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Lower Bounds on the Number of Cyclic Subgroups in Finite Non-Cyclic Nilpotent Groups

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Abstract. Let G be a finite group and $\mathfrak{c}(G)$ denote the number of cyclic subgroups of G. It is known that the minimal value of $\mathfrak{c}$ on the set of groups of order n, where n is a positive integer, will occur at the cyclic group Z_n. In this paper, for non-cyclic nilpotent groups G of order n, the lower bounds of $\mathfrak{c}(G)$ are established.

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1 Introduction

Throughout this paper all groups are finite. For a group G of order n, let $\mathfrak{c}(G)$ denote the number of cyclic subgroups of G and $d(n)$ denote the number of divisors of n. A well-known result on group theory says that a cyclic group of order n has a unique subgroup of order d, for any divisor of n, so a cyclic group of order n has exactly $d(n)$ (necessarily cyclic) subgroups. Richard [14] proved that $\mathfrak{c}(G) \geq d(n)$, with equality if and only if G is a cyclic group. Another basic result of group theory states that $\mathfrak{c}(G) = |G|$ if and only if G is an elementary abelian 2-group. Tărnăuceanu [16, 17] described the finite groups with $\mathfrak{c}(G) = |G| - r$ $(r = 1,2)$. Regarding the results about $\mathfrak{c}(G) = |G| - r$. Belshoff, Dillstrom and Reid [2, 3] established a more remarkable bound. They showed that $|G| \leq 8r$. Cocke and Jensen [4] proved that if G is not a 2-group then $|G| \leq 6r$. Jafari and Madadi [9] proved that for any a divisor m of $|G|$, G has at least $d(m)$ cyclic subgroups whose orders divide m. Garonzi and Lima [5] studied the function $\alpha(G) = \frac{\mathfrak{c}(G)}{|G|}$. They explored basic properties of $\alpha(G)$ and pointed out a connection with the probability of commutation.

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Let $\mathfrak{s}(G)$ denote the number of subgroups of G. It's well-known that if G is a p-group of order p^n, then $\mathfrak{s}(G)\leq\mathfrak{s}(Z_p^n)$. Qu [13] proved that if p is odd and G is non-elementary abelian p-group, then

$$\mathfrak{s}(G)\leq\mathfrak{s}(M_p\times Z_p^{n-3}),$$

where $M_p=\langle a,b|a^p=b^p=c^p=1,[a,b]=c,[a,c]=[b,c]=1\rangle$. Tărnăuceanu [18] showed that if G is a non-elementary abelian 2-group of order 2^n, then

$$\mathfrak{s}(G)\leq\mathfrak{s}(D_8\times Z_2^{n-3}).$$

Aivazidis and Müller [1] determined the structure of those finite non-cyclic p-groups whose number of subgroups is minimal. Recently, we [12] generalized the results of Aivazidis and Müller on all finite non-cyclic nilpotent groups.

In the light of above investigations, it is a natural question that to ask for a given order which non-cyclic groups have the minimal number of cyclic subgroups. In this paper, this question is answered among all non-cyclic nilpotent groups. In fact, we obtain the lower bounds of $\mathfrak{c}(G)$, where G is a non-cyclic nilpotent of order n. Our main results are the following theorems.

Theorem 1.1. *Let p be a prime, G a non-cyclic p-group of order p^n.*

(1) *If $p^n=2^3$, then $\mathfrak{c}(G)\geq 5$, with equality if and only if $G\cong Q_8$.*

(2) *If $p^n\neq 2^3$, then $\mathfrak{c}(G)\geq(n-1)p+2$, with equality if and only if $G\cong Z_{p^{n-1}}\times Z_p$, M_{p^n} or Q_{16}.*

Theorem 1.2. *Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be a positive integer and $s=\min\{i\in\{1,\cdots,k\}|\alpha_i>1\}$, where $p_1<p_2<\cdots<p_k$ are distinct primes. Suppose G is a non-cyclic nilpotent group of order n, then there exists a suitable $q\in\pi(n)$, such that Q is non-cyclic and $p_s\leq q\leq 3p_s-2$, where $Q\in Syl_q(G)$. Furthermore,*

(1) *If $q^\lambda=2^3$, then $\mathfrak{c}(G)\geq 5\cdot d(\frac{n}{8})$, with equality if and only if $G\cong Q_8\times Z_{\frac{n}{8}}$.*

(2) *If $q^\lambda\neq 2^3$, then $\mathfrak{c}(G)\geq[(\lambda-1)q+2]\cdot d(\frac{n}{q^\lambda})$, with equality if and only if $G\cong Z_q\times Z_{\frac{n}{q}}$, $M_{q^\lambda}\times Z_{\frac{n}{q^\lambda}}$ or Q_{16}.*

All unexplained notations and terminologies are standard and can be found in [6,8,15]. In addition, $\pi(n)$, the set of the prime divisors of n; Z_n, the cyclic group of order n; Q_{2^n}, the generalized quaternion of order 2^n; Z_p^n, the elementary abelian group of order p^n; $M_{p^\lambda}=\langle a,b|a^{p^{\lambda-1}}=b^p=1,a^b=a^{1+p^{\lambda-2}}\rangle$. $A\times B$ means a direct product of A and B.

2 Preliminaries

Lemma 2.1. ([7]) *Let p be an odd prime, G a p-group of order p^n with $exp(G)=p^{n-\alpha}(n\geq 3)$. If $\alpha\geq 1$, then $\mathfrak{c}_k(G)\equiv 0$ mod p, where $2\leq k\leq n-\alpha$.*

Lemma 2.2. ([15]) *Let p be a prime, G a p-group of order p^n. If $\exp(G)=p^{n-1}$, then one of the following statements holds:*

(1) $G\cong Z_{p^{n-1}}\times Z_p$ *is abelian of type* (p^{n-1},p).

(2) $G\cong M_{p^n}=\langle a,b\big|a^{p^{n-1}}=b^p=1,b^{-1}ab=a^{1+p^{n-2}}\rangle$, $n\geq 3$.

(3) $G\cong Q_{2^n}=\langle a,b\big|a^{2^{n-1}}=1,b^2=a^{2^{n-2}},b^{-1}ab=a^{-1}\rangle$, $n\geq 3$.

(4) $G\cong D_{2^n}=\langle a,b\big|a^{2^{n-1}}=b^2=1,b^{-1}ab=a^{-1}\rangle$, $n\geq 3$.

(5) $G\cong SD_{2^n}=\langle a,b\big|a^{2^{n-1}}=b^2=1,b^{-1}ab=a^{-1+2^{n-2}}\rangle$, $n\geq 4$.

Lemma 2.3. Let G be a 2-group of order 2^n. If $\exp(G)=2^{n-1}$, then the following table holds.

	G	$\mathfrak{c}(G)$
(1)	$Z_{2^{n-2}}\times Z_2(n\geq 2)$	$2n$
(2)	$M_{2^n}(n\geq 4)$	$2n$
(3)	$Q_{2^n}(n\geq 3)$	$2^{n-2}+n$
(4)	$D_{2^n}(n\geq 3)$	$2^{n-1}+n$
(5)	$SD_{2^n}(n\geq 4)$	$3\cdot 2^{n-3}+n$

Proof. (1) Let $G=\langle a,b\big|a^{2^{n-1}}=b^2=1,ab=ba\rangle$. It is easy seen that the subgroups $\langle a^{2^i}\rangle$ and $\langle a^{2^i}b\rangle$ for all $1\leq i\leq n-1$, which are all cyclic subgroups of G. Therefore, $\mathfrak{c}(G)=2n$.

(2) Let $G=\langle a,b\big|a^{2^{n-1}}=b^2=1,b^{-1}ab=a^{1+2^{n-2}}\rangle$. It is easily seen that

$$o(a^kb)=o(a^k)\quad\text{for all}\quad 1\leq k\leq 2^{n-1}-1.$$

Thus, the subgroups

$$\langle a^{2^i}\rangle\quad\text{and}\quad\langle a^{2^i}b\rangle\ (1\leq i\leq n-1)$$

are all cyclic subgroups of G. Therefore, $\mathfrak{c}(G)=2n$.

(3) Let $G=\langle a,b\big|a^{2^{n-1}}=1,b^2=a^{2^{n-2}},b^{-1}ab=a^{-1}\rangle$. It is easily seen that

$$o(a^kb)=4\quad\text{for all}\quad 1\leq k\leq 2^{n-1}.$$

Thus, the subgroups

$$\langle a^{2^i}\rangle,\quad 0\leq i\leq n-1\quad\text{and}\quad\langle a^jb\rangle,\quad 1\leq j\leq 2^{n-2}$$

are all cyclic subgroups of G. Therefore, $\mathfrak{c}(G)=2^{n-2}+n$.

(4) Let $G=\langle a,b\big|a^{2^{n-1}}=b^2=1,b^{-1}ab=a^{-1}\rangle$. It is easily seen that

$$o(a^jb)=2\quad\text{for all}\quad 1\leq j\leq 2^{n-1}.$$

Thus, the subgroups

$$\langle a^{2^i}\rangle,\quad 0\le i\le n-1 \quad \text{and} \quad \langle a^j b\rangle,\quad 1\le j\le 2^{n-1}$$

are all cyclic subgroups of G. Therefore, $\mathfrak{c}(G)=2^{n-1}+n$.

(5) Let $G=\langle a,b\,|\,a^{2^{n-1}}=b^2=1, b^{-1}ab=a^{-1+2^{n-2}}\rangle$. For any $1\le k\le 2^{n-1}$, we have

$$o(a^k b)=2 \text{ if } k \text{ is even;} \quad o(a^k b)=4 \text{ if } k \text{ is odd.}$$

Thus, the subgroups

$$\langle a^{2^i}\rangle(0\le i\le n-1),\quad \langle a^{2k}b\rangle(1\le k\le 2^{n-2}) \quad \text{and} \quad \langle a^{2j+1}b\rangle(1\le j\le 2^{n-3})$$

are all cyclic subgroups of G. Therefore, $\mathfrak{c}(G)=n+2^{n-2}+2^{n-3}=3\cdot 2^{n-3}+n$. □

Lemma 2.4. ([5])
Let A and B be groups and $gcd(|A|,|B|)=1$. Then $\mathfrak{c}(A\times B)=\mathfrak{c}(A)\cdot\mathfrak{c}(B)$.

Lemma 2.5. ([14]) *Let G be a group of order n. Then $\mathfrak{c}(G)\ge d(n)$, with equality if and only if $G\cong Z_n$.*

Lemma 2.6. *Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be a positive integer, then*

$$d(n)=\prod_{i=1}^{k} d(p_i^{\alpha_i})=\prod_{i=1}^{k}(\alpha_i+1).$$

Proof. The proof is straightforward. □

Lemma 2.7. *Let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be a positive integer, then*

$$d(n)=\mathfrak{c}(Z_n)=\mathfrak{c}(Z_{p_i^{\alpha_i}})\cdot\mathfrak{c}(Z_{\frac{n}{p_i^{\alpha_i}}}) \quad \textit{for any} \quad i\in\{1,2,\cdots,k\}.$$

Proof. It follows from Lemmas 2.4–2.6. □

3 The proof of Theorem 1.1

Theorem 3.1. *Let p be an odd prime, G a non-cyclic p-group of order p^n. Then $\mathfrak{c}(G)\ge(n-1)p+2$, with equality if and only if $G\cong Z_{p^{n-1}}\times Z_p$ or M_{p^n}.*

Proof. Let p be an odd prime. Given a non-cyclic p-group G, recall that $\mathfrak{s}_k(G)$ is the number of subgroups of order p^k of G. A well-known theorem due to Kulakoff [10] asserts that

$$\mathfrak{s}_k(G)\equiv p+1 \bmod (p^2)$$

for all k such that $1\le k\le n-1$. Thus, in particular, $\mathfrak{c}_1(G)=\mathfrak{s}_1(G)\ge p+1$.

Suppose that $\exp(G)=p^{n-\alpha}$, then $\alpha\ge 1$. By Lemma 2.1, we know that

$$\mathfrak{c}_k(G)\equiv 0 \bmod (p)$$

for all k such that $2\le k\le n-\alpha$. In particular, $\mathfrak{c}_k(G)\ge p$, and therefore

$$\begin{aligned}\mathfrak{c}(G)&=\sum_{k=0}^{n-\alpha}\mathfrak{c}_k(G)=\mathfrak{c}_0(G)+\mathfrak{c}_1(G)+\sum_{k=2}^{n-\alpha}\mathfrak{c}_k(G)\\&\ge 1+(p+1)+\sum_{k=2}^{n-\alpha}p=(n-\alpha-1)p+(p+1)+1=(n-\alpha)p+2.\end{aligned}$$

So $\mathfrak{c}(G)\ge(n-1)p+2$ whenever $\alpha=1$.

Suppose that $\alpha\ge 2$, then G has a maximal subgroup M such that

$$\exp(M)=p^{n-\alpha}=p^{(n-1)-(\alpha-1)}.$$

By induction on α, we get $\mathfrak{c}(M)\ge[(n-1)-1]p+2=(n-2)p+2$. Observing

$$|G|-|M|=p^n-p^{n-1}>p^{n-1}-p=p(p^{n-2}-1)\ge p(p^{n-\alpha}-1).$$

We can choose p elements of G, say $a_1,a_2\cdots,a_p$, such that

$$a_1\in G-M,\ a_2\in G-M\bigcup\langle a_1\rangle,\ \cdots,\ a_p\in G-M\bigcup_{k=1}^{p-1}\langle a_k\rangle.$$

Since $\exp(G)=p^{n-\alpha}$, we have $o(a_i)\le p^{n-\alpha}$ for any $i\in\{1,2\cdots,p\}$. So G has at least p cyclic subgroups $\langle a_i\rangle(i=1,2\cdots,p)$, which are not contained in M. So we get

$$\mathfrak{c}(G)>\mathfrak{c}(M)+p\ge(n-2)p+2+p=(n-1)p+2.$$

This proves the first part of our assertion.

Now, we may assume that $n\ge 3$ and $\mathfrak{c}(G)=(n-1)p+2$. By the above argument, the equality implies $\alpha=1$. So we have $G\cong Z_{p^{n-1}}\times Z_p$ or M_{p^n} by Lemma 2.2.

In the following, let $G=Z_{p^{n-1}}:Z_p$ (the implied action of Z_p on $Z_{p^{n-1}}$ may well be trivial; we only require that G is a split extension), then $\mathfrak{s}_k(G)=p+1$, for all $1\le k\le n-1$ by a result of Lindenberg [11]. Applying Lemma 2.1, we get $\mathfrak{c}_1(G)=p+1$ $\mathfrak{c}_k(G)=p$ for all $2\le k\le n-1$, and thus $\mathfrak{c}(G)=(n-1)p+2$. The proof is complete. $\square$

Theorem 3.2. *Let G be a non-cyclic 2-group of order $2^n(n\ge 3)$.*

(1) *If $n=3$, then $\mathfrak{c}(G)\ge 5$, with equality if and only if $G\cong Q_8$.*

(2) *If $n=4$, then $\mathfrak{c}(G)\ge 8$, with equality if and only if $G\cong Q_{16}$, M_{16} or $Z_8\times Z_2$*

(3) *If $n\ge 5$, then $\mathfrak{c}(G)\ge 2n$, with equality if and only if $G\cong Z_{2^{n-1}}\times Z_2$ or M_{2^n}.*

Proof. There are 5 groups of order 2^3, and 14 groups of order 2^4. We use Magma to obtain a full list of the isomorphism classes of groups in each case, and ask Magma for the total number of cyclic subgroups of each group in the list. Our claim for $n=3$ and $n=4$ is now a simple matter of inspection.

In the following, we can assume that $n\geq 5$ and $\exp(G)=2^{n-\alpha}$. Since G is non-cyclic, then $\alpha\geq 1$. If $\alpha=1$, then $\mathfrak{c}(G)\geq 2n$ in Lemma 2.3.

Suppose that $\alpha\geq 2$. Then G has a maximal subgroup M such that

$$\exp(M)=2^{n-\alpha}=2^{(n-1)-(\alpha-1)}.$$

By induction on α, we have $\mathfrak{c}(M)\geq 2(n-1)$. Observing

$$|G|-|M|=2^n-2^{n-1}=2^{n-1}>2(2^{n-2}-1)\geq 2(2^{n-\alpha}-1).$$

We can choose two elements $a_1,a_2\in G$ such that

$$a_1\in G-M,\quad a_2\in G-M\bigcup\langle a_1\rangle.$$

Since $\exp(G)=2^{n-\alpha}$, we get $o(a_i)\leq 2^{n-\alpha}$ for any $i\in\{1,2\}$. Thus we find there at least 2 cyclic subgroups of G, say $\langle a_1\rangle$ and $\langle a_2\rangle$, which are not contained in M. So we get

$$c(G)\geq c(M)+2\geq 2(n-1)+2=2n.$$

Furthermore, we can get that $\mathfrak{c}(G)=2n$ if and only if $G\cong Z_{2^{n-1}}\times Z_2$ or M_{2^n} by the above arguments and Lemma 2.3. The proof is complete. □

Now Theorem 1.1 follows from Theorems 3.1 and 3.2.

4 The proof of Theorem 1.2

In this section, let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be a positive integer and

$$\Omega=\{i\in\{1,\cdots,k\}\,\big|\,\alpha_i>1\},$$

where $p_1<p_2<\cdots<p_k$ are distinct primes. Suppose that G is a finite group with the second minimal value of $\mathfrak{c}$ on the set of nilpotent groups of order n, we know that G is a non-cyclic nilpotent group by Lemma 2.5.

Let $G=P_1\times P_2\times\cdots\times P_k$, where $P_i\in Syl_{p_i}(G)(i=1,\cdots,k)$. By Lemma 2.4, we have

$$\mathfrak{c}(G)=\mathfrak{c}(P_1)\cdot\mathfrak{c}(P_2)\cdots\mathfrak{c}(P_k).$$

Proposition 4.1. *G has a unique non-cyclic Sylow subgroup.*

Proof. Since G is non-cyclic, there exits at least one of Sylow subgroups, say P_i is not cyclic and hence $\alpha_i > 1$. Suppose that P_j $(j \neq i)$ is another non-cyclic Sylow subgroup of G. By Lemma 2.4, we have

$$\mathfrak{c}(G) = \mathfrak{c}(P_i) \cdot \mathfrak{c}(P_j) \cdot \mathfrak{c}(\prod_{l \neq i,j} P_l).$$

Applying Lemma 2.5, we know $\mathfrak{c}(P_l) \geq \mathfrak{c}(Z_{p^{\alpha_l}}) = d(p^{\alpha_l})$, with equality iff $P_l \cong Z_{p^{\alpha_l}}$. Let $H = P_i \times Z_{\frac{n}{p^{\alpha_i}}}$, then $\mathfrak{c}(G) > \mathfrak{c}(H) > \mathfrak{c}(Z_n)$. It contradicts the fact that $\mathfrak{c}(G)$ is the second minimal value of $\mathfrak{s}$ on the set of groups of order n. So G has a unique non-cyclic Sylow subgroup. $\square$

By Proposition 4.1, we can assume that $Q \in Syl_q(G)$ is a unique non-cyclic Sylow subgroup of G. Thus $G = Q \times Z_{\frac{n}{q^\lambda}}$, where $|Q| = q^\lambda$. By hypothesis and Theorem 1.1, we know that $\mathfrak{c}(Q) = (\lambda-1)q+2$ or 5. Furthermore, we have $\lambda > 1$ and hence $\Omega \neq \emptyset$. Write $s = \min \Omega$. In particular, when $|\Omega| = 1$, we can get the Proposition 4.2 as follows.

Proposition 4.2. *Suppose $|\Omega| = 1$, then $q = p_s$.*

Proof. It is obvious. $\square$

In the following, we always suppose that $|\Omega| \geq 2$.

Proposition 4.3. *Let $n = 2^3 p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then $q = 2$.*

Proof. Let T be a Sylow 2-subgroup of G. We only need show that T is non-cyclic. Suppose that T is cyclic, then $q \geq 3$. Since $G = Q \times Z_{\frac{n}{q^\lambda}} = T \times Q \times Z_{\frac{n}{8q^\lambda}}$, by Lemma 2.4, we get

$$\mathfrak{c}(G) = \mathfrak{c}(T) \cdot \mathfrak{c}(Q) \cdot \mathfrak{c}(Z_{\frac{n}{8q^\lambda}}).$$

Furthermore, applying Lemmas 2.5, 2.6 and 2.7, we have

$$\begin{aligned}\mathfrak{c}(G) &= (3+1)[(\lambda-1)q+2] \cdot \mathfrak{c}(Z_{\frac{n}{8q^\lambda}}) \geq (12\lambda-4) \cdot \mathfrak{c}(Z_{\frac{n}{8q^\lambda}}) \\ &> 5(\lambda+1) \cdot \mathfrak{c}(Z_{\frac{n}{8q^\lambda}}) = \mathfrak{c}(Q_8) \cdot \mathfrak{c}(Z_{q^\lambda}) \cdot \mathfrak{c}(Z_{\frac{n}{8q^\lambda}}) = \mathfrak{c}(Q_8 \times Z_{\frac{n}{8}}) > \mathfrak{c}(Z_n).\end{aligned}$$

It contradicts the fact that $\mathfrak{c}(G)$ is the second minimal value of $\mathfrak{c}$ on the set of nilpotent groups of order n. So $Q = T$ is non-cyclic. $\square$

Proposition 4.4. *Let $n = 2^\alpha 3^\beta p_3^{\alpha_3} \cdots p_k^{\alpha_k}$, where $\alpha \geq 2$ and $\alpha \neq 3$.*

(1) *Suppose $\beta \neq 2$, then $q = 2$.*

(2) *Suppose $\beta = 2$.*

(2.1) *If $\alpha \leq 5$, then $q = 2$.*

(2.2) *If $\alpha > 5$, then $q = 3$.*

Proof. (1) Suppose that $\beta\neq 2$ and T is a Sylow 2-subgroup of G, we only need show that T is non-cyclic.

Suppose that T is cyclic. Now we claim that $\beta\geq 3$. Suppose $\beta\leq 1$, then the Sylow 3-subgroup of G is cyclic and hence $q\geq 5$. Since $G=Q\times Z_{\frac{n}{q^\lambda}}=T\times Q\times Z_{\frac{n}{2^\alpha q^\lambda}}$, we get

$$\mathfrak{c}(G)=\mathfrak{c}(T)\cdot\mathfrak{c}(Q)\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha q^\lambda}})\geq(\alpha+1)[(\lambda-1)q+2]\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha q^\lambda}}).$$

Observing

$$\frac{(\lambda-1)q+2}{\lambda+1}\geq\frac{5\lambda-3}{\lambda+1}\geq\frac{5\cdot 2-3}{2+1}=\frac{7}{3}>2>\frac{2\alpha}{\alpha+1},$$

We have

$$\mathfrak{c}(G)>2\alpha(\lambda+1)\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha q^\lambda}})=\mathfrak{c}(M_{2^\alpha})\cdot\mathfrak{c}(Z_{q^\lambda})\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha q^\lambda}})=\mathfrak{c}(M_{2^\alpha}\times Z_{\frac{n}{2^\alpha}})>\mathfrak{c}(Z_n).$$

It contradicts the fact that $\mathfrak{c}(G)$ is the second minimal value of $\mathfrak{c}$ on the set of nilpotent groups of order n. So we get $\beta\geq 3$.

We now assume that P is a Sylow 3-subgroup of G. We claim P is non-cyclic. If P is cyclic, then $q\geq 5$. Similar to above argument, we know that T is non-cyclic, a contradiction. So we get P is non-cyclic and hence $q=3$.

By the above arguments, we have

$$\mathfrak{c}(G)=\mathfrak{c}(T)\cdot\mathfrak{c}(Q)\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha q^\lambda}})=(\alpha+1)(3\lambda-1)\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha q^\lambda}}).$$

Since $\frac{3\lambda-1}{\lambda+1}\geq\frac{3\cdot 3-1}{3+1}=2>\frac{2\alpha}{\alpha+1}$, we get

$$\mathfrak{c}(G)>2\alpha(\lambda+1)\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha q^\lambda}})=\mathfrak{c}(M_{2^\alpha})\cdot\mathfrak{c}(Z_{q^\lambda})\cdot\mathfrak{s}(Z_{\frac{n}{2^\alpha q^\lambda}})=\mathfrak{c}(M_{2^\alpha}\times Z_{\frac{n}{2^\alpha}})>\mathfrak{c}(Z_n).$$

This is a final contradiction. So T is non-cyclic and hence $Q=T$, the conclusion (1) holds.

(2) Suppose that $\beta=2$ and P is cyclic. Similar to the proof of (1), we can get T is non-cyclic. So $Q=T$ and $G=T\times Z_{3^2}$. Furthermore, we have

$$\mathfrak{c}(G)=\mathfrak{c}(T)\cdot\mathfrak{c}(Z_{3^2})\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha.3^2}})=2\alpha\cdot(2+1)\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha.3^2}})=6\alpha\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha.3^2}}).$$

Let $H=Z_{2^\alpha}\times Z_3\times Z_3\times Z_{\frac{n}{2^\alpha.3^2}}$, then

$$\mathfrak{c}(H)=\mathfrak{c}(Z_{2^\alpha})\cdot\mathfrak{c}(Z_3\times Z_3)\cdot\mathfrak{c}(Z_{\frac{n}{2^\alpha.3^2}})=5(\alpha+1)\mathfrak{c}(Z_{\frac{n}{2^\alpha.3^2}}).$$

By hypothesis, $\mathfrak{c}(H)\geq\mathfrak{c}(G)$ implies that $5(\alpha+1)\geq 6\alpha$, which leads to $\alpha\leq 5$. So the conclusions (2) holds. □

Corollary 4.1. *Let $n=2^\alpha p_3^{\alpha_3}\cdots p_k^{\alpha_k}$, where $\alpha\geq 2$. If $3^2\nmid n$, then $q=2$.*

Proof. It follows from Propositions 4.3 and 4.4. □

Proposition 4.5. *Let* $n=2p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. *Then* $p_s \le q \le 3p_s-2$.

Proof. Since $s=\min\{i\in\{1,\cdots,k\}|\alpha_i>1\}$, we get $s\ge 2$ and hence $p_s\ge 3$. It is obviously that $q\ge p_s$. Suppose $q>3p_s-2$. Since $G=Q\times Z_{\frac{n}{q^\lambda}}=Q\times Z_{p_s^{\alpha_s}}\times Z_{\frac{n}{p_s^{\alpha_s}q^\lambda}}$, we have

$$\mathfrak{c}(G)=[(\lambda-1)q+2](\alpha_s+1)\mathfrak{c}(Z_{\frac{n}{p_s^{\alpha_s}q^\lambda}}).$$

As $\frac{(\lambda-1)q+2}{\lambda+1}\ge\frac{q+2}{3}>\frac{3p_s-2+2}{3}=p_s>\frac{(\alpha_s-1)p_s+2}{\alpha_s+1}$, we get

$$\begin{aligned}\mathfrak{c}(G)&>[(\alpha_s-1)p_s+2](\lambda+1)\mathfrak{c}(Z_{\frac{n}{p_s^{\alpha_s}q^\lambda}})\\ &=\mathfrak{c}(M_{p_s^{\alpha_s}})\cdot\mathfrak{c}(Z_{q^\lambda})\cdot\mathfrak{c}(Z_{\frac{n}{p_s^{\alpha_s}q^\lambda}})=\mathfrak{c}(M_{p_s^{\alpha_s}}\times Z_{\frac{n}{p_s^{\alpha_s}}})>\mathfrak{c}(Z_n).\end{aligned}$$

It contradicts the fact that $\mathfrak{c}(G)$ is the second minimal value of $\mathfrak{c}$ on the set of nilpotent groups of order n. So $q\le 3p_s-2$. □

Now Theorem 1.2 follows from Propositions 4.1–4.4, 4.6 and Theorem 1.1.

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