((|\omega_{xx}|+|\hat{A}_x|)|\hat{A}_x|+|\hat{A}_{xx}|\big)|\omega_{xt}|\mathrm{d}x\\
\le&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_{\mathbb{R}}\left[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right]\omega_{xx}^2\mathrm{d}x\\
&+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xx}(t)\|^2+C\delta e^{-ct}.
\end{aligned}\tag{3.34}
$$

Then by direct calculation, we get

$$
\begin{aligned}
\int_{\mathbb{R}}F_{2x}\omega_{xt}\mathrm{d}x\le& C\int_{\mathbb{R}}\big(|\omega_x|+|\hat{A}|\big)|\hat{m}_x||\omega_{xt}|\mathrm{d}x\\
&+C\int_{\mathbb{R}}\big(|\omega_{xx}|+|\hat{A}_x|\big)|\hat{m}||\omega_{xt}|\mathrm{d}x\\
&+C\int_{\mathbb{R}}\big(|\omega_x|+|\hat{A}|\big)\big(|\omega_{xx}|+|\hat{A}_x|\big)|\hat{m}||\omega_{xt}|\mathrm{d}x\\
\le&\frac{\mu\|\omega_{xt}(t)\|^2}{12A_\infty}+Ce^{-ct}\|\omega_{xx}(t)\|^2+C\delta e^{-ct}.
\end{aligned}\tag{3.35}
$$

Finally, we deal with the last two terms of the righthand side of (3.32). Noticing

$$
\begin{aligned}
F_{3x}=&-\frac{2\alpha m}{A}\omega_{xxt}-\frac{\alpha m^2}{A^2}\omega_{xxx}\\
&+\mathcal{O}(1)\big(m_x^2+m\hat{m}_{xx}+mm_xA_x+m^2\hat{A}_{xx}+m^2A_x^2\big),
\end{aligned}\tag{3.36}
$$

$$
\begin{aligned}
F_{4x}=&\frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxxt}-\frac{\nu}{4\rho A_r\sqrt{A}}A_t\omega_{xxx}\\
&+\nu\mathcal{O}(1)\big(\hat{A}_{xxt}+A_t\hat{A}_{xx}+A_x^2A_t\big),
\end{aligned}\tag{3.37}
$$

we can compute that

$$
\begin{aligned}
\int_{\mathbb{R}}F_{3x}\omega_{xt}\mathrm{d}x=&-\int_{\mathbb{R}}\frac{2\alpha m}{A}\omega_{xxt}\omega_{xt}\mathrm{d}x-\int_{\mathbb{R}}\frac{\alpha m^2}{A^2}\omega_{xxx}\omega_{xt}\mathrm{d}x\\
&+C\int_{\mathbb{R}}\Big(|\hat{m}_{xx}|+|\hat{A}_{xx}|+\big(\omega_{xt}^2+\hat{m}_x^2\big)\\
&\qquad+\big(|\omega_t|+|\hat{m}|\big)\big(|\omega_{xt}|+|\hat{m}_x|\big)\big(|\omega_{xx}|+|\hat{A}_x|\big)\\
&\qquad+\big(\omega_t^2+\hat{m}^2\big)\big(\omega_{xx}^2+\hat{A}_x^2\big)\Big)|\omega_{xt}|\mathrm{d}x\\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{\alpha m^2}{2A^2}\omega_{xx}^2\mathrm{d}x+\frac{\mu\|\omega_{xt}(t)\|^2}{12A_\infty}\\
&+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xx}(t)\|^2+Ce^{-ct}\|\omega_{xx}(t)\|^2+C\delta e^{-ct},
\end{aligned}\tag{3.38}
$$

$$\begin{aligned}\int_{\mathbb{R}} F_{4x}\omega_{xt}\mathrm{d}x=&\int_{\mathbb{R}}\frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxxt}\omega_{xt}\mathrm{d}x-\int_{\mathbb{R}}\frac{\nu}{4\rho A_r\sqrt{A}}A_t\omega_{xxx}\omega_{xt}\mathrm{d}x\\&+C\int_{\mathbb{R}}\Big(|\hat{A}_{xxt}|+|\hat{A}_{xx}|+\big(\omega_{xx}^2+\hat{A}_x^2\big)\big(|\omega_{xt}|+|\hat{A}_t|\big)\Big)|\omega_{xt}|\mathrm{d}x\\\le&-\int_{\mathbb{R}}\frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xxt}^2\mathrm{d}x+\frac{\mu\|\omega_{xt}(t)\|^2}{12A_\infty}+C\delta e^{-ct}.\end{aligned}\tag{3.39}$$

Putting (3.33)-(3.35) and (3.38)-(3.39) into (3.32) yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[\omega_{xt}^2+\left(P'(A_\infty)-\frac{\alpha m^2}{A^2}\right)\omega_{xx}^2\right]\mathrm{d}x+c\int_{\mathbb{R}}\big(\omega_{xt}^2+\nu\omega_{xxt}^2\big)\mathrm{d}x\\\le&-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\big]\frac{\omega_{xx}^2}{2}\mathrm{d}x\\&+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xx}(t)\|^2+Ce^{-ct}\|\omega_{xx}(t)\|^2+C\delta e^{-ct}.\end{aligned}\tag{3.40}$$

We have from (3.31) $\times\lambda$ + (3.40) $(0<\lambda\ll 1)$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[\omega_{xt}^2+\frac{\lambda\mu\omega_x^2}{\omega_x+A_\infty+\hat{A}}+2\lambda\omega_x\omega_{xt}+\left(P'(A_\infty)+\frac{\lambda\nu\sqrt{A}}{2\rho A_r}-\frac{\alpha m^2}{A^2}\right)\omega_{xx}^2\right]\mathrm{d}x\\&\quad+c\int_{\mathbb{R}}\big(\omega_{xx}^2+\omega_{xt}^2+\nu\omega_{xxt}^2\big)\mathrm{d}x\\\le&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\big]\omega_{xx}^2\mathrm{d}x\\&\quad+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_x(t)\|^2+Ce^{-ct}\|\omega_x(t)\|_1^2+C\delta e^{-ct}.\end{aligned}\tag{3.41}$$

In order to claim $\|\omega_{xx}(\tau)\|^2$ is a good term, we require

$$P'(A_\infty)-\frac{\alpha m^2}{A^2}>0,\tag{3.42}$$

this can be obtained by (2.16), where we have taken ε and δ small enough.

Integrating (3.41) over $[0,t]$ and noticing (3.42), we can obtain

$$\begin{aligned}&\|\omega_x(t)\|_1^2+\|\omega_{xt}(t)\|^2+\int_0^t\Big(\|\omega_{xx}(\tau)\|^2+\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\Big)\mathrm{d}\tau\\\le&C\Big(\|\omega_0\|_2^2+\|z_0\|_1^2+\delta\Big)+C\varepsilon\int_0^t\|\omega_x(\tau)\|^2\mathrm{d}\tau+C\int_0^t e^{-c\tau}\|\omega_x(\tau)\|_1^2\mathrm{d}\tau.\end{aligned}\tag{3.43}$$

It follows from (3.24) and (3.43) that

$$\|\omega(t)\|_2^2+\|\omega_t(t)\|_1^2+\int_0^t\Big(\|\omega_x(\tau)\|_1^2+\|\omega_t(\tau)\|_1^2+\nu\|\omega_{xxt}(\tau)\|^2\Big)\mathrm{d}\tau$$

$$\leq C\Big(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\Big)+C\int_0^t e^{-c\tau}\|\omega(\tau)\|_2^2\mathrm{d}\tau. \tag{3.44}$$

By using Gronwall's inequality, we obtain

$$\sup_{0\leq t\leq T}\Big\{\|\omega(t)\|_2^2+\|\omega_t(t)\|_1^2\Big\}\leq C\Big(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\Big). \tag{3.45}$$

Arranging (3.44) and (3.45), we get the desired inequality (3.7). This completes the proof of Lemma 3.3. $\square$

Lemma 3.4. *Suppose that all the conditions in Proposition* 3.1 *hold. If ε and δ are sufficiently small, then for $0\leq t\leq T$, we have*

$$\begin{aligned}
&(1+t)\big(\|\omega_x(t)\|^2+\|\omega_t(t)\|^2\big)+\int_0^t(1+\tau)\|\omega_t(\tau)\|^2\mathrm{d}\tau\\
\leq\,& C\Big(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\Big),
\end{aligned}\tag{3.46}$$

$$\begin{aligned}
&(1+t)^2\big(\|\omega_{xx}(t)\|^2+\|\omega_{xt}(t)\|^2\big)\\
&\quad+\int_0^t\Big[(1+\tau)\|\omega_{xx}(\tau)\|^2+(1+\tau)^2\Big(\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\Big)\Big]\mathrm{d}\tau\\
\leq\,& C\Big(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\Big).
\end{aligned}\tag{3.47}$$

Proof. Multiplying (3.22) by $(1+t)$ and integrating the resulting equation with respect to t over $[0,t]$, it is easy to obtain that

$$\begin{aligned}
&\frac{1}{2}(1+t)\int_{\mathbb{R}}\Big[\omega_t^2+P'(A_\infty)\omega_x^2\Big]\mathrm{d}x+c\int_0^t\int_{\mathbb{R}}(1+t)\big(\omega_t^2+\nu\omega_{xt}^2\big)\mathrm{d}x\\
\leq\,& C\int_0^t\Big(\|\omega_x(t)\|_1^2+\|\omega_t(t)\|_1^2\Big)+C\Big(\|\omega_0\|_1^2+\|\omega_1\|^2+\delta\Big).
\end{aligned}\tag{3.48}$$

The above inequality together with Lemma 3.3 implies (3.46).

The same process applied to (3.41) deduces that

$$\begin{aligned}
&(1+t)\Big(\|\omega_x(t)\|_1^2+\|\omega_{xt}(t)\|^2\Big)\\
&\quad+\int_0^t\Big[(1+\tau)\big(\|\omega_{xx}(\tau)\|^2+\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\big)\Big]\mathrm{d}\tau\\
\leq\,& C\Big(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\Big).
\end{aligned}\tag{3.49}$$

Moreover, multiplying (3.40) by $(1+t)^2$ and integrating it over $[0,t]$, applying Lemma 3.3 and (3.49), we get

$$\begin{aligned}&(1+t)^2\left(\|\omega_{xx}(t)\|^2+\|\omega_{xt}(t)\|^2\right)\\&\quad+\int_0^t(1+\tau)^2\left(\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\right)\mathrm{d}\tau\\&\leq C\left(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\right).\end{aligned}\tag{3.50}$$

Arranging (3.49) and (3.50), we have (3.47). This completes the proof. $\square$

Lemma 3.5. *Suppose that all the conditions in Proposition* 3.1 *hold. If ε and δ are sufficiently small, then for $0\leq t\leq T$, we have*

$$\begin{aligned}&(1+t)^3\left(\|\omega_{xxx}(t)\|^2+\|\omega_{xxt}(t)\|^2\right)\\&\quad+\int_0^t\left[(1+\tau)^2\|\omega_{xxx}(\tau)\|^2+(1+\tau)^3\left(\|\omega_{xxt}(\tau)\|^2+\nu\|\omega_{xxxt}(\tau)\|^2\right)\right]\mathrm{d}\tau\\&\leq C\left(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\right).\end{aligned}\tag{3.51}$$

Proof. Differentiating Eq. (2.14a) in x, we multiply the resulting equation by $-\omega_{xxx}$ and integrating it with respect to x over $\mathbb{R}$, after integration by parts, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega_{xx}^2}{2(\omega_x+A_\infty+\hat{A})}+\omega_{xx}\omega_{xxt}\right)\mathrm{d}x+\int_{\mathbb{R}}P'(A_\infty)\omega_{xxx}^2\mathrm{d}x\\&=\int_{\mathbb{R}}\omega_{xxt}^2\mathrm{d}x-\mu\int\frac{(\omega_{xt}+\hat{A}_t)\omega_{xx}^2}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x\\&\quad+\mu\int_{\mathbb{R}}\frac{(\omega_{xt}\omega_{xx}-\omega_t\omega_{xxx})(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x-\sum_{i=1}^4\int_{\mathbb{R}}F_{ix}\omega_{xxx}\mathrm{d}x.\end{aligned}\tag{3.52}$$

By using (3.3), (3.4), Lemma 3.1 and Cauchy-Schwarz's inequality, it is easy to deduce

$$\begin{aligned}&-\mu\int_{\mathbb{R}}\frac{(\omega_{xt}+\hat{A}_t)\omega_{xx}^2}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x+\mu\int_{\mathbb{R}}\frac{(\omega_{xt}\omega_{xx}-\omega_t\omega_{xxx})(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x\\&\leq C\int_{\mathbb{R}}\left(|\omega_{xt}|+|\hat{A}_t|\right)\omega_{xx}^2\mathrm{d}x+C\int_{\mathbb{R}}\left(|\omega_{xt}||\omega_{xx}|+|\omega_t||\omega_{xxx}|\right)\left(|\omega_{xx}|+|\hat{A}_x|\right)\mathrm{d}x\\&\leq\frac{P'(A_\infty)}{20}\|\omega_{xxx}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xx}(t)\|^2+C\delta e^{-ct}.\end{aligned}\tag{3.53}$$

As shown in [20], we can estimate

$$
\begin{aligned}
-\int_{\mathbb{R}} F_{1x}\omega_{xxx}\mathrm{d}x &\leq C\int_{\mathbb{R}}\Big(\omega_{xx}^2+\hat{A}_x^2+|\hat{A}_{xx}|+|\omega_{xxx}|\big(|\omega_x|+|\hat{A}|\big)\Big)|\omega_{xxx}|\mathrm{d}x \\
&\leq \frac{P'(A_\infty)}{20}\|\omega_{xxx}(t)\|^2+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{xx}(t)\|+C\delta e^{-ct}. \qquad (3.54)
\end{aligned}
$$

Then by performing some similar computations as (3.35)-(3.39), we can get

$$
\begin{aligned}
-\int_{\mathbb{R}} F_{2x}\omega_{xxx}\mathrm{d}x \leq & C\int_{\mathbb{R}}\big(|\omega_x|+|\hat{A}|\big)|\hat{m}_x||\omega_{xxx}|\mathrm{d}x \\
&+C\int_{\mathbb{R}}\big(|\omega_{xx}|+|\hat{A}_x|\big)|\hat{m}||\omega_{xxx}|\mathrm{d}x \\
&+C\int_{\mathbb{R}}\big(|\omega_x|+|\hat{A}|\big)\big(|\omega_{xx}|+|\hat{A}_x|\big)|\hat{m}||\omega_{xxx}|\mathrm{d}x \\
\leq & \frac{P'(A_\infty)}{20}\|\omega_{xxx}(t)\|^2+Ce^{-ct}\|\omega_{xx}(t)\|^2+C\delta e^{-ct}, \qquad (3.55)
\end{aligned}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}} F_{3x}\omega_{xxx}\mathrm{d}x = & \int_{\mathbb{R}}\frac{2\alpha m}{A}\omega_{xxt}\omega_{xxx}\mathrm{d}x+\int_{\mathbb{R}}\frac{\alpha m^2}{A^2}\omega_{xxx}^2\mathrm{d}x \\
&+C\int_{\mathbb{R}}\Big(|\hat{m}_{xx}|+|\hat{A}_{xx}|+\big(\omega_{xt}^2+\hat{m}_x^2\big) \\
&\qquad +\big(|\omega_t|+|\hat{m}|\big)\big(|\omega_{xt}|+|\hat{m}_x|\big)\big(|\omega_{xx}|+|\hat{A}_x|\big) \\
&\qquad +\big(\omega_t^2+\hat{m}^2\big)\big(\omega_{xx}^2+\hat{A}_x^2\big)\Big)|\omega_{xxx}|\mathrm{d}x \\
\leq & \frac{P'(A_\infty)}{20}\|\omega_{xxx}(t)\|^2+C\|\omega_{xxt}(t)\|^2+C\varepsilon(1+t)^{-6}\|\omega_{xx}(t)\|^2 \\
&+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+Ce^{-ct}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct}, \qquad (3.56)
\end{aligned}
$$

$$
\begin{aligned}
-\int_{\mathbb{R}} F_{4x}\omega_{xxx}\mathrm{d}x = & -\int_{\mathbb{R}}\frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxxt}\omega_{xxx}\mathrm{d}x+\int_{\mathbb{R}}\frac{\nu}{4\rho A_r\sqrt{A}}A_t\omega_{xxx}^2\mathrm{d}x \\
&+C\int_{\mathbb{R}}\Big(|\hat{A}_{xxt}|+|\hat{A}_{xx}|+\big(\omega_{xx}^2+\hat{A}_x^2\big)\big(|\omega_{xt}|+|\hat{A}_t|\big)\Big)|\omega_{xxx}|\mathrm{d}x \\
\leq & -\frac{\nu}{4\rho A_r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big(\sqrt{A}\omega_{xxx}^2\big)\mathrm{d}x+\frac{P'(A_\infty)}{20}\|\omega_{xxx}(t)\|^2 \\
&+C\varepsilon(1+t)^{-5}\|\omega_{xt}(t)\|^2+C\delta e^{-ct}. \qquad (3.57)
\end{aligned}
$$

Putting (3.53)-(3.57) into (3.52), we can conclude that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega_{xx}^2}{2(\omega_x+A_\infty+\hat{A})}+\omega_{xx}\omega_{xxt}+\frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xxx}^2\right)\mathrm{d}x+\frac{3}{4}\int_{\mathbb{R}}P'(A_\infty)\omega_{xxx}^2\mathrm{d}x
$$

$$\begin{aligned}
&\leq C\|\omega_{xxt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C(1+t)^{-\frac{7}{4}}\|\omega_{xx}(t)\|^2\\
&\quad +Ce^{-ct}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct}.
\end{aligned}\tag{3.58}$$

Differentiating (2.14a) twice in x, we multiply the resulting equation by ω_{xxt} and integrate the corresponding product in x over R, thus obtaining

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_{xxt}^2+P'(A_\infty)\omega_{xxx}^2\right)\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu\omega_{xxt}^2}{\omega_x+A_\infty+\hat{A}}\mathrm{d}x \\
=&\mu\int_{\mathbb{R}}\left(\frac{\omega_{xt}(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}+\left(\frac{\omega_t(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}\right)_x\right)\omega_{xxt}\mathrm{d}x+\sum_{i=1}^4\int_{\mathbb{R}}F_{ixx}\omega_{xxt}\mathrm{d}x.
\end{aligned}\tag{3.59}$$

By employing (3.3), (3.4), Lemma 3.1, and Cauchy-Schwarz's inequality. The first term on the right-hand side of (3.59) can be estimated as

$$\begin{aligned}
&\mu\int_{\mathbb{R}}\left(\frac{\omega_{xt}(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}+\left(\frac{\omega_t(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}\right)_x\right)\omega_{xxt}\mathrm{d}x \\
\leq& C\int_{\mathbb{R}}\Big(|\omega_{xt}|\big(|\omega_{xx}|+|\hat{A}_x|\big)+|\omega_t|\big(|\omega_{xxx}|+|\hat{A}_{xx}|\big)+|\omega_t|\big(\omega_{xx}^2+\hat{A}_x^2\big)\Big)|\omega_{xxt}|\mathrm{d}x \\
\leq& \frac{\mu}{15A_\infty}\|\omega_{xxt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xx}(t)\|^2+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct}.
\end{aligned}\tag{3.60}$$

Next, as shown in [20], we have

$$\begin{aligned}
\int_{\mathbb{R}}F_{1xx}\omega_{xxt}\mathrm{d}x\leq& \frac{\mu}{15A_\infty}\|\omega_{xxt}(t)\|^2-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big(P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\big)\omega_{xxx}^2\mathrm{d}x \\
&+C\varepsilon(1+t)^{-5}\|\omega_{xx}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct}.
\end{aligned}\tag{3.61}$$

Then we will go on to treat the last three terms. Noticing that

$$\begin{aligned}
F_{2xx}=&-\mu\left[\frac{\hat{m}_x(\omega_x+\hat{A})+\hat{m}(\omega_{xx}+\hat{A}_x)}{AA_\infty}-\frac{\hat{m}(\omega_x+\hat{A})A_x}{A^2A_\infty}\right]_x \\
=&\mathcal{O}(1)\Big(\hat{m}_{xx}(\omega_x+\hat{A})+\hat{m}_x(\omega_{xx}+\hat{A}_x)+m(\omega_{xxx}+\hat{A}_{xx}) \\
&\qquad +\hat{m}_x(\omega_x+\hat{A})A_x+\hat{m}(\omega_{xx}+\hat{A}_x)A_x \\
&\qquad +\hat{m}(\omega_x+\hat{A})A_{xx}+\hat{m}(\omega_x+\hat{A})A_x^2\Big),
\end{aligned}\tag{3.62}$$

$$F_{3xx}=-\frac{2\alpha m}{A}\omega_{xxxt}-\frac{\alpha m^2}{A^2}\omega_{xxxx}+\mathcal{O}(1)\Big(m_xm_{xx}+m\hat{m}_{xxx}+m_x^2A_x+mm_{xx}A_x$$

$$+mm_xA_{xx}+mm_xA_x^2+m^2\hat{A}_{xxx}+m^2A_xA_{xx}+m^2A_x^3\Big), \tag{3.63}$$

$$\begin{aligned}F_{4xx}=&\frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxxxt}-\frac{\nu}{4\rho A_r\sqrt{A}}(A_t\omega_{xxxx}+A_{xx}A_{xt})\\&+\nu\mathcal{O}(1)\big(A_tA_x^3+A_{xt}A_x^2+A_tA_xA_{xx}+A_t\hat{A}_{xxx}+A_xA_{xxt}+\hat{A}_{xxxt}\big),\end{aligned} \tag{3.64}$$

again applying (3.3), (3.4), Lemma 3.1 and Cauchy-Schwarz's inequality, we have

$$\int_{\mathbb{R}}F_{2xx}\omega_{xxt}\mathrm{d}x\le\frac{\mu}{15A_\infty}\|\omega_{xxt}(t)\|^2+Ce^{-ct}\|\omega_{xx}(t)\|_1^2+C\delta e^{-ct}, \tag{3.65}$$

$$\begin{aligned}\int_{\mathbb{R}}F_{3xx}\omega_{xxt}\mathrm{d}x\le&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{\alpha m^2}{2A^2}\omega_{xxx}^2\mathrm{d}x+\frac{\mu}{15A_\infty}\|\omega_{xxt}(t)\|^2+C\varepsilon(1+t)^{-7}\|\omega_{xx}(t)\|^2\\&+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xxx}(t)\|^2+Ce^{-ct}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct},\end{aligned} \tag{3.66}$$

$$\begin{aligned}\int_{\mathbb{R}}F_{4xx}\omega_{xxt}\mathrm{d}x\le&-\int_{\mathbb{R}}\frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xxxt}^2\mathrm{d}x+\frac{\mu}{15A_\infty}\|\omega_{xxt}(t)\|^2+C\varepsilon(1+t)^{-\frac{17}{2}}\|\omega_{xx}(t)\|^2\\&+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct}.\end{aligned} \tag{3.67}$$

Putting (3.60)-(3.61), (3.65)-(3.67) into (3.59), we can conclude that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[\omega_{xxt}^2+\left(P'(A_\infty)-\frac{\alpha m^2}{A^2}\right)\omega_{xxx}^2\right]\mathrm{d}x+c\int_{\mathbb{R}}(\omega_{xxt}^2+\nu\omega_{xxxt}^2)\mathrm{d}x\\\le&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right]\omega_{xxx}^2\mathrm{d}x\\&+C(1+t)^{-\frac{7}{2}}\|\omega_{xx}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xxx}(t)\|^2\\&+Ce^{-ct}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct}.\end{aligned} \tag{3.68}$$

We have from (3.59) $\times\lambda$ + (3.68) $(0<\lambda\ll 1)$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[\omega_{xxt}^2+\frac{\lambda\mu\omega_{xx}^2}{\omega_x+A_\infty+\hat{A}}+2\lambda\omega_{xx}\omega_{xxt}+\left(P'(A_\infty)+\frac{\lambda\nu\sqrt{A}}{2\rho A_r}-\frac{\alpha m^2}{A^2}\right)\omega_{xxx}^2\right]\mathrm{d}x\\&\quad+c\int_{\mathbb{R}}(\omega_{xxx}^2+\omega_{xxt}^2+\nu\omega_{xxxt}^2)\mathrm{d}x\\\le&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right]\omega_{xxx}^2\mathrm{d}x+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2\\&\quad+C(1+t)^{-\frac{7}{4}}\|\omega_{xx}(t)\|^2+Ce^{-ct}\|\omega_{xxx}(t)\|^2+C\delta e^{-ct}.\end{aligned} \tag{3.69}$$

Integrating (3.69) over $[0,t]$ and using Lemma 3.4, one gets

$$\begin{aligned}&\|\omega_{xx}(t)\|_1^2+\|\omega_{xxt}(t)\|^2+\int_0^t\Big(\|\omega_{xxx}(\tau)\|^2+\|\omega_{xxt}(\tau)\|^2+\nu\|\omega_{xxxt}(\tau)\|^2\Big)\mathrm{d}\tau\\ \leq& C\Big(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\Big)+C\int_0^t e^{-c\tau}\|\omega_{xxx}(\tau)\|^2\mathrm{d}\tau,\end{aligned}\tag{3.70}$$

and the Gronwall's inequality implies

$$\begin{aligned}&\|\omega_{xx}(t)\|_1^2+\|\omega_{xxt}(t)\|^2+\int_0^t\Big(\|\omega_{xxx}(\tau)\|^2+\|\omega_{xxt}(\tau)\|^2+\nu\|\omega_{xxxt}(\tau)\|^2\Big)\mathrm{d}\tau\\ \leq& C\Big(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\Big).\end{aligned}\tag{3.71}$$

Multiplying (3.69) by $(1+t)^k$ $(k=1,2)$ respectively and integrating the resulting equations over $[0,t]$, by applying Lemma 3.4 and (3.71), we have

$$\begin{aligned}&(1+t)^2\Big(\|\omega_{xx}(t)\|_1^2+\|\omega_{xxt}(t)\|^2\Big)\\ &\quad+\int_0^t(1+\tau)^2\Big(\|\omega_{xxx}(\tau)\|^2+\|\omega_{xxt}(\tau)\|^2+\nu\|\omega_{xxxt}(\tau)\|^2\Big)\mathrm{d}\tau\\ \leq& C\Big(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\Big).\end{aligned}\tag{3.72}$$

Moreover multiplying $(1+t)^3\cdot$ (3.68) and integrating the resulting equations over $[0,t]$, together with Lemma 3.4 and (3.72), it is easy to obtain that

$$\begin{aligned}&(1+t)^3\Big(\|\omega_{xxx}(t)\|^2+\|\omega_{xxt}(t)\|^2\Big)\\ &\quad+\int_0^t(1+\tau)^3\Big(\|\omega_{xxt}(\tau)\|^2+\nu\|\omega_{xxxt}(\tau)\|^2\Big)\mathrm{d}\tau\\ \leq& C\Big(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\Big).\end{aligned}\tag{3.73}$$

Arranging (3.71)-(3.73), we have (3.51). This completes the proof. □

Lemma 3.6. *Suppose that all the conditions in Proposition* 3.1 *hold. If* ε *and* δ *are sufficiently small, then for* $0\leq t\leq T$, *we have*

$$\begin{aligned}&(1+t)^2\|\omega_t(t)\|^2+(1+t)^3\Big(\|\omega_{tt}(t)\|^2+\|\omega_{xt}(t)\|^2\Big)\\ &\quad+\int_0^t\Big[(1+\tau)^2\|\omega_{xt}(\tau)\|^2+(1+\tau)^3\big(\|\omega_{tt}(\tau)\|^2+\nu\|\omega_{xtt}\|^2\big)\Big]\mathrm{d}\tau\\ \leq& C\Big(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\Big).\end{aligned}\tag{3.74}$$

Proof. Differentiating the Eq. (2.14a) in t, we multiply the resulting equation by $-\omega_t$ and integrating it with respect to x over $\mathbb{R}$, after integration by parts, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega_t^2}{2(\omega_x+A_\infty+\hat{A})}+\omega_t\omega_{tt}\right)\mathrm{d}x+\int_{\mathbb{R}}P'(A_\infty)\omega_{xt}^2\mathrm{d}x\\&=\int_{\mathbb{R}}\omega_{tt}^2\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu\omega_t^2(\omega_{xt}+\hat{A}_t)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x+\sum_{i=1}^4\int_{\mathbb{R}}F_{it}\omega_t\mathrm{d}x.\end{aligned}\tag{3.75}$$

By using (3.3), (3.4), Lemma 3.1 and Cauchy-Schwarz's inequality. The first three terms on the right-hand side of (3.75) can be estimated as

$$\int_{\mathbb{R}}\frac{\mu\omega_t^2(\omega_{xt}+\hat{A}_t)}{2(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x\le C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_t(t)\|^2+C\delta e^{-ct},\tag{3.76}$$

$$\int_{\mathbb{R}}F_{1t}\omega_t\mathrm{d}x\le\frac{P'(A_\infty)}{16}\|\omega_{xt}(t)\|^2+C\delta e^{-ct},\tag{3.77}$$

$$\begin{aligned}\int_{\mathbb{R}}F_{2t}\omega_t\mathrm{d}x\le{}&C\int_{\mathbb{R}}\Big(|\hat{m}_t|\big(|\omega_x|+|\hat{A}|\big)+|\hat{m}|\big(|\omega_{xt}|+|\hat{A}_t|\big)\\&+|\hat{m}|\big(|\omega_x|+|\hat{A}|\big)\big(|\omega_{xt}|+|\hat{A}_t|\big)\Big)|\omega_t|\mathrm{d}x\\\le{}&\frac{P'(A_\infty)}{16}\|\omega_{xt}(t)\|^2+Ce^{-ct}\Big(\|\omega_t(t)\|^2+\|\omega_x(t)\|^2\Big)+C\delta e^{-ct}.\end{aligned}\tag{3.78}$$

Then, noticing that

$$\begin{aligned}F_{3t}={}&-\frac{2\alpha m}{A}\omega_{xtt}-\frac{\alpha m^2}{A^2}\omega_{xxt}+\mathcal{O}(1)\big(m_tm_x+m\hat{m}_{xt}+mm_xA_t\\&+mm_tA_x+m^2\hat{A}_{xt}+m^2A_xA_t\big),\end{aligned}\tag{3.79}$$

$$F_{4t}=\frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxtt}+\nu\mathcal{O}(1)\big(A_t^2A_x+A_xA_{tt}+\hat{A}_{xtt}\big),\tag{3.80}$$

again applying (3.3), (3.4), Lemma 3.1 and Cauchy-Schwarz's inequality, the other terms on the right-hand side of (3.75) can be bounded as

$$\begin{aligned}\int_{\mathbb{R}}F_{3t}\omega_t\mathrm{d}x\le{}&\frac{P'(A_\infty)}{16}\|\omega_{xt}(t)\|^2+C\|\omega_{tt}(t)\|^2+C(1+t)^{-\frac{7}{2}}\|\omega_{xx}(t)\|^2\\&+C(1+t)^{-\frac{7}{2}}\|\omega_t(t)\|^2+Ce^{-ct}\|\omega_{xt}(t)\|^2+C\delta e^{-ct},\end{aligned}\tag{3.81}$$

$$\begin{aligned}\int_{\mathbb{R}}F_{4t}\omega_t\mathrm{d}x\le{}&-\frac{\nu}{4\rho A_r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big(\sqrt{A}\omega_{xt}^2\big)\mathrm{d}x+\frac{P'(A_\infty)}{16}\|\omega_{xt}(t)\|^2+C\nu\|\omega_{xtt}(t)\|^2\\&+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{xx}(t)\|^2+C\delta e^{-ct}.\end{aligned}\tag{3.82}$$

Putting (3.76)-(3.78), (3.81)-(3.82) into (3.75), we can conclude that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega_t^2}{2(\omega_x+A_\infty+\hat{A})}+\omega_t\omega_{tt}+\frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xt}^2\right)\mathrm{d}x+\frac{3}{4}\int_{\mathbb{R}}P'(A_\infty)\omega_{xt}^2\mathrm{d}x\\ \leq\, & C\nu\|\omega_{xtt}\|^2+C\|\omega_{tt}(t)\|^2+C(1+t)^{-\frac{5}{2}}\|\omega_{xx}\|^2+C(1+t)^{-\frac{7}{4}}\|\omega_t(t)\|^2\\ &+Ce^{-ct}\left(\|\omega_{xt}(t)\|^2+\|\omega_x(t)\|^2\right)+C\delta e^{-ct}. \end{aligned}\tag{3.83}$$

Differentiating Eq. (2.14a) in t, we multiply the resulting equation by ω_{tt} and obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_{tt}^2+P'(A_\infty)\omega_{xt}^2\right)\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu\omega_{tt}^2}{\omega_x+A_\infty+\hat{A}}\mathrm{d}x\\ =\,&\mu\int_{\mathbb{R}}\frac{\omega_t\omega_{tt}(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x+\sum_{i=1}^{4}\int_{\mathbb{R}}F_{it}\omega_{tt}\mathrm{d}x.\end{aligned}\tag{3.84}$$

By direct computations, we have

$$\int_{\mathbb{R}}\frac{\omega_t\omega_{tt}(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\mathrm{d}x\leq\frac{\mu}{15A_\infty}\|\omega_{tt}(t)\|^2+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{xt}(t)\|^2+C\delta e^{-ct},\tag{3.85}$$

$$\begin{aligned}\int_{\mathbb{R}}F_{1t}\omega_{tt}\mathrm{d}x\leq\,&\frac{\mu}{15A_\infty}\|\omega_{tt}(t)\|^2-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right)\omega_{xt}^2\mathrm{d}x\\ &+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xt}(t)\|^2+C\delta e^{-ct},\end{aligned}\tag{3.86}$$

$$\begin{aligned}\int_{\mathbb{R}}F_{2t}\omega_{tt}\mathrm{d}x\leq\,& C\int_{\mathbb{R}}\Big(|\hat{m}_t|\left(|\omega_x|+|\hat{A}|\right)+|\hat{m}|\left(|\omega_{xt}|+|\hat{A}_t|\right)\\ &\qquad+|\hat{m}|\left(|\omega_x|+|\hat{A}|\right)\left(|\omega_{xt}|+|\hat{A}_t|\right)\Big)|\omega_{tt}|\mathrm{d}x\\ \leq\,&\frac{\mu}{15A_\infty}\|\omega_{tt}(t)\|^2+Ce^{-ct}\left(\|\omega_{xt}(t)\|^2+\|\omega_x\|^2\right)+C\delta e^{-ct}.\end{aligned}\tag{3.87}$$

Applying Eqs. (3.3), (3.4), (3.79), (3.80), together with Lemma 3.1 and Cauchy-Schwarz's inequality yields that

$$\begin{aligned}\int_{\mathbb{R}}F_{3t}\omega_{tt}\mathrm{d}x\leq\,&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{\alpha m^2}{2A^2}\omega_{xt}^2\mathrm{d}x+\frac{\mu}{15A_\infty}\|\omega_{tt}(t)\|^2\\ &+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C(1+t)^{-7}\|\omega_{xx}(t)\|^2\\ &+Ce^{-ct}\|\omega_{xt}(t)\|^2+C\delta e^{-ct},\end{aligned}\tag{3.88}$$

$$\begin{aligned}\int_{\mathbb{R}} F_{4t}\omega_{tt}\mathrm{d}x \leq& -\int_{\mathbb{R}} \frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xtt}^2\mathrm{d}x + \frac{\mu}{15A_\infty}\|\omega_{tt}(t)\|^2 \\ &+C\varepsilon(1+t)^{-7}\|\omega_{xx}(t)\|^2 + C\delta e^{-ct}. \end{aligned} \tag{3.89}$$

Putting (3.85)-(3.89) into (3.84), we can conclude that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_{tt}^2 + \left(P'(A_\infty) - \frac{\alpha m^2}{A^2}\right)\omega_{xt}^2\right)\mathrm{d}x + c\int_{\mathbb{R}}\left(\omega_{tt}^2 + \nu\omega_{xtt}^2\right)\mathrm{d}x \\ \leq& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[P'(\omega_x + A_\infty + \hat{A}) - P'(A_\infty)\right]\omega_{xt}^2\mathrm{d}x \\ &+C(1+t)^{-7}\|\omega_{xx}(t)\|^2 + C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xt}(t)\|^2 \\ &+Ce^{-ct}\left(\|\omega_{xt}(t)\|^2 + \|\omega_x(t)\|^2\right) + C\delta e^{-ct}. \end{aligned} \tag{3.90}$$

We have from (3.83) $\times\lambda$ + (3.90) $(0<\lambda\ll 1)$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_{tt}^2 + \frac{\lambda\mu\omega_t^2}{\omega_x + A_\infty + \hat{A}} + 2\lambda\omega_t\omega_{tt} + \left(P'(A_\infty) + \frac{\lambda\nu\sqrt{A}}{2\rho A_r} - \frac{\alpha m^2}{A^2}\right)\omega_{xt}^2\right)\mathrm{d}x \\ &+c\int_{\mathbb{R}}\left(\omega_{xt}^2 + \omega_{tt}^2 + \nu\omega_{xtt}^2\right)\mathrm{d}x \\ \leq& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[P'(\omega_x + A_\infty + \hat{A}) - P'(A_\infty)\right]\omega_{xt}^2\mathrm{d}x + C(1+t)^{-\frac{5}{2}}\|\omega_{xx}\|^2 \\ &+C(1+t)^{-\frac{7}{4}}\|\omega_t(t)\|^2 + Ce^{-ct}\left(\|\omega_{xt}(t)\|^2 + \|\omega_x(t)\|^2\right) + C\delta e^{-ct}. \end{aligned} \tag{3.91}$$

Multiplying (3.91) by $(1+t)^k$ $(k=0,1,2)$ respectively and integrating the resulting equations over $[0,t]$, by applying Lemmas 3.3-3.4, we obtain

$$\begin{aligned}&(1+t)^2\left(\|\omega_t(t)\|_1^2 + \|\omega_{tt}(t)\|^2\right) \\ &\quad+\int_0^t (1+\tau)^2\left(\|\omega_{tt}(\tau)\|^2 + \|\omega_{xt}(\tau)\|^2 + \nu\|\omega_{xtt}(\tau)\|^2\right)\mathrm{d}\tau \\ \leq& C\left(\|\omega_0\|_2^2 + \|\omega_1\|_1^2 + \delta\right). \end{aligned} \tag{3.92}$$

Moreover, multiplying Eq. (3.90) by $(1+t)^3$ and integrating the resulting equations over $[0,t]$, together with Lemmas 3.3-3.4 and (3.92), it is easy to deduce

$$\begin{aligned}&(1+t)^3\left(\|\omega_{xt}(t)\|^2 + \|\omega_{tt}(t)\|^2\right) \\ &\quad+\int_0^t (1+\tau)^3\left(\|\omega_{tt}(\tau)\|^2 + \nu\|\omega_{xtt}(\tau)\|^2\right)\mathrm{d}\tau \\ \leq& C\left(\|\omega_0\|_2^2 + \|\omega_1\|_1^2 + \delta\right). \end{aligned} \tag{3.93}$$

Arranging (3.92) and (3.93), we can obtain (3.74). This completes the proof of Lemma 3.6. $\square$

Lemma 3.7. *Suppose that all the conditions in Proposition* 3.1 *hold. If ε and δ are sufficiently small, then for $0\le t\le T$, we have*

$$\begin{aligned}&(1+t)^4\Big(\|\omega_{xtt}(t)\|^2+\|\omega_{xxt}(t)\|^2\Big)\\&\quad+\int_0^t(1+\tau)^4\Big(\|\omega_{xtt}(\tau)\|^2+\nu\|\omega_{xxtt}(\tau)\|^2\Big)\mathrm{d}\tau\\&\le C\Big(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\Big).\end{aligned}\tag{3.94}$$

Proof. Mutiplying the derivative ∂_{xt} of Eq. (2.14a) by ω_{xtt} and integrating equation in x over R, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\Big(\omega_{xtt}^2+P'(A_\infty)\omega_{xxt}^2\Big)\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu\omega_{xtt}^2}{\omega_x+A_\infty+\hat{A}}\mathrm{d}x\\&=\mu\int_{\mathbb{R}}\left[\frac{\omega_{xt}(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}+\left(\frac{\omega_t(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}\right)_t\right]\omega_{xtt}\mathrm{d}x+\sum_{i=1}^4\int_{\mathbb{R}}F_{ixt}\omega_{xtt}\mathrm{d}x.\end{aligned}\tag{3.95}$$

Now, we deal with the terms on the right-hand side of (3.95). By direct calculations, we first have

$$\begin{aligned}&\mu\int_{\mathbb{R}}\left[\frac{\omega_{xt}(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}+\left(\frac{\omega_t(\omega_{xx}+\hat{A}_x)}{(\omega_x+A_\infty+\hat{A})^2}\right)_t\right]\omega_{xtt}\mathrm{d}x\\&\le C\int_{\mathbb{R}}\Big(|\omega_{xt}|\big(|\omega_{xt}|+|\hat{A}_t|\big)+|\omega_{tt}|\big(|\omega_{xx}|+|\hat{A}_x|\big)+|\omega_t|\big(|\omega_{xxt}|+|\hat{A}_{xt}|\big)\\&\qquad+|\omega_t|\big(|\omega_{xx}|+|\hat{A}_x|\big)\big(|\omega_{xt}|+|\hat{A}_t|\big)\Big)|\omega_{xtt}|\mathrm{d}x\\&\le\frac{\mu}{15A_\infty}\|\omega_{xtt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{tt}(t)\|^2\\&\quad+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{xxt}(t)\|^2+C\delta e^{-ct},\end{aligned}\tag{3.96}$$

and

$$\begin{aligned}\int_{\mathbb{R}}F_{1xt}\omega_{xtt}\mathrm{d}x\le&\frac{\mu}{15A_\infty}\|\omega_{xtt}(t)\|^2-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\big(P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\big)\omega_{xxt}^2\mathrm{d}x\\&+C\varepsilon(1+t)^{-5}\|\omega_{xt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xxx}(t)\|^2\\&+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xxt}(t)\|^2+C\delta e^{-ct}.\end{aligned}\tag{3.97}$$

Secondly, we consider the last three terms. Owing to

$$\begin{aligned}
F_{2xt} &= -\mu\left[\frac{\hat{m}_x(\omega_x+\hat{A})+\hat{m}(\omega_{xx}+\hat{A}_x)}{AA_\infty}-\frac{\hat{m}(\omega_x+\hat{A})A_x}{A^2A_\infty}\right]_t \\
&= \mathcal{O}(1)\Big(\hat{m}_{xt}(\omega_x+\hat{A})+\hat{m}_x(\omega_{xt}+\hat{A}_t)+\hat{m}_t(\omega_{xx}+\hat{A}_x) \\
&\qquad +\hat{m}(\omega_{xxt}+\hat{A}_{xt})+\hat{m}_x(\omega_x+\hat{A})A_t \\
&\qquad +\hat{m}(\omega_{xx}+\hat{A}_x)A_t+\hat{m}_t(\omega_x+\hat{A})A_x \\
&\qquad +\hat{m}(\omega_{xt}+\hat{A}_t)A_x+\hat{m}(\omega_x+\hat{A})A_{xt}+\hat{m}(\omega_x+\hat{A})A_xA_t\Big),
\end{aligned} \tag{3.98}$$

$$\begin{aligned}
F_{3xt} &= -\frac{2\alpha m}{A}\omega_{xxtt}-\frac{\alpha m^2}{A^2}\omega_{xxxt} \\
&\quad +\mathcal{O}(1)\Big(m_xm_{xt}+m_tm_{xx}+m\hat{m}_{xxt}+m_x^2A_t+mm_{xx}A_t \\
&\qquad +m_tm_xA_x+mm_{xt}A_x+mm_xA_{xt} \\
&\qquad +mm_xA_xA_t+mm_tA_{xx}+m^2\hat{A}_{xxt} \\
&\qquad +m^2A_{xx}A_t+mm_tA_x^2+m^2A_xA_{xt}+m^2A_x^2A_t\Big),
\end{aligned} \tag{3.99}$$

$$\begin{aligned}
F_{4xt} &= \frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxxtt}-\frac{\nu}{4\rho A_r\sqrt{A}}A_{xx}A_{tt} \\
&\quad +\nu\mathcal{O}(1)\Big(A_t^2A_x^2+A_{tt}A_x^2+A_tA_xA_{xt}+A_t^2A_{xx}+\hat{A}_{xxtt}\Big),
\end{aligned} \tag{3.100}$$

we can compute that

$$\begin{aligned}
\int_{\mathbb{R}} F_{2xt}\omega_{xtt}\mathrm{d}x \le & \frac{\mu}{15A_\infty}\|\omega_{xtt}(t)\|^2+Ce^{-ct}\Big(\|\omega_{xxt}(t)\|^2+\|\omega_{xx}(t)\|^2\Big) \\
& +C\delta e^{-ct},
\end{aligned} \tag{3.101}$$

$$\begin{aligned}
\int_{\mathbb{R}} F_{3xt}\omega_{xtt}\mathrm{d}x \le & \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{\alpha m^2}{2A^2}\omega_{xxt}^2\mathrm{d}x+\frac{\mu}{15A_\infty}\|\omega_{xtt}(t)\|^2 \\
& +C\varepsilon(1+t)^{-\frac{19}{2}}\|\omega_{xx}(t)\|^2+C(1+t)^{-7}\|\omega_{xxx}(t)\|^2 \\
& +C\varepsilon(1+t)^{-7}\|\omega_{xt}(t)\|^2+C(1+t)^{-\frac{7}{2}}\|\omega_{xxt}(t)\|^2+C\delta e^{-ct},
\end{aligned} \tag{3.102}$$

$$\begin{aligned}
\int_{\mathbb{R}} F_{4xt}\omega_{xtt}\mathrm{d}x \le & -\int_{\mathbb{R}}\frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xxtt}^2\mathrm{d}x+\frac{\mu}{15A_\infty}\|\omega_{xtt}(t)\|^2 \\
& +C\varepsilon(1+t)^{-\frac{17}{2}}\|\omega_{xt}(t)\|^2+C\varepsilon(1+t)^{-7}\|\omega_{xxx}(t)\|^2 \\
& +C\varepsilon(1+t)^{-6}\|\omega_{xxt}(t)\|^2+C\delta e^{-ct}.
\end{aligned} \tag{3.103}$$

Putting (3.96)-(3.97) and (3.101)-(3.103) into (3.95), we can conclude that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[\omega_{xtt}^2+\left(P'(A_\infty)-\frac{\alpha m^2}{A^2}\right)\omega_{xxt}^2\right]\mathrm{d}x+c\int_{\mathbb{R}}(\omega_{xtt}^2+\nu\omega_{xxtt}^2)\mathrm{d}x\\ \le&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right]\omega_{xxt}^2\mathrm{d}x\\ &+C(1+t)^{-\frac{19}{2}}\|\omega_{xx}(t)\|^2+C(1+t)^{-\frac{7}{2}}\|\omega_{xxx}(t)\|^2\\ &+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{tt}(t)\|^2\\ &+C(1+t)^{-\frac{7}{4}}\|\omega_{xxt}(t)\|^2+C\delta e^{-ct}. \end{aligned}\tag{3.104}$$

Multiplying (3.104) by $(1+t)^k$ $(k=0,1,2,3,4)$ respectively and integrating the resulting equations over $[0,t]$, by applying Lemmas 3.4-3.6, we obtain the desired estimate (3.94). This completes the proof of Lemma 3.7. □

Lemma 3.8. *Suppose that all the conditions in Proposition* 3.1 *hold. If ε and δ are sufficiently small, then for $0\le t\le T$, we have*

$$\begin{aligned}&(1+t)^4\|\omega_{tt}(t)\|^2+(1+t)^5\left(\|\omega_{ttt}(t)\|^2+\|\omega_{xtt}(t)\|^2\right)\\ &\quad+\int_0^t(1+\tau)^5\left(\|\omega_{ttt}(\tau)\|^2+\nu\|\omega_{xttt}(\tau)\|^2\right)\mathrm{d}\tau\\ &\le C\left(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\right). \end{aligned}\tag{3.105}$$

Proof. Differentiating (2.14a) twice in t, we multiply the resulting equation by ω_{tt} and integrate the corresponding product in x over R, thus obtaining

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega_{tt}^2}{2(\omega_x+A_\infty+\hat{A})}+\omega_{tt}\omega_{ttt}\right)\mathrm{d}x+\int_{\mathbb{R}}P'(A_\infty)\omega_{xtt}^2\mathrm{d}x\\ =&\int_{\mathbb{R}}\omega_{ttt}^2\mathrm{d}x+\mu\int_{\mathbb{R}}\frac{(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\frac{\omega_{tt}^2}{2}\mathrm{d}x\\ &+\mu\int_{\mathbb{R}}\left(\frac{\omega_t(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\right)_t\omega_{tt}\mathrm{d}x+\sum_{i=1}^4\int_{\mathbb{R}}F_{itt}\omega_{tt}\mathrm{d}x. \end{aligned}\tag{3.106}$$

By performing some similar computations as the work of Lemma 3.7, we can get

$$\mu\int_{\mathbb{R}}\frac{(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\frac{\omega_{tt}^2}{2}\mathrm{d}x+\mu\int_{\mathbb{R}}\left(\frac{\omega_t(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\right)_t\omega_{tt}\mathrm{d}x \tag{3.107}$$

$$\leq \frac{P'(A_\infty)}{20}\|\omega_{xtt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{tt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C\delta e^{-ct},$$

$$\int_{\mathbb{R}} F_{1tt}\omega_{tt}\mathrm{d}x \leq \frac{P'(A_\infty)}{20}\|\omega_{xtt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C\delta e^{-ct}. \tag{3.108}$$

Notice that

$$\begin{aligned} F_{2tt}=&-\mu\left[\frac{\hat{m}_t(\omega_x+\hat{A})+\hat{m}(\omega_{xt}+\hat{A}_t)}{AA_\infty}-\frac{\hat{m}(\omega_x+\hat{A})A_t}{A^2A_\infty}\right]_t\\ =&\mathcal{O}(1)\Big(\hat{m}_{tt}(\omega_x+\hat{A})+\hat{m}_t(\omega_{xt}+\hat{A}_t)+\hat{m}(\omega_{xtt}+\hat{A}_{tt})\\ &\qquad+\hat{m}_t(\omega_x+\hat{A})A_t+\hat{m}(\omega_{xt}+\hat{A}_t)A_t\\ &\qquad+m(\omega_x+\hat{A})A_{tt}+\hat{m}(\omega_x+\hat{A})A_t^2\Big), \end{aligned} \tag{3.109}$$

$$\begin{aligned} F_{3tt}=&-\frac{2\alpha m}{A}\omega_{xttt}-\frac{\alpha m^2}{A^2}\omega_{xxtt}\\ &+\mathcal{O}(1)\Big(m_{tt}m_x+m_tm_{xt}+m\hat{m}_{xtt}+m_tm_xA_t+mm_{xt}A_t\\ &\qquad+mm_xA_{tt}+mm_xA_t^2+m_t^2A_x\\ &\qquad+mm_{tt}A_x+mm_tA_{xt}+m^2\hat{A}_{xtt}\\ &\qquad+mm_tA_xA_t+m^2A_{xt}A_t+m^2A_xA_{tt}+m^2A_xA_t^2\Big), \end{aligned} \tag{3.110}$$

$$\begin{aligned} F_{4tt}=&\frac{\nu\sqrt{A}}{2\rho A_r}\omega_{xxttt}-\frac{\nu}{4\rho A_r\sqrt{A}}A_{tt}A_{xt}\\ &+\nu\mathcal{O}(1)\Big(A_t^3A_x+A_tA_{tt}A_x+A_t^2A_{xt}+A_{ttt}A_x+A_tA_{xtt}+\hat{A}_{xttt}\Big). \end{aligned} \tag{3.111}$$

After some detailed calculations, we obtain

$$\begin{aligned} \int_{\mathbb{R}} F_{2tt}\omega_{tt}\mathrm{d}x\leq&\frac{P'(A_\infty)}{20}\|\omega_{xtt}(t)\|^2+Ce^{-ct}\Big(\|\omega_x(t)\|^2+\|\omega_{xt}(t)\|^2+\|\omega_{tt}(t)\|^2\Big)\\ &+C\delta e^{-ct}, \end{aligned} \tag{3.112}$$

$$\begin{aligned} \int_{\mathbb{R}} F_{3tt}\omega_{tt}\mathrm{d}x\leq&\frac{P'(A_\infty)}{20}\|\omega_{xtt}(t)\|^2+C\|\omega_{ttt}(t)\|^2+C(1+t)^{-\frac{9}{2}}\|\omega_{xt}(t)\|^2\\ &+C(1+t)^{-\frac{9}{2}}\|\omega_{xxt}(t)\|^2+C(1+t)^{-\frac{5}{2}}\|\omega_{tt}(t)\|^2\\ &+Ce^{-ct}\Big(\|\omega_{xx}(t)\|^2+\|\omega_{xtt}(t)\|^2\Big)+C\delta e^{-ct}, \end{aligned} \tag{3.113}$$

$$\int_{\mathbb{R}} F_{4tt}\omega_{tt}\mathrm{d}x\leq-\frac{\nu}{4\rho A_r}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\sqrt{A}\omega_{xtt}^2\mathrm{d}x+\frac{P'(A_\infty)}{20}\|\omega_{xtt}(t)\|^2$$

$$+C\nu\|\omega_{xttt}(t)\|^2+C\nu\|\omega_{xxtt}(t)\|^2+C\varepsilon(1+t)^{-\frac{9}{2}}\|\omega_{xxt}(t)\|^2$$
$$+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{tt}(t)\|^2+C\delta e^{-ct}. \tag{3.114}$$

Putting (3.107)-(3.108), (3.112)-(3.114) into (3.106), we can conclude that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\frac{\mu\omega_{tt}^2}{2(\omega_x+A_\infty+\hat{A})}+\omega_{tt}\omega_{ttt}+\frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xtt}^2\right)\mathrm{d}x+\frac{3}{4}\int_{\mathbb{R}}P'(A_\infty)\omega_{xtt}^2\mathrm{d}x\\
\le{}& C\|\omega_{ttt}(t)\|^2+C\nu\|\omega_{xttt}(t)\|^2+C\nu\|\omega_{xxtt}(t)\|^2+C(1+t)^{-\frac{9}{2}}\|\omega_{xxt}(t)\|^2\\
&+C(1+t)^{-\frac{7}{2}}\|\omega_{xt}(t)\|^2+C(1+t)^{-\frac{7}{4}}\|\omega_{tt}(t)\|^2\\
&+Ce^{-ct}\left(\|\omega_x(t)\|_1^2+\|\omega_{xtt}(t)\|^2\right)+C\delta e^{-ct}.
\end{aligned}\tag{3.115}$$

Differentiating (2.14a) twice in t, we multiply the resulting equation by ω_{ttt} and integrate the corresponding product in x over R, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_{ttt}^2+P'(A_\infty)\omega_{xtt}^2\right)\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu\omega_{ttt}^2}{\omega_x+A_\infty+\hat{A}}\mathrm{d}x\\
={}&\mu\int_{\mathbb{R}}\left[\frac{\omega_{tt}(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}+\left(\frac{\omega_t(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\right)_t\right]\omega_{ttt}\mathrm{d}x+\sum_{i=1}^4\int_{\mathbb{R}}F_{itt}\omega_{ttt}\mathrm{d}x.
\end{aligned}\tag{3.116}$$

In a similar fashion to the preceding estimates, we have

$$\begin{aligned}
&\mu\int_{\mathbb{R}}\left[\frac{\omega_{tt}(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}+\left(\frac{\omega_t(\omega_{xt}+\hat{A}_t)}{(\omega_x+A_\infty+\hat{A})^2}\right)_t\right]\omega_{ttt}\mathrm{d}x\\
\le{}& C\int_{\mathbb{R}}\Big(|\omega_{tt}|\big(|\omega_{xt}|+|\hat{A}_t|\big)+|\omega_t|\big(|\omega_{xtt}|+|\hat{A}_{tt}|\big)+|\omega_t|\big(|\omega_{xt}|^2+|\hat{A}_t|^2\big)\Big)|\omega_{ttt}|\mathrm{d}x\\
\le{}&\frac{\mu}{15A_\infty}\|\omega_{ttt}(t)\|^2+C\varepsilon(1+t)^{-6}\|\omega_{xt}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{tt}(t)\|^2\\
&+C\varepsilon(1+t)^{-\frac{5}{2}}\|\omega_{xtt}(t)\|^2+C\delta e^{-ct},
\end{aligned}\tag{3.117}$$

and

$$\begin{aligned}
\int_{\mathbb{R}}F_{1tt}\omega_{ttt}\mathrm{d}x\le{}&\frac{\mu}{15A_\infty}\|\omega_{ttt}(t)\|^2-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right)\omega_{xtt}^2\mathrm{d}x\\
&+C\varepsilon(1+t)^{-7}\|\omega_{xx}(t)\|^2+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xxt}(t)\|^2\\
&+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xtt}(t)\|^2+C\delta e^{-ct},
\end{aligned}\tag{3.118}$$

$$\begin{aligned}
\int_{\mathbb{R}}F_{2tt}\omega_{ttt}\mathrm{d}x\le{}&\frac{\mu}{15A_\infty}\|\omega_{ttt}(t)\|^2+Ce^{-ct}\left(\|\omega_x(t)\|^2+\|\omega_{xt}(t)\|^2+\|\omega_{xtt}(t)\|^2\right)\\
&+C\delta e^{-ct},
\end{aligned}\tag{3.119}$$

$$\begin{aligned}\int_{\mathbb{R}} F_{3tt}\omega_{ttt}\mathrm{d}x\leq&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{\alpha m^2}{2A^2}\omega_{xtt}^2\mathrm{d}x+\frac{\mu}{15A_\infty}\|\omega_{ttt}(t)\|^2+C(1+t)^{-9}\|\omega_{xx}(t)\|^2\\&+C(1+t)^{-8}\|\omega_{xt}(t)\|^2+C(1+t)^{-7}\|\omega_{xxt}(t)\|^2\\&+C\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xtt}(t)\|^2+Ce^{-ct}\|\omega_{xtt}(t)\|^2+C\delta e^{-ct},\end{aligned}\tag{3.120}$$

$$\begin{aligned}\int_{\mathbb{R}} F_{4tt}\omega_{ttt}\mathrm{d}x\leq&-\int_{\mathbb{R}}\frac{\nu\sqrt{A}}{4\rho A_r}\omega_{xttt}^2\mathrm{d}x+\frac{\mu}{15A_\infty}\|\omega_{ttt}(t)\|^2+C\varepsilon(1+t)^{-\frac{21}{2}}\|\omega_{xx}(t)\|^2\\&+C\varepsilon(1+t)^{-7}\|\omega_{xxt}(t)\|^2+C\nu\varepsilon(1+t)^{-\frac{7}{2}}\|\omega_{xxtt}(t)\|^2\\&+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xtt}(t)\|^2+C\delta e^{-ct}.\end{aligned}\tag{3.121}$$

Putting (3.117)-(3.121) into (3.116), we can conclude that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[\omega_{ttt}^2+\left(P'(A_\infty)-\frac{\alpha m^2}{A^2}\right)\omega_{xtt}^2\right]\mathrm{d}x+c\int_{\mathbb{R}}(\omega_{ttt}^2+\nu\omega_{xttt}^2)\mathrm{d}x\\\leq&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right]\omega_{xtt}^2\mathrm{d}x+C(1+t)^{-7}\|\omega_{xx}(t)\|^2\\&+C(1+t)^{-6}\|\omega_{xt}(t)\|^2+C(1+t)^{-\frac{7}{2}}\|\omega_{xxt}(t)\|^2\\&+C\varepsilon(1+t)^{-\frac{7}{2}}\left(\|\omega_{tt}(t)\|^2+\nu\|\omega_{xxtt}(t)\|^2\right)+C\varepsilon(1+t)^{-\frac{7}{4}}\|\omega_{xtt}(t)\|^2\\&+Ce^{-ct}\left(\|\omega_x(t)\|^2+\|\omega_{xtt}(t)\|^2\right)+C\delta e^{-ct}.\end{aligned}\tag{3.122}$$

We have from (3.115)$\times\lambda$ + (3.122) $(0<\lambda\ll 1)$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(\omega_{ttt}^2+\frac{\lambda\mu\omega_{tt}^2}{\omega_x+A_\infty+\hat{A}}+2\lambda\omega_{tt}\omega_{ttt}+\left(P'(A_\infty)+\frac{\lambda\nu\sqrt{A}}{2\rho A_r}-\frac{\alpha m^2}{A^2}\right)\omega_{xtt}^2\right)\mathrm{d}x\\&\quad+c\int_{\mathbb{R}}(\omega_{xtt}^2+\omega_{ttt}^2+\nu\omega_{xttt}^2)\mathrm{d}x\\\leq&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left[P'(\omega_x+A_\infty+\hat{A})-P'(A_\infty)\right]\omega_{xtt}^2\mathrm{d}x+C\nu\|\omega_{xxtt}\|^2\\&\quad+C(1+t)^{-7}\|\omega_{xx}(t)\|^2+C(1+t)^{-\frac{7}{2}}\left(\|\omega_{xt}(t)\|^2+\|\omega_{xxt}\|^2\right)\\&\quad+C(1+t)^{-\frac{7}{4}}\|\omega_{tt}(t)\|^2+Ce^{-ct}\left(\|\omega_x(t)\|^2+\|\omega_{xtt}(t)\|^2\right)+C\delta e^{-ct}.\end{aligned}\tag{3.123}$$

Multiplying (3.123) by $(1+t)^k$ $(k=0,1,2,3,4)$ respectively and integrating the resulting equations over $[0,t]$, by applying Lemmas 3.3-3.7, it follows that

$$(1+t)^4\left(\|\omega_{ttt}(t)\|^2+\|\omega_{tt}(t)\|_1^2\right)$$

$$
\begin{aligned}
&+\int_0^t (1+\tau)^4 \left(\|\omega_{xtt}(\tau)\|^2+\|\omega_{ttt}(\tau)\|^2+\nu\|\omega_{xttt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq & C\left(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\right).
\end{aligned} \tag{3.124}
$$

Then the integration of $(1+t)^5\cdot$(3.122) over $[0,t]$ yields

$$
\begin{aligned}
&(1+t)^5\left(\|\omega_{ttt}(t)\|^2+\|\omega_{xtt}(t)\|^2\right) \\
&\quad+\int_0^t (1+\tau)^5\left(\|\omega_{ttt}(\tau)\|^2+\nu\|\omega_{xttt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq & C\left(\|\omega_0\|_3^2+\|\omega_1\|_2^2+\delta\right).
\end{aligned} \tag{3.125}
$$

Arranging (3.124) and (3.125), we can obtain the desired estimate (3.105). This completes the proof of Lemma 3.8. □

Hence, from Lemmas 3.3-3.8, we can close the a priori estimate and obtain (2.17)-(2.18).

3.2 Proof of (2.19)-(2.20)

In this subsection, assume further that $(\omega_0,\omega_1)(x)\in(L^1\times L^1)(\mathbb{R})$. We are going to obtain a better decay rate for $\omega(x,t)$ by using Green's function and time-weighted energy method.

Firstly, (2.14) can be rewritten as follows:

$$
\begin{cases}
\omega_t-\dfrac{A_\infty P'(A_\infty)}{\mu}\omega_{xx} \\
=\dfrac{(\omega_x+\hat{A})P'(A_\infty)}{\mu}\omega_{xx}-\dfrac{(\omega_x+A_\infty+\hat{A})}{\mu}\omega_{tt} \\
\qquad+\dfrac{(\omega_x+A_\infty+\hat{A})}{\mu}(F_1+F_2+F_3+F_4), \\
(\omega,\omega_t)|_{t=0}=(\omega_0,\omega_1)(x).
\end{cases} \tag{3.126}
$$

Using the Green function of heat equation, $\omega(x,t)$ has the following integral representation:

$$
\begin{aligned}
\omega(x,t)=&\int_{\mathbb{R}} G(x-y,t)\omega_0(y)\mathrm{d}y \\
&+\frac{P'(A_\infty)}{\mu}\int_0^t\int_{\mathbb{R}} G(x-y,t-\tau)\left[(\omega_y+\hat{A})\omega_{yy}\right](y,\tau)\mathrm{d}y\mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\mu}\int_0^t\int_{\mathbb{R}} G(x-y,t-\tau)(\omega_y+A_\infty+\hat{A})F_1(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{1}{\mu}\int_{\frac{t}{2}}^t\int_{\mathbb{R}} G(x-y,t-\tau)\left[(\omega_y+A_\infty+\hat{A})(-\omega_{\tau\tau}+F_2+F_3+F_4)\right](y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{1}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\left[(\omega_y+\hat{A})(-\omega_{\tau\tau}+F_2+F_3+F_4)\right](y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&-\frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\omega_{\tau\tau}(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)F_2(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)F_3(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)F_4(y,\tau)\mathrm{d}y\mathrm{d}\tau := \sum_{k=1}^{9} J_k(x,t),
\end{aligned}
\tag{3.127}
$$

where

$$
G(x,t)=\frac{\sqrt{\mu}}{\sqrt{4\pi A_\infty P'(A_\infty)t}}\exp\left\{\frac{-\mu x^2}{4A_\infty P'(A_\infty)t}\right\}.
$$

Utilizing (2.17)-(2.18), Lemma 3.2 and (3.127), we first prove the following lemma.

Lemma 3.9. *Suppose that all the conditions of Theorem* 2.1 *hold, it holds that*

$$
\|\omega(t)\|\le C(1+t)^{-\frac{1}{4}}. \tag{3.128}
$$

Proof. We have from (2.17)-(2.18), (3.4), Lemma 3.2 and Hausdorff-Young's inequality that

$$
\|J_1(t)\|\le \|G(t)\|\|\omega_0\|_{L^1}\le Ct^{-\frac{1}{4}}, \tag{3.129}
$$

$$
\begin{aligned}
\|J_2(t)\| &\le C\int_0^t \|G(t-\tau)\|(\|\omega_y(\tau)\|\|\omega_{yy}(\tau)\|+\|\hat{A}(\tau)\|_{L^1})\mathrm{d}\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{4}}(1+\tau)^{-\frac{3}{2}}\mathrm{d}\tau\le Ct^{-\frac{1}{4}},
\end{aligned}
\tag{3.130}
$$

$$
\begin{aligned}
\|J_3(t)\| &\le C\int_0^t \|G(t-\tau)\|\|F_1(\tau)\|_{L^1}\mathrm{d}\tau \\
&\le C\int_0^t \|G(t-\tau)\|(\|\omega_y(\tau)\|\|\omega_{yy}(\tau)\|+\|\hat{A}(\tau)\|_{L^1}+\|\hat{A}_y(\tau)\|_{L^1})\mathrm{d}\tau \\
&\le C\int_0^t (t-\tau)^{-\frac{1}{4}}(1+\tau)^{-\frac{3}{2}}\mathrm{d}\tau\le Ct^{-\frac{1}{4}},
\end{aligned}
\tag{3.131}
$$

$$
\begin{aligned}
\|J_4(t)\| &\leq C\int_{\frac{t}{2}}^{t}\|G(t-\tau)\|_{L^1}\big(\|\omega_{\tau\tau}(\tau)\|+\|F_2(\tau)\|+\|F_3(\tau)\|+\|F_4(\tau)\|\big)\mathrm{d}\tau\\
&\leq C\int_{\frac{t}{2}}^{t}(1+\tau)^{-2}\mathrm{d}\tau\leq Ct^{-1}, \qquad (3.132)\\
\|J_5(t)\| &\leq C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\big(\|\omega_y(\tau)\|+\|\hat{A}(\tau)\|\big)\\
&\qquad\times\big(\|\omega_{\tau\tau}(\tau)\|+\|F_2(\tau)\|+\|F_3(\tau)\|+\|F_4(\tau)\|\big)\mathrm{d}\tau\\
&\leq C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{4}}(1+\tau)^{-\frac{5}{2}}\mathrm{d}\tau\leq Ct^{-\frac{1}{4}}. \qquad (3.133)
\end{aligned}
$$

From (3.127) and (3.129)-(3.133), to conclude the proof of (3.128), we only need to get similar estimates on $J_6(x,t)$-$J_9(x,t)$. Indeed to deal with these terms, we need to integrate them by parts of variables τ and y.

$$
\begin{aligned}
J_6=&-\frac{A_\infty}{\mu}\int_{\mathbb{R}}G(x-y,t-\tau)\omega_\tau(y,\tau)\mathrm{d}y\Big|_{\tau=0}^{\tau=\frac{t}{2}}\\
&-\frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}}G_t(x-y,t-\tau)\omega_\tau(y,\tau)\mathrm{d}y\mathrm{d}\tau\\
=&-\frac{A_\infty}{\mu}\int_{\mathbb{R}}G\left(x-y,\frac{t}{2}\right)\omega_\tau\left(y,\frac{t}{2}\right)\mathrm{d}y\\
&+\frac{A_\infty}{\mu}\int_{\mathbb{R}}G(x-y,t)\omega_1(y)\mathrm{d}y\\
&-\frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}}G_t(x-y,t-\tau)\omega_\tau(y,\tau)\mathrm{d}y\mathrm{d}\tau,\\
J_7=&-\int_0^{\frac{t}{2}}\int_{\mathbb{R}}G(x-y,t-\tau)\left(\frac{\hat{m}\omega_y}{A}\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau\\
&-\int_0^{\frac{t}{2}}\int_{\mathbb{R}}G(x-y,t-\tau)\left(\frac{\hat{m}\hat{A}}{A}\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau\\
=&\int_0^{\frac{t}{2}}\int_{\mathbb{R}}G(x-y,t-\tau)\left[\left(\frac{\hat{m}}{A}\right)_y\omega\right](y,\tau)\mathrm{d}y\mathrm{d}\tau\\
&-\int_0^{\frac{t}{2}}\int_{\mathbb{R}}G_x(x-y,t-\tau)\left(\frac{\hat{m}\omega}{A}\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau\\
&-\int_0^{\frac{t}{2}}\int_{\mathbb{R}}G(x-y,t-\tau)\Big(\frac{\hat{m}\hat{A}}{A}\Big)(y,\tau)\mathrm{d}y\mathrm{d}\tau,
\end{aligned}
$$

$$
\begin{aligned}
J_8 =& -\frac{\alpha A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\left(\frac{2\hat{m}\omega_{y\tau}}{A}+\frac{\hat{m}^2\omega_{yy}}{A^2}\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{\alpha A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\frac{2(-\omega_\tau m_y+\hat{m}\hat{m}_y)}{A}(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&-\frac{\alpha A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\frac{(\omega_\tau^2-2\hat{m}\omega_\tau)A_y+\hat{m}^2\hat{A}_y}{A^2}(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
=& J_8^1+J_8^2+J_8^3,
\end{aligned}
$$

where

$$
\begin{aligned}
J_8^1 =& -\frac{\alpha A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G_x(x-y,t-\tau)\left(\frac{2\hat{m}}{A}\omega_\tau+\frac{\hat{m}^2}{A^2}\omega_y\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{\alpha A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\left[\left(\frac{2\hat{m}}{A}\right)_y\omega_\tau+\left(\frac{\hat{m}^2}{A^2}\right)_y\omega_y\right](y,\tau)\mathrm{d}y\mathrm{d}\tau, \\
J_9 =& \frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\left(\frac{\nu\sqrt{A}}{2\rho A_r}\omega_{yy\tau}\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&+\frac{A_\infty}{\mu}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau) \\
&\times\left(\frac{\nu\sqrt{A}}{2\rho A_r}\hat{A}_{y\tau}-\frac{\nu}{4\rho A_r\sqrt{A}}(\omega_{y\tau}+\hat{A}_\tau)(\omega_{yy}+\hat{A}_y)\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
=& J_9^1+J_9^2,
\end{aligned}
$$

where

$$
\begin{aligned}
J_9^1 =& \frac{\nu A_\infty}{2\mu\rho A_r}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G_x(x-y,t-\tau)(\sqrt{A}\omega_{y\tau})(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&-\frac{\nu A_\infty}{2\mu\rho A_r}\int_0^{\frac{t}{2}}\int_{\mathbb{R}} G(x-y,t-\tau)\big((\sqrt{A})_y\omega_{y\tau}\big)(y,\tau)\mathrm{d}y\mathrm{d}\tau.
\end{aligned}
$$

Now, we estimate $J_6(x,t)$-$J_9(x,t)$ in detail in the following. By applying (2.17)-(2.18), (3.4), Lemma 3.2 and Hausdorff-Young's inequality, we can easily deduce that

$$
\begin{aligned}
\|J_6(t)\| \le& C\|G(t/2)\|_{L^1}\|\omega_t(t/2)\|+C\|G(t)\|\|\omega_1\|_{L^1} \\
&+C\int_0^{\frac{t}{2}}\|G_t(t-\tau)\|_{L^1}\|\omega_\tau(\tau)\|\mathrm{d}\tau
\end{aligned}
$$

$$\le Ct^{-1}+Ct^{-\frac{1}{4}}+C\int_0^{\frac{t}{2}}(t-\tau)^{-1}(1+\tau)^{-1}\mathrm{d}\tau\le Ct^{-\frac{1}{4}}, \tag{3.134}$$

$$\begin{aligned}\|J_7(t)\|\le& C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\Big(\|\hat{m}(\tau)\|_{L^\infty}\|\omega_{yy}(\tau)\|\|\omega(\tau)\|\\&\qquad+\|\hat{m}_y(\tau)\|_{L^1}+\|\hat{A}_y(\tau)\|_{L^1}\Big)\mathrm{d}\tau\\&+C\int_0^{\frac{t}{2}}\|G_x(t-\tau)\|_{L^1}\|\hat{m}(\tau)\|_{L^\infty}\|\omega(\tau)\|\mathrm{d}\tau\\&+C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\|\hat{m}(\tau)\|_{L^\infty}\|\hat{A}(\tau)\|_{L^1}\mathrm{d}\tau\\\le& C\int_0^{\frac{t}{2}}\Big((t-\tau)^{-\frac{1}{2}}+(t-\tau)^{-\frac{1}{4}}\Big)e^{-c\tau}\mathrm{d}\tau\le Ct^{-\frac{1}{4}},\end{aligned} \tag{3.135}$$

$$\begin{aligned}\big\|J_8^1(t)\big\|=& C\int_0^{\frac{t}{2}}\|G_x(t-\tau)\|_{L^1}\Big(\|\hat{m}(\tau)\|_{L^\infty}\|\omega_\tau(\tau)\|+\|\hat{m}(\tau)\|_{L^\infty}^2\|\omega_y(\tau)\|\Big)\mathrm{d}\tau\\&+C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\Big(\|\hat{m}_y(\tau)\|_{L^1}+\|\hat{A}_y(\tau)\|_{L^1}\\&\qquad+\|\hat{m}(\tau)\|_{L^\infty}\|\omega_\tau(\tau)\|\|\omega_{yy}(\tau)\|\\&\qquad+\|\hat{m}(\tau)\|_{L^\infty}^2\|\omega_y(\tau)\|\|\omega_{yy}(\tau)\|\Big)\mathrm{d}\tau\\\le& C\int_0^{\frac{t}{2}}\Big((t-\tau)^{-\frac{1}{2}}+(t-\tau)^{-\frac{1}{4}}\Big)e^{-c\tau}\mathrm{d}\tau\le Ct^{-\frac{1}{4}},\end{aligned} \tag{3.136}$$

and

$$\begin{aligned}\big\|J_8^2(t)\big\|+\big\|J_8^3(t)\big\|=& C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\Big(\|\omega_\tau(\tau)\|\|\omega_{y\tau}(\tau)\|\\&\qquad+\|\omega_\tau(\tau)\|^{\frac{3}{2}}\|\omega_{y\tau}(\tau)\|^{\frac{1}{2}}\|\omega_{yy}(\tau)\|\Big)\mathrm{d}\tau\\&+C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\Big(\|\hat{m}(\tau)\|_{L^\infty}\|\omega_\tau(\tau)\|\|\omega_{yy}(\tau)\|\\&\qquad+\|\hat{m}_y(\tau)\|_{L^1}+\|\hat{A}_y(\tau)\|_{L^1}\Big)\mathrm{d}\tau\\\le& C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{4}}\Big((1+\tau)^{-\frac{5}{2}}+e^{-c\tau}\Big)\mathrm{d}\tau\le Ct^{-\frac{1}{4}},\end{aligned} \tag{3.137}$$

$$\begin{aligned}\big\|J_9^1(t)\big\|\le& C\int_0^{\frac{t}{2}}\|G_x(t-\tau)\|_{L^1}\|\omega_{y\tau}(\tau)\|\mathrm{d}y\mathrm{d}\tau\\&+C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\Big(\|\omega_{yy}(\tau)\|\|\omega_{y\tau}(\tau)\|+\|\hat{A}_y(\tau)\|_{L^1}\Big)\mathrm{d}y\mathrm{d}\tau\end{aligned}$$

$$
\begin{aligned}
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}}\mathrm{d}\tau\\
&\quad +C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{4}}\left((1+\tau)^{-\frac{5}{2}}+e^{-c\tau}\right)\mathrm{d}\tau\le Ct^{-\frac{1}{4}}, \qquad (3.138)\\
\left\|J_9^2(t)\right\|&\le C\int_0^{\frac{t}{2}}\|G(t-\tau)\|\Big(\|\omega_{y\tau}(\tau)\|\|\omega_{yy}(\tau)\|+\|\hat{A}_{y\tau}(\tau)\|_{L^1}\\
&\qquad\qquad +\|\hat{A}_\tau(\tau)\|_{L^1}+\|\hat{A}_y(\tau)\|_{L^1}\Big)\mathrm{d}\tau\\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{1}{4}}\left((1+\tau)^{-\frac{5}{2}}+e^{-c\tau}\right)\mathrm{d}\tau\le Ct^{-\frac{1}{4}}. \qquad (3.139)
\end{aligned}
$$

Combing (3.129)-(3.139), which implies (3.128). Now we have completed the proof of Lemma 3.9. □

Lemma 3.10. *Suppose that all the conditions of Theorem* 2.1 *hold. Then $\omega(x,t)$ satisfies the following decay estimates:*

$$
\begin{aligned}
&\left\|\partial_x^k\partial_t^l\omega(t)\right\|\le C(1+t)^{-\frac{1}{4}-\frac{k}{2}-l},\quad 0\le k+l\le 3,\quad 0\le l\le 2, \qquad (3.140)\\
&\left\|\partial_t^3\omega(t)\right\|\le C(1+t)^{-\frac{11}{4}}. \qquad (3.141)
\end{aligned}
$$

Proof. For any fixed $0<\epsilon_0<1/2$, multiplying (3.22) by $(1+t)^{\epsilon_0+1/2}$ and integrating the resulting equation over $[0,t]$, it follows that

$$
\begin{aligned}
&\frac{1}{2}(1+t)^{\epsilon_0+\frac{1}{2}}\int\left[\omega_t^2+\frac{\lambda\mu\omega^2}{\omega_x+A_\infty+\hat{A}}+2\lambda\omega\omega_t+\left(P'(A_\infty)+\frac{\nu\sqrt{A}}{2\rho A_r}\right)\omega_x^2\right]\mathrm{d}x\\
&\quad +c\int_0^t(1+\tau)^{\epsilon_0+\frac{1}{2}}\left(\|\omega_x(\tau)\|^2+\|\omega_t(\tau)\|^2+\nu\|\omega_{xt}(\tau)\|^2\right)\mathrm{d}\tau\\
&\le C\int_0^t(1+\tau)^{\epsilon_0-\frac{1}{2}}\left(\|\omega(\tau)\|_1^2+\|\omega_t(\tau)\|^2\right)\mathrm{d}\tau\\
&\quad +C\int_0^t(1+\tau)^{\epsilon_0+\frac{1}{2}}\left(\|\omega_{xx}(\tau)\|^2+\|\omega_{xt}(\tau)\|^2\right)\mathrm{d}\tau\\
&\quad +C\int_0^t(1+\tau)^{\epsilon_0+\frac{1}{2}}e^{-\tau}\|\omega(\tau)\|^2\mathrm{d}\tau+C\int_0^t(1+\tau)^{\epsilon_0+\frac{1}{2}}e^{-\tau}\|\omega_x(\tau)\|^2\mathrm{d}\tau\\
&\quad +C\left(\|\omega_0\|_1^2+\|\omega_1\|^2+\delta\right). \qquad (3.142)
\end{aligned}
$$

By using (2.17)-(2.18) and Lemma 3.9, we can treat the right-hand of (3.142) as follows:

$$
\int_0^t(1+\tau)^{\epsilon_0-\frac{1}{2}}\left(\|\omega(\tau)\|_1^2+\|\omega_t(\tau)\|^2\right)\mathrm{d}\tau\le C\int_0^t(1+\tau)^{\epsilon_0-1}\mathrm{d}\tau\le C(1+t)^{\epsilon_0},
$$

$$\int_0^t (1+\tau)^{\epsilon_0+\frac{1}{2}}\left(\|\omega_{xx}(\tau)\|^2+\|\omega_{xt}(\tau)\|^2\right)\mathrm{d}\tau \le C\int_0^t (1+\tau)^{\epsilon_0-\frac{3}{2}}\mathrm{d}\tau \le C,$$

and

$$\begin{aligned}
&C\int_0^t (1+\tau)^{\epsilon_0+\frac{1}{2}}e^{-\tau}\|\omega(\tau)\|^2\mathrm{d}\tau+\int_0^t (1+\tau)^{\epsilon_0+\frac{1}{2}}e^{-\tau}\|\omega_x(\tau)\|^2\mathrm{d}\tau\\
\le& C\sup_{0\le\tau\le t}\|\omega(\tau)\|^2\int_0^t (1+\tau)^{\epsilon_0+\frac{1}{2}}e^{-\tau}\mathrm{d}\tau+C\int_0^t \|\omega_x(\tau)\|^2\mathrm{d}\tau\\
\le& C\left(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\right).
\end{aligned}$$

Notice that ε and δ are small enough, then it follows that

$$\begin{aligned}
&(1+t)^{\epsilon_0+\frac{1}{2}}\left(\|\omega(t)\|_1^2+\|\omega_t(t)\|^2\right)\\
&\quad+\int_0^t (1+\tau)^{\epsilon_0+\frac{1}{2}}\left(\|\omega_x(\tau)\|^2+\|\omega_t(\tau)\|^2+\nu\|\omega_{xt}(\tau)\|^2\right)\mathrm{d}\tau\\
\le& C(1+t)^{\epsilon_0}.
\end{aligned} \tag{3.143}$$

Next, multiplying (3.22) by $(1+t)^{\epsilon_0+3/2}$ and integrating the resulting equation over $[0,t]$, we obtain

$$\begin{aligned}
&\frac{1}{2}(1+t)^{\epsilon_0+\frac{3}{2}}\int\left(\omega_t^2+P'(A_\infty)\omega_x^2\right)\mathrm{d}x\\
&\quad+c\int_0^t (1+\tau)^{\epsilon_0+\frac{3}{2}}\left(\|\omega_t(\tau)\|^2+\nu\|\omega_{xt}(\tau)\|^2\right)\mathrm{d}\tau\\
\le& C\int_0^t (1+\tau)^{\epsilon_0+\frac{1}{2}}\left(\|\omega_x(\tau)\|^2+\|\omega_t(\tau)\|^2\right)\mathrm{d}\tau+C\varepsilon\int_0^t \|\omega_{xx}(\tau)\|^2\mathrm{d}\tau\\
&\quad+C\int_0^t (1+\tau)^{\epsilon_0+\frac{3}{2}}e^{-\tau}\left(\|\omega_{xt}(\tau)\|^2+\|\omega_x(\tau)\|_1^2\right)\mathrm{d}\tau+C\left(\|\omega_0\|_1^2+\|\omega_1\|_1^2+\delta\right).
\end{aligned}$$

By using (2.17)-(2.18) and (3.143), we have

$$\begin{aligned}
&(1+t)^{\epsilon_0+\frac{3}{2}}\left(\|\omega_x(t)\|^2+\|\omega_t(t)\|^2\right)\\
&\quad+\int_0^t (1+\tau)^{\epsilon_0+\frac{3}{2}}\left(\|\omega_t(\tau)\|^2+\nu\|\omega_{xt}(\tau)\|^2\right)\mathrm{d}\tau\\
\le& C(1+t)^{\epsilon_0}.
\end{aligned} \tag{3.144}$$

Then, multiplying (3.41) by $(1+t)^{\epsilon_0+3/2}$ and integrating the resulting equation over $[0,t]$, we obtain

$$
\begin{aligned}
&\frac{1}{2}(1+t)^{\epsilon_0+\frac{3}{2}}\int\Bigg[\omega_{xt}^2+\frac{\lambda\mu\omega_x^2}{\omega_x+A_\infty+\hat{A}}+2\lambda\omega_x\omega_{xt}\\
&\qquad\qquad+\left(P'(A_\infty)+\frac{\lambda\nu\sqrt{A}}{2\rho A_r}-\frac{\alpha m^2}{A^2}\right)\omega_{xx}^2\Bigg]\mathrm{d}x\\
&\quad+c\int_0^t(1+\tau)^{\epsilon_0+\frac{3}{2}}\left(\|\omega_{xx}(\tau)\|^2+\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\right)\mathrm{d}\tau\\
&\le C\int_0^t(1+\tau)^{\epsilon_0+\frac{1}{2}}\left(\|\omega_x(\tau)\|_1^2+\|\omega_{xt}(\tau)\|^2\right)\mathrm{d}\tau\\
&\quad+C(\varepsilon+\delta)(1+t)^{\epsilon_0+\frac{3}{2}}\|\omega_{xx}(t)\|^2e^{-\tau}\|\omega_x(\tau)\|_1^2\mathrm{d}\tau+C\left(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\right).
\end{aligned}
$$

By applying (2.17)-(2.18) and (3.143), we can derive

$$
\begin{aligned}
&(1+t)^{\epsilon_0+\frac{3}{2}}\left(\|\omega_x(t)\|_1^2+\|\omega_{xt}(t)\|^2\right)\\
&\quad+\int_0^t(1+\tau)^{\epsilon_0+\frac{3}{2}}\left(\|\omega_{xx}(\tau)\|^2+\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\right)\mathrm{d}\tau\\
&\le C(1+t)^{\epsilon_0}.
\end{aligned}\tag{3.145}
$$

Multiplying (3.40) by $(1+t)^{\epsilon_0+5/2}$ and integrating the resulting equation over $[0,t]$, we obtain

$$
\begin{aligned}
&\frac{1}{2}(1+t)^{\epsilon_0+\frac{5}{2}}\int\left[\omega_{xt}^2+\left(P'(A_\infty)-\frac{\alpha m^2}{A^2}\right)\omega_{xx}^2\right]\mathrm{d}x\\
&\quad+c\int_0^t(1+\tau)^{\epsilon_0+\frac{5}{2}}\left(\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\right)\mathrm{d}\tau\\
&\le C\int_0^t(1+\tau)^{\epsilon_0+\frac{3}{2}}\left(\|\omega_{xt}(\tau)\|^2+\|\omega_{xx}(\tau)\|^2\right)\mathrm{d}\tau\\
&\quad+C(\varepsilon+\delta)(1+t)^{\epsilon_0+\frac{5}{2}}\|\omega_{xx}(t)\|^2\mathrm{d}\tau\\
&\quad+C\int_0^t(1+\tau)^{\epsilon_0+\frac{5}{2}}e^{-\tau}\|\omega_{xx}(\tau)\|^2\mathrm{d}\tau\\
&\quad+C\left(\|\omega_0\|_2^2+\|\omega_1\|_1^2+\delta\right).
\end{aligned}
$$

By using (2.17)-(2.18) and (3.145), one can immediately obtain

$$
(1+t)^{\epsilon_0+\frac{5}{2}}\left(\|\omega_{xx}(t)\|^2+\|\omega_{xt}(t)\|^2\right)
$$

$$
\begin{aligned}
&+\int_0^t (1+\tau)^{\epsilon_0+\frac{5}{2}}\left(\|\omega_{xt}(\tau)\|^2+\nu\|\omega_{xxt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq\, & C(1+t)^{\epsilon_0}. \qquad (3.146)
\end{aligned}
$$

Analogously, multiplying Eq. (3.69) by $(1+t)^{\epsilon_0+5/2}$ and Eq. (3.68) by $(1+t)^{\epsilon_0+7/2}$ and integrating the resulting equations over $(0,t)$, we obtain

$$
\begin{aligned}
&(1+t)^{\epsilon_0+\frac{5}{2}}\left(\|\omega_{xx}(t)\|_1^2+\|\omega_{xxt}(t)\|^2\right) \\
&\quad+\int_0^t (1+\tau)^{\epsilon_0+\frac{5}{2}}\left(\|\omega_{xxx}(\tau)\|^2+\|\omega_{xxt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq\, & C(1+t)^{\epsilon_0}, \qquad (3.147) \\
&(1+t)^{\epsilon_0+\frac{7}{2}}\left(\|\omega_{xxx}(t)\|^2+\|\omega_{xxt}(t)\|^2\right) \\
&\quad+\int_0^t (1+\tau)^{\epsilon_0+\frac{7}{2}}\left(\|\omega_{xxt}(\tau)\|^2+\nu\|\omega_{xxxt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq\, & C(1+t)^{\epsilon_0}. \qquad (3.148)
\end{aligned}
$$

Mutiplying Eq. (3.91) by $(1+t)^{\epsilon_0+5/2}$ and Eq. (3.90) by $(1+t)^{\epsilon_0+7/2}$, and integrating the resulting equations over $(0,t)$, we obtain

$$
\begin{aligned}
&(1+t)^{\epsilon_0+\frac{5}{2}}\left(\|\omega_{t}(t)\|_1^2+\|\omega_{tt}(t)\|^2\right) \\
&\quad+\int_0^t (1+\tau)^{\epsilon_0+\frac{5}{2}}\left(\|\omega_{xt}(\tau)\|^2+\|\omega_{tt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq\, & C(1+t)^{\epsilon_0}, \qquad (3.149) \\
&(1+t)^{\epsilon_0+\frac{7}{2}}\left(\|\omega_{xt}(t)\|^2+\|\omega_{tt}(t)\|^2\right) \\
&\quad+\int_0^t (1+\tau)^{\epsilon_0+\frac{7}{2}}\left(\|\omega_{tt}(\tau)\|^2+\nu\|\omega_{xtt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq\, & C(1+t)^{\epsilon_0}. \qquad (3.150)
\end{aligned}
$$

Mutiplying Eq. (3.104) by $(1+t)^{\epsilon_0+9/2}$ and integrating the resulting equations over $(0,t)$, we obtain

$$
\begin{aligned}
&(1+t)^{\epsilon_0+\frac{9}{2}}\left(\|\omega_{xxt}(t)\|^2+\|\omega_{xtt}(t)\|^2\right) \\
&\quad+\int_0^t (1+\tau)^{\epsilon_0+\frac{9}{2}}\left(\|\omega_{xtt}(\tau)\|^2+\nu\|\omega_{xxtt}(\tau)\|^2\right)\mathrm{d}\tau \\
\leq\, & C(1+t)^{\epsilon_0}. \qquad (3.151)
\end{aligned}
$$

Mutiplying Eq. (3.123) by $(1+t)^{\epsilon_0+9/2}$ and Eq. (3.122) by $(1+t)^{\epsilon_0+11/2}$ and integrating the resulting equations over $(0,t)$, we obtain

$$\begin{aligned}
&(1+t)^{\epsilon_0+\frac{9}{2}}\left(\|\omega_{tt}(t)\|_1^2+\|\omega_{ttt}(t)\|^2\right)\\
&\quad+\int_0^t(1+\tau)^{\epsilon_0+\frac{9}{2}}\left(\|\omega_{xtt}(\tau)\|^2+\|\omega_{ttt}(\tau)\|^2\right)\mathrm{d}\tau\\
\leq\, &C(1+t)^{\epsilon_0}, \qquad (3.152)\\
&(1+t)^{\epsilon_0+\frac{11}{2}}\left(\|\omega_{xtt}(t)\|^2+\|\omega_{ttt}(t)\|^2\right)\\
&\quad+\int_0^t(1+\tau)^{\epsilon_0+\frac{11}{2}}\left(\|\omega_{ttt}(\tau)\|^2+\nu\|\omega_{xttt}(\tau)\|^2\right)\mathrm{d}\tau\leq C(1+t)^{\epsilon_0}. \qquad (3.153)
\end{aligned}$$

Arranging (3.142)-(3.153), we obtain (3.140)-(3.141). This completes the proof of Lemma 3.10. □

Lemmas 3.9 and 3.10 lead to the desired estimates (2.18)-(2.19). Thus the proof of Theorem 2.1 is complete.

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