

Vanishing Viscosity Limit to Planar Rarefaction Wave with Vacuum for 3D Compressible Navier-Stokes Equations

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Abstract. The vanishing viscosity limit of the three-dimensional (3D) compressible and isentropic Navier-Stokes equations is proved in the case that the corresponding 3D inviscid Euler equations admit a planar rarefaction wave solution connected with vacuum states. Moreover, a uniform convergence rate with respect to the viscosity coefficients is obtained. Compared with previous results on the zero dissipation limit to planar rarefaction wave away from vacuum states [27, 28], the new ingredients and main difficulties come from the degeneracy of vacuum states in the planar rarefaction wave in the multi-dimensional setting. Suitable cut-off techniques and some delicate estimates are needed near the vacuum states. The inviscid decay rate around the planar rarefaction wave with vacuum is determined by the cut-off parameter and the nonlinear advection flux terms of 3D compressible Navier-Stokes equations.

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1 Introduction and main results

In the paper, we investigate the vanishing viscosity limit of the three-dimensional (3D) compressible and isentropic Navier-Stokes equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \mu_1 \Delta \mathbf{u} + (\mu_1 + \lambda_1) \nabla \operatorname{div} \mathbf{u}, \end{cases} \quad (1.1)$$

where $\rho = \rho(t, x) \geq 0$, $\mathbf{u} = \mathbf{u}(t, x) = (u_1, u_2, u_3)(t, x)$ and $p = p(\rho(t, x))$ represent the fluid density, velocity and pressure, respectively, with the spatial variable $x = (x_1, x_2, x_3) \in \Omega \subset \mathbb{R}^3$ and the time variable $t \geq 0$. The pressure $p = p(\rho)$ is given by the well-known γ -law

$$p(\rho) = A\rho^\gamma \quad (1.2)$$

with $\gamma > 1$ the adiabatic constant and $A > 0$ the fluid constant. The two constants μ_1 and λ_1 denote the shear and the bulk viscosity coefficients satisfying the physical restrictions

$$\mu_1 > 0, \quad 2\mu_1 + 3\lambda_1 \geq 0,$$

and we take

$$\mu_1 = \mu' \varepsilon, \quad \lambda_1 = \lambda' \varepsilon,$$

where $\varepsilon > 0$ is the vanishing parameter, and μ', λ' are the prescribed uniform-in- ε constants.

We consider the viscous system (1.1) in the spatial domain $\Omega = \mathbb{R} \times \mathbb{T}^2$ with $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ denoting the two-dimensional unit flat torus, subject to the following initial conditions:

$$(\rho, \mathbf{u})(0, x) = (\rho_0, \mathbf{u}_0)(x) = (\rho_0, u_{10}, u_{20}, u_{30})(x) \quad (1.3)$$

with the far fields condition of solutions imposed in the x_1 -direction

$$(\rho, u_1, u_2, u_3)(t, x) \rightarrow (\rho_\pm, u_{1\pm}, 0, 0) \quad \text{as } x_1 \rightarrow \pm\infty, \quad (1.4)$$

where $\rho_\pm \geq 0$, $u_{1\pm}$ are the prescribed constants. In the present paper, we are concerned with the case $\rho_- = 0$, $\rho_+ > 0$ such that the end state (ρ_+, u_{1+}) is connected with the vacuum state $\rho_- = 0$ through the 2-rarefaction wave.

Formally speaking, as $\varepsilon \rightarrow 0$, the solutions to the 3D compressible Navier-Stokes equations (1.1)-(1.3) with the far fields condition (1.4) converge to the solutions to the 3D compressible Euler equations

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, & x \in \Omega, \quad t \geq 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = 0 \end{cases} \quad (1.5)$$

with planar Riemann initial data

$$(\rho_0^r, u_0^r)(x) = (\rho_0^r, u_0^r)(x_1) = \begin{cases} (\rho_-, u_{1-}, 0, 0), & x_1 < 0, \\ (\rho_+, u_{1+}, 0, 0), & x_1 > 0. \end{cases} \quad (1.6)$$

However, the solutions to the multi-dimensional compressible Euler equations (1.5) and (1.6) develop the singularities, for example, shocks or vacuum, generally, which make the above vanishing viscosity limit problem is a singular limit process.

Note that the 3D planar Riemann problem (1.5)-(1.6) is closely related to the classical 1D Riemann problem (see Riemann's pioneer work [34] in 1860)

$$\begin{cases} \rho_t + (\rho u_1)_{x_1} = 0, & x_1 \in \mathbb{R}, \quad t > 0, \\ (\rho u_1)_t + (\rho u_1^2 + p(\rho))_{x_1} = 0 \end{cases} \quad (1.7)$$

with the initial data

$$(\rho_0^r, u_{10}^r)(0, x_1) = \begin{cases} (\rho_-, u_{1-}), & x_1 < 0, \\ (\rho_+, u_{1+}), & x_1 > 0. \end{cases} \quad (1.8)$$

It is well-known that 1D entropic Riemann solutions, i.e., shock, rarefaction wave and their linear superpositions, to (1.7)-(1.8) is a solution to the multi-dimensional planar Riemann problem (1.5)-(1.6). However, for planar shock Riemann initial data, besides the 1D entropic shock Riemann solution, it is proved by Chiodaroli *et al.* [8] and Chiodaroli and Kreml [9] that there exist infinitely many bounded admissible weak solutions to the multi-dimensional (2D/3D) planar Riemann problem to Euler equations satisfying the entropy condition by using the convex integration method as in De Lellis and Székelyhidi [11]. Then, the results in [8, 9] are extended by [3, 32] to the multi-dimensional planar Riemann initial data with planar shock composited by contact discontinuity or rarefaction wave. On the other hand, the uniqueness of the uniformly bounded admissible weak solution was proved in [4, 13, 14] for the multi-dimensional planar Riemann data containing only planar rarefaction wave even connected with vacuum states, which is in sharp contrast with the multi-dimensional planar shock. At the same time, the results in [4, 13, 14] hint that the planar rarefaction wave even with vacuum is more stable than the planar shock for the multi-dimensional compressible Euler equations (1.5), at least for the planar Riemann data, which promote us to justify mathematically the vanishing viscosity limit of the 3D compressible Navier-Stokes equations (1.1)-(1.4) towards the planar rarefaction wave to the 3D Riemann problem of the compressible Euler equations (1.5)-(1.6). Very recently,

Li *et al.* [27, 28] proved the zero dissipation limit to the planar rarefaction wave for the 2D compressible isentropic Navier–Stokes equations and 3D full Navier–Stokes–Fourier equations when the planar rarefaction wave is uniformly away from the vacuum states. The goal of the present paper is to prove rigorously the vanishing viscosity limit to the planar rarefaction wave connected continuously with the vacuum states for 3D compressible Navier–Stokes equations (1.1)–(1.4), that is, $\rho_- \equiv 0$ for planar 2-rarefaction wave, and the new ingredients and main difficulties come from the degeneracy of vacuum states for the planar rarefaction wave in the multi-dimensional setting.

Precisely, when $\rho_- \equiv 0$, the 2-rarefaction wave with vacuum is the unique entropic solution to the 1D Riemann problem, i.e., the one-dimensional (1D) Euler equations (1.7) with the one-side vacuum Riemann initial data

$$(\rho_0^r, u_{10}^r)(0, x_1) = \begin{cases} \rho_0^r \equiv 0, & x_1 < 0, \\ (\rho_+, u_{1+}), & x_1 > 0, \end{cases} \quad (1.9)$$

which is also the unique entropic admissible and bounded weak solution to the 3D planar Riemann problem (1.5)–(1.6) with $\rho_- \equiv 0$, that is, the one-side vacuum planar Riemann initial data

$$(\rho_0^r, \mathbf{u}_0^r)(0, x) = \begin{cases} \rho_0^r \equiv 0, & x_1 < 0, \\ (\rho_+, u_{1+}, 0, 0), & x_1 > 0. \end{cases} \quad (1.10)$$

The zero dissipation limit to the basic wave patterns, including rarefaction and shock waves and contact discontinuity in 1D case, has been studied extensively in literature and one can refer to [2, 7, 15–24, 31, 35–38] for the 1D compressible Navier–Stokes equations or 1D hyperbolic conservation laws with artificial viscosities, and [1, 5, 6, 10, 12, 26, 33] for the other related results. However, compared with the 1D case, the multi-dimensional vanishing dissipation limit has been less developed and more challenging difficulties are encountered, such that there exist only a few results [25, 27–29] on the vanishing dissipation limit to compressible Navier–Stokes equations as far as we know.

As pointed out by Liu and Smoller [30], only the rarefaction wave can be connected to the vacuum states, among the two nonlinear waves, i.e., shock and rarefaction waves to the 1D compressible isentropic Euler equations (1.7). Huang *et al.* [18] proved the zero viscosity limit of 1D compressible isentropic Navier–Stokes to the rarefaction wave connected by vacuum states and the inviscid decay rate is obtained with respect to the viscosity. On the other hand, Li *et al.* [27, 28] proved the zero dissipation limit to the planar rarefaction wave for the 2D compressible isentropic Navier–Stokes equations and 3D full Navier–Stokes–Fourier

equations when the planar rarefaction wave is uniformly away from the vacuum states. Motivated by [18, 27, 28], the goal of this paper is to prove the vanishing viscosity limit to the planar rarefaction wave with vacuum states for 3D compressible Navier-Stokes (1.1) and obtain the inviscid decay rate. Remark that due to the multi-dimensional perturbations around the planar rarefaction wave with vacuum states, there are essential differences from the one-dimensional inviscid limit in [18], the hyperbolic waves are crucially needed to recover the viscous terms of the 3D compressible Navier-Stokes equations due to the inviscid rarefaction wave profile and the insufficient decay rate (with respect to the viscosities) of the error terms, which is originated from [21] in 1D case and [27, 28] for multi-dimensional case. However, the hyperbolic waves here are degenerate near the vacuum states of the planar rarefaction wave and the cut-off of the vacuum states are needed.

Now we give a detailed description of the planar rarefaction wave connected by the vacuum to the 3D compressible Euler equations (1.5), (1.10), which is a 1D solution to the compressible Euler equations (1.7), (1.9). The Euler system (1.7) is hyperbolic for $\rho \geq 0$ and has two real eigenvalues

$$\lambda_1(\rho, u_1) = u_1 - \sqrt{p'(\rho)}, \quad \lambda_2(\rho, u_1) = u_1 + \sqrt{p'(\rho)}$$

with corresponding right eigenvectors denoted by $r_1(\rho, u_1), r_2(\rho, u_1)$, such that the both two characteristic fields are genuinely nonlinear, i.e.,

$$\nabla_{(\rho, u_1)} \lambda_i(\rho, u_1) \cdot r_i(\rho, u_1) \neq 0, \quad i = 1, 2$$

for $\rho > 0$. The i -Riemann invariant ($i = 1, 2$) can be defined by

$$\sum_i(\rho, u_1) = u_1 + (-1)^{i+1} \int^\rho \frac{\sqrt{p'(s)}}{s} ds, \quad (1.11)$$

such that

$$\nabla_{(\rho, u_1)} \sum_i(\rho, u_1) \cdot r_i(\rho, u_1) \equiv 0, \quad i = 1, 2$$

for $\rho > 0$. Without loss of generality, we only consider the 2-rarefaction wave with vacuum. Since 2-Riemann invariant $\sum_2(\rho, u_1)$ is constant along the 2-rarefaction wave curve, we can get $u_{1-} = \sum_2(\rho_+, u_{1+})$ being the velocity from the vacuum state to non-vacuum region. The 2-rarefaction wave connecting the vacuum $\rho_- = 0$ to (ρ_+, u_{1+}) is the self-similar solution $(\rho^{r_2}, u_1^{r_2})(\xi), (\xi = x_1/t)$ of the Euler equations (1.7) defined by

$$\lambda_2(\rho^{r_2}, u_1^{r_2}) = \begin{cases} \rho^{r_2}(\xi) \equiv 0, & \xi < \lambda_2(0, u_{1-}) = u_{1-}, \\ \xi, & u_{1-} \leq \xi \leq \lambda_2(\rho_+, u_{1+}), \\ \lambda_2(\rho_+, u_{1+}), & \xi > \lambda_2(\rho_+, u_{1+}), \end{cases} \quad (1.12)$$

and

$$\sum_2(\rho^{r_2}(\xi), u_1^{r_2}(\xi)) = \sum_2(0, u_{1-}) = \sum_2(\rho_+, u_{1+}). \quad (1.13)$$

Define the momentum of 2-rarefaction wave by

$$m_1^{r_2}(\xi) := \begin{cases} \rho^{r_2}(\xi) u_1^{r_2}(\xi), & \rho^{r_2} > 0, \\ 0, & \rho^{r_2} = 0. \end{cases} \quad (1.14)$$

Note that in the vacuum region, the fluid velocity can not be defined, but the momentum is uniquely determined as 0. Since the Riemann problem (1.7)-(1.8) has 2-rarefaction wave $(\rho^{r_2}, m_1^{r_2})(x_1/t)$, then the planar rarefaction wave solution to the 3D compressible Euler equations (1.5)-(1.6) can be defined by $(\rho^{r_2}, m_1^{r_2}, 0, 0)(x_1/t)$.

Theorem 1.1. *Let $(\rho^{r_2}, m_1^{r_2}, 0, 0)(x_1/t)$ be the planar 2-rarefaction wave to the 3D Euler system (1.5) defined by (1.12)-(1.14) with one-side vacuum state and $T > 0$ be any fixed time. Assume that $1 < \gamma < 19$ in (1.2), then there exists a small positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, the 3D compressible Navier-Stokes equations with initial values (3.2)-(3.3) has a family of 3D smooth solutions $(\rho^\varepsilon, \mathbf{m}^\varepsilon := \rho^\varepsilon \mathbf{u}^\varepsilon = (m_1^\varepsilon, m_2^\varepsilon, m_3^\varepsilon))$ up to time T satisfying*

$$\begin{cases} (\rho^\varepsilon - \rho^{r_2}, m_1^\varepsilon - m_1^{r_2}, m_2^\varepsilon, m_3^\varepsilon) \in C^0((0, T]; L^2(\Omega)), \\ (\nabla \rho^\varepsilon, \nabla \mathbf{m}^\varepsilon) \in C^0([0, T]; H^1(\Omega)), \\ \nabla^3 \mathbf{m}^\varepsilon \in L^2(0, T; L^2(\Omega)). \end{cases}$$

Furthermore, for any given positive constant h , there exist positive constants $C_{h,T}$ and C_T independent of ε , such that

$$\begin{aligned} \sup_{h \leq t \leq T} \left\| (\rho^\varepsilon, m_1^\varepsilon)(t, x_1, x_2, x_3) - (\rho^{r_2}, m_1^{r_2})\left(\frac{x_1}{t}\right) \right\|_\infty &\leq C_{h,T} \varepsilon^{a_1} |\ln \varepsilon|, \\ \sup_{0 \leq t \leq T} \left\| (m_1^\varepsilon, m_2^\varepsilon)(t, x_1, x_2, x_3) \right\|_\infty &\leq C_T \varepsilon^{a_2} |\ln \varepsilon|^{-\frac{7\gamma+1}{4}} \end{aligned}$$

with the positive constants a_1, a_2 given by

$$a_1 = \begin{cases} \frac{3}{3\gamma+26}, & 1 < \gamma \leq \frac{5}{3}, \\ \frac{1}{2\gamma+7}, & \frac{5}{3} < \gamma \leq 2, \\ \frac{3}{7\gamma+19}, & 2 < \gamma \leq \frac{5}{2}, \\ \frac{3}{9\gamma+14}, & \frac{5}{2} < \gamma \leq 3, \\ \frac{3}{11\gamma+8}, & 3 < \gamma < 19, \end{cases} \quad (1.15)$$

and

$$a_2 = \frac{5}{4} - \frac{7\gamma + 18}{4} a_1. \quad (1.16)$$

As the viscosities approach zero, i.e. $\varepsilon \rightarrow 0+$, $(\rho^\varepsilon, \mathbf{m}^\varepsilon) = (\rho^\varepsilon, m_1^\varepsilon, m_2^\varepsilon, m_3^\varepsilon)$ converges to the planar 2-rarefaction wave with one-side vacuum $(\rho^{r_2}, m_1^{r_2}, 0, 0)$ pointwisely except at the original point $(0, 0)$, i.e.

$$(\rho^\varepsilon, m_1^\varepsilon, m_2^\varepsilon, m_3^\varepsilon)(t, x_1, x_2, x_3) \rightarrow (\rho^{r_2}, m_1^{r_2}, 0, 0) \left(\frac{x_1}{t} \right), \quad a.e \quad \text{in } \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}^2.$$

Remark 1.1. Theorem 1.1 is proved in the case $\gamma \in (1, 19)$, which including the most physical cases of the γ -law fluids. Moreover, the vanishing viscosity limit can also be justified for $\gamma \geq 19$ by the using the similar methods, but the inviscid decay rate might be slower.

To prove Theorem 1.1, we first employ the cut-off to the planar 2-rarefaction wave around the vacuum states to overcome the degeneracies. More precisely, for any $\nu := \varepsilon^{a_1} |\ln \varepsilon| > 0$ to be determined, the cut-off 2-rarefaction wave will connect $(\nu, u_{1\nu})$ and (ρ_+, u_{1+}) , where $u_{1\nu}$ is uniquely determined along the 2-rarefaction curve. In the current framework, a_1 given by (1.15) is optimal. Since the cut-off 2-rarefaction wave is only Lipschitz continuous, we need to make a smooth approximation to the cut-off 2-rarefaction wave as in [18, 37]. Next, the hyperbolic waves are needed to recover the physical viscosities for the inviscid approximate wave profile as in Huang *et al.* [21] and Li *et al.* [27, 28]. Here the estimates of the hyperbolic wave depend on the cut-off parameter ν , which means that the hyperbolic waves also have the degeneracies around the vacuum states. Then we will carry out the uniform (in viscosities) estimates for the 3D compressible Navier-Stokes equations around the solution profile consisting the approximate cut-off rarefaction wave and hyperbolic wave in the original non-scaled spatial and time variables. Our final inviscid decay rate in terms of viscosities is determined by the cut-off parameter and the nonlinear convection flux terms for the 3D limit.

The rest of the paper is organized as follows. In Section 2, we first conduct the cut-off to the planar 2-rarefaction wave around the vacuum states and construct the approximate cut-off 2-rarefaction wave and hyperbolic waves, and then present the solution profile. In Section 3, we reformulate the compressible Navier-Stokes equations around the perturbation of the solution profile and prove Theorem 1.1 based on the uniform a priori estimates. Finally, in Section 4, the a priori estimates will be justified by the fundamental energy methods.

The following notations will be used in the paper. Denote by $H^l(\Omega)$ ($l \geq 0, l \in \mathbb{Z}$) the usual Sobolev space with the norm $\|\cdot\|_l$, and $L^2(\Omega) := H^0(\Omega)$ with the norm $\|\cdot\| := \|\cdot\|_0$. We denote by C a generic positive constant that does not depend on

$\varepsilon, \nu, \delta, T$, denote by C_T a positive constant that does not depend on ε, ν, δ but may depend on T .

2 Construction of the solution profile

In this section, we first conduct the cut-off 2-rarefaction wave and construct its smooth approximation, the hyperbolic waves and finally get the solution profile.

2.1 Cut-off 2-rarefaction wave

As mentioned in the introduction, we will cut off the 2-rarefaction wave with vacuum along the rarefaction wave curve to overcome the degeneracies near the vacuum states. Precisely, for any $\nu > 0$, the cut-off 2-rarefaction wave will connect $(\nu, u_{1\nu})$ and (ρ_+, u_{1+}) , where $u_{1\nu}$ is uniquely determined by the fact that 2-Riemann invariant is constant along the rarefaction curve. Then the cut-off 2-rarefaction wave $(\rho_v^{r_2}, u_{1\nu}^{r_2})(\xi = x_1/t)$ to the Euler system (1.7) is expressed explicitly by

$$\lambda_2(\rho_v^{r_2}, u_{1\nu}^{r_2}) = \begin{cases} \lambda_2(\nu, u_{1\nu}), & \xi < \lambda_2(\nu, u_{1\nu}), \\ \xi, & \lambda_2(\nu, u_{1\nu}) < \xi < \lambda_2(\rho_+, u_{1+}), \\ \lambda_2(\rho_+, u_{1+}), & \xi > \lambda_2(\rho_+, u_{1+}), \end{cases} \quad (2.1)$$

and

$$\sum_2(\rho_v^{r_2}(\xi), u_{1\nu}^{r_2}(\xi)) = \sum_2(\nu, u_{1\nu}) = \sum_2(\rho_+, u_{1+}), \quad (2.2)$$

where $\sum_2$ is 2-Riemann invariant defined in (1.11). Correspondingly we can define the momentum of 2-rarefaction wave by $m_{1\nu}^{r_2}(\xi) = \rho_v^{r_2}(\xi) u_{1\nu}^{r_2}(\xi)$. Note that the cut-off 2-rarefaction wave $(\rho_v^{r_2}, m_{1\nu}^{r_2})(x/t)$ converges to the original 2-rarefaction wave with vacuum $(\rho^{r_2}, m_1^{r_2})(x/t)$ in the sup-norm as the cut-off parameter ν tends to zero.

Lemma 2.1 ([18]). *There exists a constant $\nu_0 \in (0, 1)$ such that for $\nu \in (0, \nu_0)$, $t > 0$,*

$$\|(\rho_v^{r_2}, m_{1\nu}^{r_2})(\cdot/t) - (\rho^{r_2}, m_1^{r_2})(\cdot/t)\|_{L^\infty} \leq C\nu. \quad (2.3)$$

2.2 Approximate cut-off 2-rarefaction wave

We will construct a smooth approximate cut-off 2-rarefaction wave to the Euler system (1.7), since the cut-off 2-rarefaction wave is only Lipschitz continuous.

Consider the Riemann problem for the inviscid Burgers' equation

$$\begin{cases} \omega_t + \omega \omega_{x_1} = 0, \\ \omega(0, x_1) = \begin{cases} \omega_-, & x_1 < 0, \\ \omega_+, & x_1 > 0. \end{cases} \end{cases} \quad (2.4)$$

If $\omega_- < \omega_+$, then the self-similar rarefaction wave fan $\omega^r(t, x_1) = \omega^r(x_1/t)$ is given by

$$\omega^r\left(\frac{x_1}{t}\right) = \begin{cases} \omega_-, & \frac{x_1}{t} < \omega_-, \\ \frac{x_1}{t}, & \omega_- < \frac{x_1}{t} < \omega_+, \\ \omega_+, & \frac{x_1}{t} > \omega_+. \end{cases} \quad (2.5)$$

As in [18,37], the approximate rarefaction wave can be constructed by the solution of the Burgers' equation

$$\begin{cases} \omega_t + \omega \omega_{x_1} = 0, \\ \omega(0, x_1) = \omega_\delta(x_1) = \omega\left(\frac{x_1}{\delta}\right) = \frac{\omega_- + \omega_+}{2} + \frac{\omega_- - \omega_+}{2} \tanh\left(\frac{x_1}{\delta}\right), \end{cases} \quad (2.6)$$

where $\delta = \varepsilon^{a_1} > 0$ is a small parameter (depending on ε) to be determined. The following properties can be proved by characteristic method, see [18,37].

Lemma 2.2. *The problem (2.6) has a unique smooth global solution $\omega_\delta^r(t, x_1)$ for any $\delta > 0$, which has the following properties:*

- (1) $\omega_- < \omega_\delta^r(t, x_1) < \omega_+$, $\partial_{x_1} \omega_\delta^r(t, x_1) > 0$ for any $x_1 \in \mathbb{R}, t \geq 0, \delta > 0$.
- (2) The following estimates hold for all $t > 0, \delta > 0, p \in [1, \infty]$:

$$\begin{aligned} \|\partial_{x_1} \omega_\delta^r(t, \cdot)\|_{L^p} &\leq C(\omega_+ - \omega_-)^{\frac{1}{p}} (\delta + t)^{-1 + \frac{1}{p}}, \\ \|\partial_{x_1 x_1} \omega_\delta^r(t, \cdot)\|_{L^p} &\leq C(\delta + t)^{-1} \delta^{-1 + \frac{1}{p}}, \\ \|\partial_{x_1 x_1 x_1} \omega_\delta^r(t, \cdot)\|_{L^p} &\leq C(\delta + t)^{-1} \delta^{-2 + \frac{1}{p}}, \\ |\partial_{x_1 x_1} \omega_\delta^r(t, x_1)| &\leq \frac{4}{\delta} \partial_{x_1} \omega_\delta^r(t, x_1). \end{aligned}$$

- (3) There exists a constant $\delta_0 \in (0, 1)$ such that $\delta \in (0, \delta_0], t > 0$,

$$\left\| \omega_\delta^r(t, \cdot) - \omega^r\left(\frac{\cdot}{t}\right) \right\|_{L^\infty} \leq C \delta t^{-1} [\ln(1+t) + |\ln \delta|].$$

Now the approximate cut-off 2-rarefaction wave $(\bar{\rho}_{v,\delta}, \bar{u}_{1v,\delta})(x_1, t)$ of the cut-off 2-rarefaction wave $(\rho_v^{r_2}, u_{1v}^{r_2})(\xi = x_1/t)$ to the compressible Navier-Stokes equations (1.1) can be defined by

$$\begin{cases} \omega_+ = \lambda_2(\rho_+, u_{1+}), & \omega_- = \lambda_2(v, u_{1v}), \\ \omega_\delta^{r_2}(t, x_1) = \lambda_2(\bar{\rho}_{v,\delta}, \bar{u}_{1v,\delta})(t, x_1), \\ \Sigma_2(\bar{\rho}_{v,\delta}, \bar{u}_{1v,\delta})(t, x_1) = \Sigma_2(v, u_{1v}) = \Sigma_2(\rho_+, u_{1+}), \end{cases} \quad (2.7)$$

where $\omega_\delta^{r_2}(t, x_1)$ is the smooth solution of the Burgers' equation in (2.6) and $\Sigma_2(\rho, u_1)$ is the 2-Riemann defined in (1.11). For simplicity, the subscript of the approximate cut-off 2-rarefaction wave $(\bar{\rho}_{v,\delta}, \bar{u}_{1v,\delta})$ will be omitted as $(\bar{\rho}, \bar{u}_1)$, which satisfies the 1D Euler system

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u}_1)_{x_1} = 0, \\ (\bar{\rho}\bar{u}_1)_t + (\bar{\rho}\bar{u}_1^2 + p(\bar{\rho}))_{x_1} = 0, \\ (\bar{\rho}, \bar{u}_1)(0, x_1) := (\bar{\rho}_0, \bar{u}_{10})(x_1). \end{cases} \quad (2.8)$$

The following properties is directly from Lemma 2.2.

Lemma 2.3. *The approximate cut-off 2-rarefaction wave $(\bar{\rho}, \bar{u}_1)$ to the Euler equations (2.8) satisfies the following properties:*

- (1) $v < \bar{\rho}(t, x_1) < \rho_+$, $\bar{u}_{1x_1}(t, x_1) = 2(\omega_\delta^{r_2})_{x_1}/(\gamma+1) > 0$ for any $x_1 \in \mathbb{R}, t \geq 0$,
 $\bar{\rho}_{x_1} = \bar{\rho}^{(3-\gamma)/2} \bar{u}_{1x_1}$ and $\bar{\rho}_{x_1 x_1} = \bar{\rho}^{(3-\gamma)/2} \bar{u}_{1x_1 x_1} + (3-\gamma)\bar{\rho}^{2-\gamma}(\bar{u}_{1x_1})^2/2$.
- (2) The following estimates hold for all $t > 0, \delta > 0, p \in [1, \infty]$:

$$\begin{aligned} \|\bar{u}_{1x_1}(t, \cdot)\|_{L^p} &\leq C(\omega_+ - \omega_-)^{\frac{1}{p}}(\delta+t)^{-1+\frac{1}{p}}, \\ \|\bar{u}_{1x_1 x_1}(t, \cdot)\|_{L^p} &\leq C(\delta+t)^{-1}\delta^{-1+\frac{1}{p}}, \\ \|\bar{u}_{1x_1 x_1 x_1}(t, \cdot)\|_{L^p} &\leq C(\delta+t)^{-1}\delta^{-2+\frac{1}{p}}, \\ \|\bar{u}_{1x_1 x_1 x_1 x_1}(t, \cdot)\|_{L^p} &\leq C(\delta+t)^{-1}\delta^{-3+\frac{1}{p}}, \\ \|\bar{u}_{1x_1 x_1 x_1 x_1 x_1}(t, \cdot)\|_{L^p} &\leq C(\delta+t)^{-1}\delta^{-4+\frac{1}{p}}. \end{aligned}$$

- (3) There exists a constant $\delta_0 \in (0, 1)$ such that $\delta \in (0, \delta_0], t > 0$,

$$\|(\bar{\rho} - \rho_v^{r_2}, \bar{u}_1 - u_{1v}^{r_2})(t, \cdot)\|_{L^\infty} \leq C\delta t^{-1}[\ln(1+t) + |\ln \delta|].$$

2.3 Hyperbolic wave

As mentioned in [27, 28], the hyperbolic waves are needed to recover the physical viscosities for the inviscid approximate cut-off 2-rarefaction wave profile in the multi-dimensional vanishing viscosity limit. The hyperbolic wave (d_1, d_2) is constructed through the linearized hyperbolic system around the approximate rarefaction wave

$$\begin{cases} d_{1t} + d_{2x_1} = 0, \\ d_{2t} + \left(-\frac{\bar{m}_1^2}{\bar{\rho}^2} d_1 + p'(\bar{\rho}) d_1 + \frac{2\bar{m}_1}{\bar{\rho}} d_2 \right)_{x_1} = (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1}, \\ (d_1, d_2)(0, x_1) = (0, 0), \end{cases} \quad (2.9)$$

where $\bar{m}_1 := \bar{\rho} \bar{u}_1$ represents the momentum of the approximate cut-off 2-rarefaction wave. Then we solve the hyperbolic equations (2.9) on the fixed time interval $[0, T]$. Diagonalize the system (2.9) as

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_t + \left(\bar{A} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right)_{x_1} = \begin{bmatrix} 0 \\ (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1} \end{bmatrix},$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -\frac{\bar{m}_1^2}{\bar{\rho}^2} + p'(\bar{\rho}) & \frac{2\bar{m}_1}{\bar{\rho}} \end{bmatrix}$$

with two eigenvalues

$$\bar{\lambda}_i(\bar{\rho}, \bar{m}_1) = \frac{\bar{m}_1}{\bar{\rho}} + (-1)^i \sqrt{p'(\bar{\rho})}, \quad i = 1, 2,$$

and the corresponding left and right eigenvectors

$$\bar{l}_i = \bar{l}_i(\bar{\rho}, \bar{m}_1) = \left(\frac{\sqrt{2}}{2} \left(-\frac{\bar{m}_1}{\bar{\rho}} + (-1)^i \sqrt{p'(\bar{\rho})} \right), \frac{\sqrt{2}}{2} \right), \quad i = 1, 2,$$

and

$$\bar{r}_i = \bar{r}_i(\bar{\rho}, \bar{m}_1) = \left(\frac{(-1)^i \sqrt{2}}{2\sqrt{p'(\bar{\rho})}}, \frac{(-1)^i \sqrt{2}}{2\sqrt{p'(\bar{\rho})}} \left(\frac{\bar{m}_1}{\bar{\rho}} + (-1)^i \sqrt{p'(\bar{\rho})} \right) \right)^\top, \quad i = 1, 2,$$

satisfying

$$\bar{L} \bar{A} \bar{R} = \text{diag}(\bar{\lambda}_1, \bar{\lambda}_2) := \bar{\Lambda}, \quad \bar{L} \bar{R} = I,$$

where $\bar{L} = (\bar{l}_1, \bar{l}_2)^\top$, $\bar{R} = (\bar{r}_1, \bar{r}_2)$ and I is the 2×2 identity matrix. Set

$$(D_1, D_2)^\top = \bar{L}(d_1, d_2)^\top,$$

then

$$(d_1, d_2)^\top = \bar{R}(D_1, D_2)^\top$$

and (D_1, D_2) satisfies the diagonalized system

$$\begin{aligned} & \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}_t + \left(\bar{A} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \right)_{x_1} \\ &= \bar{L}_t \bar{R} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + \bar{L}_{x_1} \bar{A} \bar{R} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + \bar{L} \begin{bmatrix} 0 \\ (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1} \end{bmatrix}. \end{aligned} \quad (2.10)$$

Since the 2-Riemann invariant is constant along the approximate cut-off 2-rarefaction wave curve, we have

$$\bar{L}_t = -\bar{\lambda}_2 \bar{L}_{x_1}, \quad (2.11)$$

which is utilized to decouple the strongly coupled hyperbolic system (2.10). Use the relation (2.11) to simplify the diagonalized system (2.10) as

$$\begin{cases} D_{1t} + (\bar{\lambda}_1 D_1)_{x_1} = \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1} - \frac{\gamma+1}{2} \bar{u}_{1x_1} D_1, \end{cases} \quad (2.12a)$$

$$\begin{cases} D_{2t} + (\bar{\lambda}_2 D_2)_{x_1} = \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1} - \frac{3-\gamma}{2} \bar{u}_{1x_1} D_1, \end{cases} \quad (2.12b)$$

$$\begin{cases} (D_1, D_2)(0, x_1) = (0, 0). \end{cases} \quad (2.12c)$$

The equation of D_1 is decoupled with D_2 due to the relation (2.11). Therefore, we have the following estimates for the hyperbolic wave (d_1, d_2) .

Lemma 2.4. *There exists a constant C_T independent of δ and ε such that*

$$\begin{aligned} & \|(D_1, D_2)(t, \cdot)\|^2 \leq C_T \frac{\varepsilon^2}{\delta^2}, \\ & \left\| \frac{\partial^k}{\partial x_1^k} (D_1, D_2)(t, \cdot) \right\|^2 \leq C_T \frac{\varepsilon^2}{\nu^\alpha \delta^{2(k+1)}}, \quad k = 1, 2, 3 \end{aligned}$$

with $\alpha \geq 0$ and $\alpha > (5\gamma - 23)/4$.

Remark that due to the degeneracy of the vacuum states, the estimates of the hyperbolic waves in Lemma 2.4 depend on the cut-off parameter ν .

Proof. The proof consists of the following four steps:

Step 1. Multiplying the Eq. (2.12b) by D_2 yields

$$\left(\frac{D_2^2}{2}\right)_t + \left(\bar{\lambda}_2 \frac{D_2^2}{2}\right)_{x_1} + \frac{\gamma+1}{4} \bar{u}_{1x_1} D_2^2 = \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1} D_2 - \frac{3-\gamma}{2} \bar{u}_{1x_1} D_1 D_2.$$

Integrating the above equation over $[0, t] \times \mathbb{R}$, it holds that

$$\begin{aligned} & \int_{\mathbb{R}} \frac{D_2^2}{2} dx_1 + \frac{\gamma+1}{4} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_2^2 dx_1 dt_1 \\ &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1 x_1} D_2 dx_1 dt_1 - \frac{3-\gamma}{2} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_1 D_2 dx_1 dt_1 \\ &\leq \int_0^t \int_{\mathbb{R}} \frac{D_2^2}{2} dx_1 dt_1 + C\varepsilon^2 \int_0^t \|\bar{u}_{1x_1 x_1}\|^2 dt_1 + \frac{\gamma+1}{8} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_2^2 dx_1 dt_1 \\ &\quad + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_1^2 dx_1 dt_1, \\ &\leq \int_0^t \int_{\mathbb{R}} \frac{D_2^2}{2} dx_1 dt_1 + \frac{C\varepsilon^2}{\delta^2} + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_1^2 dx_1 dt_1, \end{aligned}$$

it holds from Gronwall's inequality that

$$\|D_2\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} D_2 \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\delta^2} + C_T \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} D_1 \right\|^2 dt_1. \quad (2.13)$$

In order to estimate the second item in the right-hand side of (2.13), multiplying the Eq. (2.12a) by $\bar{\rho}^\alpha D_1$ ($\alpha \geq 0$)

$$\left(\bar{\rho}^\alpha \frac{D_1^2}{2}\right)_t + \left(\bar{\rho}^\alpha \bar{\lambda}_1 \frac{D_1^2}{2}\right)_{x_1} + \left(\alpha + \frac{\gamma+5}{4}\right) \bar{\rho}^\alpha \bar{u}_{1x_1} D_1^2 = \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1} \bar{\rho}^\alpha D_1.$$

Integrating over $[0, t] \times \mathbb{R}$,

$$\begin{aligned} & \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_1^2}{2} dx_1 + \left(\alpha + \frac{\gamma+5}{4}\right) \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_1^2 dx_1 dt_1 \\ &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1 x_1} \bar{\rho}^\alpha D_1 dx_1 dt_1 \\ &\leq C\varepsilon^2 \int_0^t \|\bar{u}_{1x_1 x_1}\|^2 dt_1 + \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_1^2}{2} dx_1 dt_1 \\ &\leq \frac{C\varepsilon^2}{\delta^2} + \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_1^2}{2} dx_1 dt_1, \end{aligned}$$

using Gronwall's inequality, we can obtain that

$$\left\| \bar{\rho}^{\frac{\alpha}{2}} D_1 \right\|^2 + \left(\alpha + \frac{\gamma+5}{4} \right) \int_0^t \left\| \bar{\rho}^{\frac{\alpha}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} D_1 \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\delta^2}, \quad (2.14)$$

due to $\alpha \geq 0$, we can take $\alpha = 0$ and combine (2.13) and (2.14),

$$\left\| (D_1, D_2) \right\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} (D_1, D_2) \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\delta^2}. \quad (2.15)$$

Step 2. Applying the operator ∂_{x_1} to the Eq. (2.12b) and multiplying the resulting equation by D_{2x_1} yield

$$\begin{aligned} & \left(\frac{D_{2x_1}^2}{2} \right)_t + \left(\bar{\lambda}_2 \frac{D_{2x_1}^2}{2} \right)_{x_1} + \frac{3(\gamma+1)}{4} \bar{u}_{1x_1} D_{2x_1}^2 \\ &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1 x_1} D_{2x_1} - \frac{\gamma+1}{2} \bar{u}_{1x_1 x_1} D_2 \cdot D_{2x_1} \\ & \quad - \frac{3-\gamma}{2} (\bar{u}_{1x_1 x_1} D_1 + \bar{u}_{1x_1} D_{1x_1}) D_{2x_1}. \end{aligned}$$

Integrating the above equality over $[0, t] \times \mathbb{R}$ implies that

$$\begin{aligned} & \int_{\mathbb{R}} \frac{D_{2x_1}^2}{2} dx_1 + \frac{3(\gamma+1)}{4} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{2x_1}^2 dx_1 dt_1 \\ &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1 x_1 x_1} D_{2x_1} dx_1 dt_1 - \frac{\gamma+1}{2} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1 x_1} D_2 \cdot D_{2x_1} dx_1 dt_1 \\ & \quad - \frac{3-\gamma}{2} \int_0^t \int_{\mathbb{R}} (\bar{u}_{1x_1 x_1} D_1 + \bar{u}_{1x_1} D_{1x_1}) \cdot D_{2x_1} dx_1 dt_1 \\ &\leq \int_0^t \int_{\mathbb{R}} \frac{D_{2x_1}^2}{2} dx_1 dt_1 + C\varepsilon^2 \int_0^t \left\| \bar{u}_{1x_1 x_1 x_1} \right\|^2 dt_1 + \frac{C}{\delta^2} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} (D_2^2 + D_1^2) dx_1 dt_1 \\ & \quad + \frac{3(\gamma+1)}{8} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{2x_1}^2 dx_1 dt_1 + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{1x_1}^2 dx_1 dt_1 \\ &\leq \int_0^t \int_{\mathbb{R}} \frac{D_{2x_1}^2}{2} dx_1 dt_1 + \frac{C_T \varepsilon^2}{\delta^4} + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{1x_1}^2 dx_1 dt_1, \end{aligned}$$

by using Gronwal's inequality, we can obtain

$$\left\| D_{2x_1} \right\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} D_{2x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\delta^4} + C_T \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} D_{1x_1} \right\|^2 dt_1. \quad (2.16)$$

Then applying the operator ∂_{x_1} to the Eq. (2.12a) and multiplying the resulting equation by $\bar{\rho}^\alpha D_{1x_1}$ ($\alpha \geq 0$) give

$$\begin{aligned} & \left(\bar{\rho}^\alpha \frac{D_{1x_1}^2}{2} \right)_t + \left(\bar{\rho}^\alpha \bar{\lambda}_1 \frac{D_{1x_1}^2}{2} \right)_{x_1} + \left(\alpha + \frac{11-\gamma}{4} \right) \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1}^2 \\ &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1 x_1} \bar{\rho}^\alpha D_{1x_1} - 2 \bar{u}_{1x_1 x_1} \bar{\rho}^\alpha D_1 D_{1x_1}. \end{aligned}$$

If $\alpha \geq 0$ and $\alpha + (11 - \gamma)/4 > 0$, then integrating the above equality over $[0, t] \times \mathbb{R}$ implies that

$$\begin{aligned} & \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1}^2}{2} dx_1 + \left(\alpha + \frac{11-\gamma}{4} \right) \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1}^2 dx_1 dt_1 \\ &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1 x_1 x_1} \bar{\rho}^\alpha D_{1x_1} dx_1 dt_1 - 2 \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1 x_1} D_1 \cdot D_{1x_1} dx_1 dt_1 \\ &\leq \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1}^2}{2} dx_1 dt_1 + C\varepsilon^2 \int_0^t \|\bar{u}_{1x_1 x_1 x_1}\|^2 dt_1 + \frac{C}{\delta^2} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_1^2 dx_1 dt_1 \\ &\quad + \frac{1}{2} \left(\alpha + \frac{11-\gamma}{4} \right) \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1}^2 dx_1 dt_1 \\ &\leq \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1}^2}{2} dx_1 dt_1 + \frac{C_T \varepsilon^2}{\delta^4}, \end{aligned}$$

using Gronwall's inequality leads to

$$\left\| \bar{\rho}^{\frac{\alpha}{2}} D_{1x_1} \right\|^2 + \left(\alpha + \frac{11-\gamma}{4} \right) \int_0^t \left\| \bar{\rho}^{\frac{\alpha}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} D_{1x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\delta^4}. \quad (2.17)$$

Combining (2.16) with (2.17), we have

$$\|(D_1, D_2)_{x_1}\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} (D_1, D_2)_{x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\nu^\alpha \delta^4}, \quad (2.18)$$

if $\alpha \geq 0$ and $\alpha > (\gamma - 11)/4$.

Step 3. Applying the operator $\partial_{x_1 x_1}$ to (2.12b) and multiplying by $D_{2x_1 x_1}$ yield

$$\begin{aligned} & \left(\frac{D_{2x_1 x_1}}{2} \right)_t + \left(\frac{\bar{\lambda}_2 D_{2x_1 x_1}^2}{2} \right)_{x_1} + \frac{5(\gamma+1)}{4} \bar{u}_{1x_1} D_{2x_1 x_1}^2 \\ &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1 x_1 x_1 x_1} D_{2x_1 x_1} - \frac{\gamma+1}{2} D_{2x_1 x_1} (\bar{u}_{1x_1 x_1 x_1} D_2 + 3 \bar{u}_{1x_1 x_1} D_{2x_1}) \\ &\quad - \frac{3-\gamma}{2} D_{2x_1 x_1} (\bar{u}_{1x_1 x_1 x_1} D_1 + 2 \bar{u}_{1x_1 x_1} D_{1x_1} + \bar{u}_{1x_1} D_{1x_1 x_1}). \end{aligned}$$

Integrating the above equality over $[0, t] \times \mathbb{R}$ implies that

$$\begin{aligned}
 & \int_{\mathbb{R}} \frac{D_{2x_1x_1}^2}{2} dx_1 + \frac{5(\gamma+1)}{4} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{2x_1x_1}^2 dx_1 dt_1 \\
 &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1x_1x_1x_1} D_{2x_1x_1} dx_1 dt_1 \\
 & \quad - \int_0^t \int_{\mathbb{R}} \left[\frac{\gamma+1}{2} D_{2x_1x_1} (\bar{u}_{1x_1x_1x_1} D_2 + 3\bar{u}_{1x_1x_1} D_{2x_1}) \right. \\
 & \quad \left. + \frac{3-\gamma}{2} D_{2x_1x_1} (\bar{u}_{1x_1x_1x_1} D_1 + 2\bar{u}_{1x_1x_1} D_{1x_1} + \bar{u}_{1x_1} D_{1x_1x_1}) \right] dx_1 dt_1 \\
 & \leq \int_0^t \int_{\mathbb{R}} \frac{D_{2x_1x_1}^2}{2} dx_1 dt_1 + C\epsilon^2 \int_0^t \|\bar{u}_{1x_1x_1x_1x_1}\|^2 dt_1 + \frac{5(\gamma+1)}{8} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{2x_1x_1}^2 dx_1 dt_1 \\
 & \quad + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{1x_1x_1}^2 dx_1 dt_1 + \frac{C}{\delta^2} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} (D_{2x_1}^2 + D_{1x_1}^2) dx_1 dt_1 \\
 & \quad + \frac{C}{\delta^4} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} (D_2^2 + D_1^2) dx_1 dt_1 \\
 & \leq \int_0^t \int_{\mathbb{R}} \frac{D_{2x_1x_1}^2}{2} dx_1 dt_1 + \frac{C_T \epsilon^2}{\nu^\alpha \delta^6} + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{1x_1x_1}^2 dx_1 dt_1,
 \end{aligned}$$

using Gronwall's inequality implies that

$$\|D_{2x_1x_1}\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} D_{2x_1x_1} \right\|^2 dt_1 \leq \frac{C_T \epsilon^2}{\nu^\alpha \delta^6} + C_T \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} D_{1x_1x_1} \right\|^2 dt_1. \quad (2.19)$$

In order to calculate the second item of right side in the above inequality, applying the partial operator $\partial_{x_1x_1}$ to (2.12)₁ and multiplying the resulting equation by $\bar{\rho}^\alpha D_{2x_1x_1}$ yields that

$$\begin{aligned}
 & \left(\bar{\rho}^\alpha \frac{D_{1x_1x_1}^2}{2} \right)_t + \left(\bar{\rho}^\alpha \bar{\lambda}_1 \frac{D_{1x_1x_1}^2}{2} \right)_{x_1} + \left(\alpha + \frac{17-3\gamma}{4} \right) \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1x_1}^2 \\
 &= \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1x_1x_1x_1} \bar{\rho}^\alpha D_{1x_1x_1} - 2\bar{\rho}^\alpha \bar{u}_{1x_1x_1x_1} D_1 D_{1x_1x_1} \\
 & \quad - \frac{11-\gamma}{2} \bar{\rho}^\alpha \bar{u}_{1x_1x_1} D_{1x_1} D_{1x_1x_1}.
 \end{aligned}$$

If $\alpha \geq 0$ and $\alpha > \max\{(\gamma-11)/4, (3\gamma-17)/4\}$, then integrating the above equality over $[0, t] \times \mathbb{R}$ implies that

$$\int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1x_1}^2}{2} dx_1 + \left(\alpha + \frac{17-3\gamma}{4} \right) \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1x_1}^2 dx_1 dt_1$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{2}(2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1x_1x_1x_1} \bar{\rho}^\alpha D_{1x_1x_1} dx_1 dt_1 \\
&\quad - \int_0^t \int_{\mathbb{R}} \left[2\bar{\rho}^\alpha \bar{u}_{1x_1x_1x_1} D_1 D_{1x_1x_1} + \frac{11-\gamma}{2} \bar{\rho}^\alpha \bar{u}_{1x_1x_1} D_{1x_1} D_{1x_1x_1} \right] dx_1 dt_1 \\
&\leq \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1x_1}^2}{2} dx_1 dt_1 + C\varepsilon^2 \int_0^t \|\bar{u}_{1x_1x_1x_1x_1}\|^2 dt_1 + \frac{C}{\delta^4} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_1^2 dx_1 dt_1 \\
&\quad + \frac{C}{\delta^2} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1}^2 dx_1 dt_1 + \frac{1}{2} \left(\alpha + \frac{17-3\gamma}{4} \right) \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1x_1}^2 dx_1 dt_1 \\
&\leq \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1x_1}^2}{2} dx_1 dt_1 + \frac{C_T \varepsilon^2}{\delta^6},
\end{aligned}$$

from Gronwall's inequality, it holds that

$$\left\| \bar{\rho}^{\frac{\alpha}{2}} D_{1x_1x_1} \right\|^2 + \left(\alpha + \frac{17-3\gamma}{4} \right) \int_0^t \left\| \bar{\rho}^{\frac{\alpha}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} D_{1x_1x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\delta^6}. \quad (2.20)$$

Combining (2.19) with (2.20), we have

$$\left\| (D_1, D_2)_{x_1x_1} \right\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} (D_1, D_2)_{x_1x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\nu^\alpha \delta^6}, \quad (2.21)$$

if $\alpha \geq 0$ and $\alpha > \max\{(\gamma-11)/4, (3\gamma-17)/4\}$.

Step 4. Applying the operator $\partial_{x_1x_1x_1}$ to (2.12b) and multiplying by $D_{2x_1x_1x_1}$ yield

$$\begin{aligned}
&\left(\frac{D_{2x_1x_1x_1}^2}{2} \right)_t + \left(\bar{\lambda}_2 \frac{D_{2x_1x_1x_1}^2}{2} \right)_{x_1} + \frac{7(\gamma+1)}{4} \bar{u}_{1x_1} D_{2x_1x_1x_1}^2 \\
&= \frac{\sqrt{2}}{2}(2\mu_1 + \lambda_1) \bar{u}_{1x_1x_1x_1x_1x_1} D_{2x_1x_1x_1} \\
&\quad - \frac{\gamma+1}{2} D_{2x_1x_1x_1} (\bar{u}_{1x_1x_1x_1x_1} D_2 + 4\bar{u}_{1x_1x_1x_1} D_{2x_1} + 6\bar{u}_{1x_1x_1} D_{2x_1x_1}) \\
&\quad - \frac{3-\gamma}{2} D_{2x_1x_1x_1} (\bar{u}_{1x_1x_1x_1x_1} D_1 + 3\bar{u}_{1x_1x_1x_1} D_{1x_1} + 3\bar{u}_{1x_1x_1} D_{1x_1x_1} + \bar{u}_{1x_1} D_{1x_1x_1x_1}).
\end{aligned}$$

Integrating the above equality over $[0, t] \times \mathbb{R}$ implies that

$$\begin{aligned}
&\int_0^t \frac{D_{2x_1x_1x_1}^2}{2} dt_1 + \frac{7(\gamma+1)}{4} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{2x_1x_1x_1}^2 dx_1 dt_1 \\
&= \frac{\sqrt{2}}{2}(2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1x_1x_1x_1x_1} D_{2x_1x_1x_1} dx_1 dt_1
\end{aligned}$$

$$\begin{aligned}
& -\frac{\gamma+1}{2} \int_0^t \int_{\mathbb{R}} D_{2x_1x_1x_1} (\bar{u}_{1x_1x_1x_1x_1} D_2 + 4\bar{u}_{1x_1x_1x_1} D_{2x_1} + 6\bar{u}_{1x_1x_1} D_{2x_1x_1}) dx_1 dt_1 \\
& -\frac{3-\gamma}{2} \int_0^t \int_{\mathbb{R}} D_{2x_1x_1x_1} (\bar{u}_{1x_1x_1x_1x_1} D_1 + 3\bar{u}_{1x_1x_1x_1} D_{1x_1} \\
& \quad + 3\bar{u}_{1x_1x_1} D_{1x_1x_1} + \bar{u}_{1x_1} D_{1x_1x_1x_1}) dx_1 dt_1 \\
& \leq \int_0^t \int_{\mathbb{R}} \frac{D_{2x_1x_1x_1}^2}{2} dx_1 dt_1 + C\varepsilon^2 \int_0^t \|\bar{u}_{1x_1x_1x_1x_1x_1}\|^2 dt_1 \\
& \quad + \frac{7(\gamma+1)}{8} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{2x_1x_1x_1}^2 dx_1 dt_1 \\
& \quad + \frac{C}{\delta^2} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} (D_{2x_1x_1}^2 + D_{1x_1x_1}^2) dx_1 dt_1 + \frac{C}{\delta^4} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} (D_{2x_1}^2 + D_{1x_1}^2) dx_1 dt_1 \\
& \quad + \frac{C}{\delta^6} \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} (D_2^2 + D_1^2) dx_1 dt_1 + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{1x_1x_1x_1}^2 dx_1 dt_1 \\
& \leq \int_0^t \int_{\mathbb{R}} \frac{D_{2x_1x_1x_1}^2}{2} dx_1 dt_1 + \frac{C_T \varepsilon^2}{\nu^\alpha \delta^8} + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{1x_1x_1x_1}^2 dx_1 dt_1,
\end{aligned}$$

it implies from Gronwal's inequality that

$$\|D_{2x_1x_1x_1}\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} D_{2x_1x_1x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\nu^\alpha \delta^8} + C \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_{1x_1x_1x_1}^2 dx_1 dt_1. \quad (2.22)$$

Then applying the operator $\partial_{x_1x_1x_1}$ to (2.12a) and multiplying the resulting equation by $\bar{\rho}^\alpha D_{1x_1x_1x_1}$ ($\alpha \geq 0$) give

$$\begin{aligned}
& \left(\bar{\rho}^\alpha \frac{D_{1x_1x_1x_1}^2}{2} \right)_t + \left(\bar{\rho}^\alpha \bar{\lambda}_1 \frac{D_{1x_1x_1x_1}^2}{2} \right)_{x_1} + \left(\alpha + \frac{23-5\gamma}{4} \right) \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1x_1x_1}^2 \\
& = \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \bar{u}_{1x_1x_1x_1x_1x_1} \bar{\rho}^\alpha D_{1x_1x_1x_1} - 2\bar{\rho}^\alpha \bar{u}_{1x_1x_1x_1x_1} D_1 D_{1x_1x_1x_1} \\
& \quad - \frac{15-\gamma}{2} \bar{\rho}^\alpha \bar{u}_{1x_1x_1x_1} D_{1x_1} D_{1x_1x_1x_1} - \frac{21-3\gamma}{2} \bar{\rho}^\alpha \bar{u}_{1x_1x_1} D_{1x_1x_1} D_{1x_1x_1x_1}.
\end{aligned}$$

If $\alpha \geq 0$ and $\alpha > \max\{(\gamma-11)/4, (3\gamma-17)/4, (5\gamma-23)/4\}$, then integrating the above equality over $[0, t] \times \mathbb{R}$ implies that

$$\begin{aligned}
& \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1x_1x_1}^2}{2} dx_1 dt_1 + \left(\alpha + \frac{23-5\gamma}{4} \right) \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1x_1x_1}^2 dx_1 dt_1 \\
& = \frac{\sqrt{2}}{2} (2\mu_1 + \lambda_1) \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1x_1x_1x_1x_1} \bar{\rho}^\alpha D_{1x_1x_1x_1} dx_1 dt_1
\end{aligned}$$

$$\begin{aligned}
& -2 \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1x_1x_1x_1} D_1 D_{1x_1x_1x_1} dx_1 dt_1 \\
& - \frac{15-\gamma}{2} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1x_1x_1} D_{1x_1} D_{1x_1x_1x_1} dx_1 dt_1 \\
& - \frac{21-3\gamma}{2} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1x_1} D_{1x_1x_1} D_{1x_1x_1x_1} dx_1 dt_1 \\
& \leq \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1x_1x_1}^2}{2} dx_1 dt_1 + C\varepsilon^2 \int_0^t \|\bar{u}_{1x_1x_1x_1x_1}\| dt_1 \\
& \quad + \frac{1}{2} \left(\alpha + \frac{23-5\gamma}{4} \right) \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1x_1x_1}^2 dx_1 dt_1 + \frac{C}{\delta^6} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_1^2 dx_1 dt_1 \\
& \quad + \frac{C}{\delta^4} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1}^2 dx_1 dt_1 + \frac{C}{\delta^2} \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \bar{u}_{1x_1} D_{1x_1x_1}^2 dx_1 dt_1 \\
& \leq \int_0^t \int_{\mathbb{R}} \bar{\rho}^\alpha \frac{D_{1x_1x_1x_1}^2}{2} dx_1 dt_1 + \frac{C_T \varepsilon^2}{\delta^8},
\end{aligned}$$

it implies from Gronwall's inequality that

$$\left\| \bar{\rho}^{\frac{\alpha}{2}} D_{1x_1x_1x_1} \right\|^2 + \left(\alpha + \frac{23-5\gamma}{4} \right) \int_0^t \left\| \bar{\rho}^{\frac{\alpha}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} D_{1x_1x_1x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\delta^8}. \quad (2.23)$$

Combining (2.22) with (2.23), we have

$$\left\| (D_1, D_2)_{x_1x_1x_1} \right\|^2 + \int_0^t \left\| \bar{u}_{1x_1}^{\frac{1}{2}} (D_1, D_2)_{x_1x_1x_1} \right\|^2 dt_1 \leq \frac{C_T \varepsilon^2}{\nu^\alpha \delta^8}, \quad (2.24)$$

if $\alpha \geq 0$ and $\alpha > \max\{(\gamma-11)/4, (3\gamma-17)/4, (5\gamma-23)/4\} = (5\gamma-23)/4$. Therefore, from (2.15), (2.18), (2.21) and (2.24), we obtain Lemma 2.4.

The proof is complete. $\square$

Lemma 2.5. *There exists a constant C_T independent of δ and ε such that*

$$\left\| \frac{\partial^k}{\partial x_1^k} (d_1, d_2)(t, \cdot) \right\|^2 \leq C_T \frac{\varepsilon^2}{(\nu^\alpha \delta^2)^{k+1}}, \quad k=0,1,2,3$$

with $\alpha = \gamma - 1$ if $1 < \gamma < 19$. Therefore, it holds that

$$\sup_{0 \leq t \leq T} \left\| \frac{\partial^k}{\partial x_1^k} (d_1, d_2)(t, \cdot) \right\|_{L^\infty} \leq C_T \frac{\varepsilon}{(\nu^\alpha \delta^2)^{\frac{2k+3}{4}}}, \quad k=0,1,2.$$

Proof. We use Lemma 2.4 to obtain the estimation of d_1, d_2 . Since

$$|(d_1, d_2)|^2 = (D_1, D_2) \bar{R}^\top \bar{R} (D_1, D_2)^\top \leq \frac{C_T}{p'(\bar{\rho})} |(D_1, D_2)|^2 \leq \frac{C_T}{\nu^{\gamma-1}} |(D_1, D_2)|^2,$$

we first have

$$\|(d_1, d_2)\|^2 \leq C_T \frac{\varepsilon^2}{\nu^{\gamma-1} \delta^2}.$$

Due to

$$\begin{aligned} \left| \frac{\partial}{\partial x_1} (d_1, d_2) \right|^2 &= \frac{\partial}{\partial x_1} (D_1, D_2) \bar{R}^\top \bar{R} \frac{\partial}{\partial x_1} (D_1, D_2)^\top + (D_1, D_2) \frac{\partial \bar{R}}{\partial x_1}^\top \frac{\partial \bar{R}}{\partial x_1} (D_1, D_2)^\top \\ &\leq \frac{C}{p'(\bar{\rho})} \left| \frac{\partial}{\partial x_1} (D_1, D_2) \right|^2 + \frac{C_T}{p'(\bar{\rho})^2 \delta^2} |(D_1, D_2)|^2 \\ &\leq \frac{C}{\nu^{\gamma-1}} \left| \frac{\partial}{\partial x_1} (D_1, D_2) \right|^2 + \frac{C_T}{\nu^{2(\gamma-1)} \delta^2} |(D_1, D_2)|^2. \end{aligned}$$

When $\alpha \geq \gamma - 1$ and $\alpha > (5\gamma - 23)/4$, i.e., take $\alpha = \gamma - 1 > (5\gamma - 23)/4$ (if $1 < \gamma < 19$), we obtain the first order derivative estimates

$$\left\| \frac{\partial}{\partial x_1} (d_1, d_2) \right\|^2 \leq C_T \frac{\varepsilon^2}{\nu^{(\gamma-1)+\alpha} \delta^4} + C_T \frac{\varepsilon^2}{\nu^{2(\gamma-1)} \delta^4} \leq \frac{\varepsilon^2}{\nu^{2\alpha} \delta^4}.$$

Similarly, for the higher order derivative estimates, we get

$$\left\| \frac{\partial^k}{\partial x_1^k} (d_1, d_2) \right\|^2 \leq C_T \frac{\varepsilon^2}{(\nu^\alpha \delta^2)^{k+1}}, \quad k = 2, 3.$$

Therefore, Lemma 2.5 is proved. $\square$

2.4 The solution profile

Now the solution profile $(\tilde{\rho}, \tilde{u}_1)$ to the compressible Navier-Stokes equations (1.1) can be defined by

$$\tilde{\rho}(t, x_1) = (\bar{\rho} + d_1)(t, x_1), \quad \tilde{m}_1(t, x_1) = (\bar{m}_1 + d_2)(t, x_1) := \tilde{\rho} \tilde{u}_1(t, x_1), \quad (2.25)$$

satisfying the following system:

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho} \tilde{u}_1)_{x_1} = 0, \\ (\tilde{\rho} \tilde{u}_1)_t + (\tilde{\rho} \tilde{u}_1^2 + p(\tilde{\rho}))_{x_1} \\ = (2\mu_1 + \lambda_1) \tilde{u}_{1x_1x_1} + (\tilde{\rho} \tilde{u}_1^2 - \bar{\rho} \bar{u}_1^2 + \bar{u}_1^2 d_1 - 2\bar{u}_1 d_2)_{x_1} \\ + (p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1} \end{cases} \quad (2.26)$$

with the initial data

$$(\tilde{\rho}, \tilde{u}_1)(0, x_1) := (\bar{\rho}_0, \bar{u}_{10})(x_1). \quad (2.27)$$

Since $\nu < \bar{\rho} < \rho_+$, $\|d_1\|_\infty \leq C_T(\varepsilon/(\nu^\alpha \delta^2)^{3/4})$ (if $1 < \gamma < 19$) and by (3.9), we can get

$$\left\| \frac{d_1}{\bar{\rho}} \right\|_\infty \leq C_T \frac{\varepsilon}{\nu^{\frac{3}{4}\alpha+1} \delta^{\frac{3}{2}}} \leq C_T \varepsilon^\beta,$$

where β is a small positive constant. Thus, $\tilde{\rho}/\bar{\rho} = 1 + d_1/\bar{\rho} \sim 1$, i.e.

$$\tilde{\rho} \sim \bar{\rho} \quad \text{and} \quad \frac{\nu}{2} < \tilde{\rho} < 2\rho_+. \quad (2.28)$$

3 Reformulation of the problem

To prove Theorem 1.1, the solution $(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ to the system (1.1) is constructed as the perturbation around the solution profile $(\tilde{\rho}, \tilde{u}_1, 0, 0)$ defined in (2.25)-(2.27). Set the perturbation $(\phi, \Psi)(t, x_1, x_2, x_3)$ by

$$\begin{aligned} \phi &:= \rho^\varepsilon(t, x_1, x_2, x_3) - \tilde{\rho}(t, x_1), \\ \Psi &:= (\psi_1, \psi_2, \psi_3)(t, x_1, x_2, x_3) \\ &= (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)^\top(t, x_1, x_2, x_3) - (\tilde{u}_1, 0, 0)^\top(t, x_1), \end{aligned} \quad (3.1)$$

where $(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ is the solution to the 3D compressible Navier-Stokes equations (1.1) with the initial data

$$(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)(0, x_1, x_2, x_3) := (\bar{\rho}_0, \bar{u}_{10}, 0, 0)(x_1) + (\phi_0, \psi_{10}, \psi_{20}, \psi_{30})(x_1, x_2, x_3), \quad (3.2)$$

and the initial perturbations $(\phi_0, \psi_{10}, \psi_{20}, \psi_{30})$ satisfying

$$\left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \nabla^i \phi_0, \bar{\rho}_0^{\frac{1}{2}} \nabla^i \Psi_0 \right) \right\|^2 = \mathcal{O}(1) \frac{\varepsilon^{4-i}}{\nu^{\frac{7}{2}\alpha+3} \delta^{7+i}}, \quad i=0,1,2. \quad (3.3)$$

For the simplicity of notation, the superscript of $(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ will be omitted as (ρ, u_1, u_2, u_3) from now on if there are no confusions. Reformulate the system for

(ϕ, Ψ) as

$$\left\{ \begin{array}{l} \phi_t + \rho \operatorname{div} \Psi + \phi \tilde{u}_{1x_1} + u \nabla \phi + \psi_1 \tilde{\rho}_{x_1} = 0, \\ \rho \Psi_t + \rho u_1 \Psi_{x_1} + \rho u_2 \Psi_{x_2} + \rho u_3 \Psi_{x_3} + (\rho \psi_1 \tilde{u}_{1x_1}, 0, 0)^\top + p'(\rho) \nabla \phi \\ \quad + \left((p'(\rho) - \frac{\rho}{\tilde{\rho}} p'(\tilde{\rho})) \tilde{\rho}_{x_1}, 0, 0 \right)^\top + \left((2\mu_1 + \lambda_1) \frac{\tilde{u}_{1x_1 x_1}}{\tilde{\rho}} \phi, 0, 0 \right)^\top \\ \quad + \left(\frac{(\tilde{\rho} \tilde{u}_1^2 - \tilde{\rho} \bar{u}_1^2 + \bar{u}_1^2 d_1 - 2\bar{u}_1 d_2)_{x_1}}{\tilde{\rho}} \phi, 0, 0 \right)^\top \\ \quad + \left(\frac{(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1}}{\tilde{\rho}} \phi, 0, 0 \right)^\top \\ = \mu_1 \Delta \Psi + (\mu_1 + \lambda_1) \nabla \operatorname{div} \Psi + \left((2\mu_1 + \lambda_1) \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1}, 0, 0 \right)^\top \\ \quad - \left((\tilde{\rho} \tilde{u}_1^2 - \tilde{\rho} \bar{u}_1^2 + \bar{u}_1^2 d_1 - 2\bar{u}_1 d_2)_{x_1}, 0, 0 \right)^\top \\ \quad - \left((p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1}, 0, 0 \right)^\top \end{array} \right. \quad (3.4a)$$

with the initial perturbation

$$(\phi_0, \Psi_0)(x_1, x_2, x_3) = (\phi_0, \psi_{10}, \psi_{20}, \psi_{30})(x_1, x_2, x_3) \quad (3.5)$$

satisfying (3.3). Note that the solution to (3.4)-(3.5) will be sought in the function space $\Pi(0, t_1)$ defined by

$$\Pi(0, t_1) = \left\{ (\phi, \Psi) \mid \left(\tilde{\rho}^{\frac{\gamma-2}{2}} \nabla^i \phi, \tilde{\rho}^{\frac{1}{2}} \nabla^i \Psi \right) \in C^0([0, t_1]; L^2(\Omega)), \right. \\ \left. \nabla^{1+i} \Psi \in L^2(0, t_1; L^2(\Omega)), i=0, 1, 2 \right\}, \quad \forall t_1 \in (0, T].$$

To carry out the stability analysis, we need to introduce more accurate a priori assumptions

$$\sup_{0 \leq t \leq t_1(\varepsilon)} \|(\phi, \Psi)\|_{L^\infty} \leq \frac{\nu}{8}, \quad (3.6a)$$

$$\sup_{0 \leq t \leq t_1(\varepsilon)} \|(\nabla \phi, \nabla \Psi)\| \leq \varepsilon^{b_1} |\ln \varepsilon|^{-1}, \quad (3.6b)$$

$$\sup_{0 \leq t \leq t_1(\varepsilon)} \|(\nabla^2 \phi, \nabla^2 \Psi)\| \leq \varepsilon^{b_2} |\ln \varepsilon|^{-1}, \quad (3.6c)$$

where $[0, t_1(\varepsilon)]$ is the time interval of existence of solutions and b_1, b_2 are positive constants to be determined. From a prior assumptions (3.6a) and (2.28), we can get $|\phi/\bar{\rho}| \leq 1/4$, and then $3/4 \leq \rho/\bar{\rho} = 1 + \phi/\bar{\rho} \leq 5/4$, therefore $\rho \sim \bar{\rho} \sim \bar{\rho}$ and $3/8\nu < \rho < 5/2\rho_+$. The bound of the density ρ with respect to ε guarantees that the global strong solution to (1.1) exists.

Note that the above a prior assumptions (3.6) with more accurate rates are crucial to control the nonlinear terms for justifying the zero viscosity limit. To prove Theorem 1.1, we need the following proposition.

Proposition 3.1 (Global existence and uniform estimates). *There exists a positive constant $\varepsilon_1 < 1$ such that if $0 < \varepsilon \leq \varepsilon_1$, then the perturbation system (3.4)-(3.5) has a unique global solution $(\phi, \Psi) \in \Pi(0, T)$ satisfying*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \phi, \bar{\rho}^{\frac{1}{2}} \Psi \right) (t, \cdot) \right\|^2 + \int_0^T \left[\left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi, \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_1 \right) \right\|^2 + \varepsilon \|\nabla \Psi\|^2 \right] dt_1 \\ & \leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \phi_0, \bar{\rho}_0^{\frac{1}{2}} \Psi_0 \right) \right\|^2 + \frac{C_T \varepsilon^4}{\nu^{\frac{7}{2}\alpha+3} \delta^7}, \\ & \sup_{0 \leq t \leq T} \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \nabla^i \phi, \bar{\rho}^{\frac{1}{2}} \nabla^i \Psi \right) (t, \cdot) \right\|^2 \\ & \quad + \int_0^T \left[\left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^i \phi \right\|^2 + \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} (\nabla^i \psi_1, \nabla^{i-1} \Psi_{x_1}) \right\|^2 + \varepsilon \|\nabla^{1+i} \Psi\|^2 \right] dt_1 \\ & \leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \nabla^i \phi_0, \bar{\rho}_0^{\frac{1}{2}} \nabla^i \Psi_0 \right) \right\|^2 + \frac{C_T \varepsilon^{4-i}}{\nu^{\frac{7}{2}\alpha+3} \delta^{7+i}}, \quad i = 1, 2 \end{aligned}$$

with $\alpha = \gamma - 1$ and $1 < \gamma < 19$ and the constant $C_T > 0$ is independent of ε, δ, ν , but may depend on T .

Once Proposition 3.1 is proved, we have

$$\|(\phi, \Psi)\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+s_0} \delta^{\frac{7}{2}}}, \quad (3.7a)$$

$$\|(\nabla \phi, \nabla \Psi)\| \leq C_T \frac{\varepsilon^{\frac{3}{2}}}{\nu^{\frac{7}{4}\alpha+s_0} \delta^4}, \quad (3.7b)$$

$$\|(\nabla^2 \phi, \nabla^2 \Psi)\| \leq C_T \frac{\varepsilon}{\nu^{\frac{7}{4}\alpha+s_0} \delta^{\frac{9}{2}}}, \quad (3.7c)$$

with

$$s_0 = \begin{cases} 2, & 1 < \gamma \leq 3, \\ \frac{3}{2} + \frac{\gamma-2}{2}, & \gamma > 3. \end{cases}$$

Note that $\nu = \varepsilon^{a_1} |\ln \varepsilon|$, $\delta = \varepsilon^{a_1}$, $\alpha = \gamma - 1$ (if $1 < \gamma < 19$) defined previously, in order to close the a priori assumptions (3.6b)-(3.6c) with (3.7b)-(3.7c) and control the nonlinear terms in (4.10) and (4.15), we obtain

$$\frac{3}{4} \leq b_1 \leq \frac{3}{2} - \left(\frac{7}{4} \alpha + 4 + s_0 \right) a_1, \quad (3.8)$$

$$\frac{1}{4} + \frac{s_1}{4} a_1 \leq b_2 \leq 1 - \left(\frac{7}{4} \alpha + \frac{9}{2} + s_0 \right) a_1 \quad (3.9)$$

with s_1 defined later in (4.15). From (3.8) and (3.9), it is necessary to establish that

$$\begin{aligned} \frac{3}{4} &\leq \frac{3}{2} - \left(\frac{7}{4} \alpha + 4 + s_0 \right) a_1, \\ \frac{1}{4} + \frac{s_1}{4} a_1 &\leq 1 - \left(\frac{7}{4} \alpha + \frac{9}{2} + s_0 \right) a_1, \end{aligned}$$

we observe that the first inequality can be derived from the second, then we obtain $a_1 \leq 3 / (7\gamma + 4s_0 + s_1 + 11)$, which means

$$a_1 \leq \begin{cases} \frac{1}{2\gamma+7}, & 1 < \gamma \leq 2, \\ \frac{3}{7\gamma+19}, & 2 < \gamma \leq 3, \\ \frac{3}{9\gamma+13}, & \gamma > 3. \end{cases} \quad (3.10)$$

To control the nonlinear terms in (4.9a) and (4.9b), it is crucial to hold that

$$-\frac{3}{5} + \frac{2}{5} b_1 + \frac{6}{5} b_2 - \frac{1}{5} s_2 \cdot a_1 \geq 0,$$

where

$$s_2 = \begin{cases} 4(2-\gamma), & 1 < \gamma \leq 2, \\ \gamma-2, & \gamma > 2. \end{cases}$$

Substitute the upper bound of b_1, b_2 from (3.8) and (3.9) into the above inequality, it follows that

$$a_1 \leq \frac{6}{14\gamma + 8s_0 + s_2 + 21},$$

which implies

$$a_1 \leq \begin{cases} \frac{6}{10\gamma+45}, & 1 < \gamma \leq 2, \\ \frac{6}{15\gamma+35}, & 2 < \gamma \leq 3, \\ \frac{6}{19\gamma+23}, & \gamma > 3. \end{cases} \quad (3.11)$$

Similarly, in order to control the terms in (4.17a) and (4.17b), it follows that

$$b_1 + b_2 - 1 - s_3 \cdot a_1 \leq 0,$$

where

$$s_3 = \begin{cases} 2(2-\gamma), & 1 < \gamma \leq 2, \\ \gamma-2, & \gamma > 2. \end{cases}$$

Substitute the upper bound of b_1, b_2 into the above inequality,

$$a_1 \leq \frac{3}{7\gamma+4s_0+2s_3+10},$$

which means

$$a_1 \leq \begin{cases} \frac{3}{3\gamma+26}, & 1 < \gamma \leq 2, \\ \frac{3}{9\gamma+14}, & 2 < \gamma \leq 3, \\ \frac{3}{11\gamma+8}, & \gamma > 3. \end{cases} \quad (3.12)$$

Therefore, combining (3.10)-(3.12), it is optimal to take a_1 defined by (1.15),

$$b_1 = \frac{3}{2} - \left(\frac{7}{4}\alpha + 4 + s_0 \right) a_1, \quad b_2 = 1 - \left(\frac{7}{4}\alpha + \frac{9}{2} + s_0 \right) a_1.$$

Finally, we check the rest of the a priori assumptions in (3.6), by Sobolev's inequality in 3D and (3.7), we have

$$\begin{aligned} \|(\phi, \Psi)\|_\infty &\leq C \left(\|(\phi, \Psi)\|^{\frac{1}{2}} \|(\nabla \phi, \nabla \Psi)\|^{\frac{1}{2}} + \|(\nabla \phi, \nabla \Psi)\|^{\frac{1}{2}} \|(\nabla^2 \phi, \nabla^2 \Psi)\|^{\frac{1}{2}} \right) \\ &\leq C_T \frac{\varepsilon^{\frac{5}{4}}}{\nu^{\frac{7}{4}\alpha+s_0} \delta^{\frac{17}{4}}} < \frac{\nu}{8}. \end{aligned}$$

Therefore, for any given positive constant h , there exists a constant $C_{h,T}$ which is independent of ε, ν, δ such that

$$\begin{aligned}
 \sup_{h \leq t \leq T} \|\rho - \rho^{r_2}\|_{L^\infty(\Omega)} &\leq \sup_{0 \leq t \leq T} \|\rho - \tilde{\rho}\|_{L^\infty(\Omega)} + \sup_{0 \leq t \leq T} \|\tilde{\rho} - \bar{\rho}\|_{L^\infty(\mathbb{R})} \\
 &\quad + \sup_{h \leq t \leq T} \|\bar{\rho} - \rho_{\nu}^{r_2}\|_{L^\infty(\mathbb{R})} + \sup_{h \leq t \leq T} \|\rho_{\nu}^{r_2} - \rho^{r_2}\|_{L^\infty(\mathbb{R})} \\
 &\leq C_T \frac{\varepsilon^{\frac{5}{4}}}{\nu^{\frac{7}{4}\alpha + s_0} \delta^{\frac{17}{4}}} + C_T \frac{\varepsilon}{\nu^{\frac{3}{4}\alpha} \delta^{\frac{3}{2}}} + Ch^{-1} \delta |\ln \delta| + C_{h,T} \nu, \\
 \sup_{h \leq t \leq T} \|m_1 - m_1^{r_2}\|_{L^\infty(\Omega)} &\leq \sup_{0 \leq t \leq T} \|m_1 - \tilde{m}_1\|_{L^\infty(\Omega)} + \sup_{0 \leq t \leq T} \|\tilde{m}_1 - \bar{m}_1\|_{L^\infty(\mathbb{R})} \\
 &\quad + \sup_{h \leq t \leq T} \|\bar{m}_1 - m_{1\nu}^{r_2}\|_{L^\infty(\mathbb{R})} + \sup_{h \leq t \leq T} \|m_{1\nu}^{r_2} - m_1^{r_2}\|_{L^\infty(\mathbb{R})} \\
 &\leq C \sup_{0 \leq t \leq T} \|\phi, \Psi\|_{L^\infty(\Omega)} + C \sup_{0 \leq t \leq T} \left\| \left(d_1, \frac{-d_1 \bar{u}_1 + d_2}{\bar{\rho}} \right) \right\|_{L^\infty(\mathbb{R})} \\
 &\quad + C \sup_{h \leq t \leq T} \|(\bar{\rho} - \rho_{\nu}^{r_2}, \bar{u}_1 - u_{1\nu}^{r_2})\|_{L^\infty(\mathbb{R})} + C_{h,T} \nu \\
 &\leq C_T \frac{\varepsilon^{\frac{5}{4}}}{\nu^{\frac{7}{4}\alpha + s_0} \delta^{\frac{17}{4}}} + C_T \frac{\varepsilon}{\nu^{\frac{3}{4}\alpha + 1} \delta^{\frac{3}{2}}} + Ch^{-1} \delta |\ln \delta| + C_{h,T} \nu \leq C_{h,T} \nu, \\
 \sup_{h \leq t \leq T} \|(m_2, m_3)\|_{L^\infty(\Omega)} &\leq C \sup_{0 \leq t \leq T} \|(\psi_2, \psi_3)\|_{L^\infty(\Omega)} \leq C_T \frac{\varepsilon^{\frac{5}{4}}}{\nu^{\frac{7}{4}\alpha + 2} \delta^{\frac{17}{4}}} \\
 &= C_T \varepsilon^{\frac{5}{4} - \frac{7\gamma + 18}{4} a_1} |\ln \varepsilon|^{-\frac{7\gamma + 1}{4}},
 \end{aligned}$$

if $1 < \gamma < 19$, which completes the proof of Theorem 1.1. $\square$

Now it remains to prove Proposition 3.1. Recalling the local existence and uniqueness of the classical solution to the hyperbolic-parabolic equations (3.4) with (3.5), which is standard and hence its proof is omitted, it is sufficient to obtain the following uniform a priori estimates.

Proposition 3.2 (A priori estimates). *Suppose that $(\phi, \Psi) \in \Pi(0, t_1(\varepsilon))$ is the solution to (3.4)-(3.5) for some $t_1(\varepsilon) \in (0, T)$, there exists a positive constant $\varepsilon_2 > 0$ which is independent of $\varepsilon, \delta, \nu, t_1(\varepsilon)$ such that $\forall 0 < \varepsilon \leq \varepsilon_2$, it holds that*

$$\begin{aligned}
 &\sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \phi, \bar{\rho}^{\frac{1}{2}} \Psi \right) (t, \cdot) \right\|^2 + \int_0^{t_1(\varepsilon)} \left[\left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi, \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_1 \right) \right\|^2 + \varepsilon \|\nabla \Psi\|^2 \right] dt_1 \\
 &\leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \phi_0, \bar{\rho}_0^{\frac{1}{2}} \Psi_0 \right) \right\|^2 + \frac{C_T \varepsilon^4}{\nu^{\frac{7}{2}\alpha + 3} \delta^7},
 \end{aligned}$$

$$\begin{aligned}
& \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \nabla^i \phi, \bar{\rho}^{\frac{1}{2}} \nabla^i \Psi \right) (t, \cdot) \right\|^2 \\
& + \int_0^{t_1(\varepsilon)} \left[\left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^i \phi \right\|^2 + \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} (\nabla^i \psi_1, \nabla^{i-1} \Psi_{x_1}) \right\|^2 + \varepsilon \|\nabla^{1+i} \Psi\|^2 \right] dt_1 \\
& \leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \nabla^i \phi_0, \bar{\rho}_0^{\frac{1}{2}} \nabla^i \Psi_0 \right) \right\|^2 + \frac{C_T \varepsilon^{4-i}}{\nu^{\frac{7}{2}\alpha+3} \delta^{7+i}}, \quad i=1,2
\end{aligned}$$

with $\alpha = \gamma - 1$ and $1 < \gamma < 19$, and the constant $C_T > 0$ is independent of ε, δ, ν , but may depend on T .

The next section is devoted to the proof of Proposition 3.2.

4 A prior estimate

We now give a proof of Proposition 3.2 for the uniform H^2 estimates. In Subsection 4.1, we derive the lower order L^2 -estimates in Lemma 4.1. Then in Subsections 4.2 and 4.3, the first-order derivative estimates and second-order derivative estimates are proved in Lemmas 4.2 and 4.3 respectively. Let $\Omega = \mathbb{R} \times \mathbb{T}^2, \alpha := \gamma - 1$.

4.1 Lower order estimates

Lemma 4.1. *Under the assumption of Proposition 3.2, there exists a positive constant C_T such that $\forall 0 \leq t \leq t_1(\varepsilon)$,*

$$\begin{aligned}
& \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \phi, \bar{\rho}^{\frac{1}{2}} \Psi \right) (t, \cdot) \right\|^2 \\
& + \int_0^{t_1(\varepsilon)} \left[\left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\|^2 + \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_1 \right\|^2 + \varepsilon \|\nabla \Psi\|^2 \right] dt \\
& \leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \phi_0, \bar{\rho}_0^{\frac{1}{2}} \Psi_0 \right) \right\|^2 + \frac{C_T \varepsilon^4}{\nu^{\frac{7}{2}\alpha+3} \delta^7}.
\end{aligned} \tag{4.1}$$

Proof. First, multiplying the Eq. (3.4b) by Ψ gives

$$\begin{aligned}
& \left(\rho \frac{|\Psi|^2}{2} \right)_t + \operatorname{div} \left(\rho u \frac{|\Psi|^2}{2} - \mu_1 \psi_i \nabla \psi_i + (\mu_1 + \lambda_1) \Psi \operatorname{div} \Psi \right) + \rho \bar{u}_{1x_1} \psi_1^2 \\
& + \mu_1 |\nabla \Psi|^2 + (\mu_1 + \lambda_1) |\operatorname{div} \Psi|^2 + p'(\rho) \nabla \phi \cdot \Psi + \left(p'(\rho) - \frac{\rho}{\bar{\rho}} p'(\bar{\rho}) \right) \bar{\rho}_{x_1} \psi_1
\end{aligned}$$

$$\begin{aligned}
&= (2\mu_1 + \lambda_1) \left(\frac{-d_1 \bar{u}_1 + d_2}{\bar{\rho}} \right)_{x_1 x_1} \psi_1 - \left(\bar{\rho} \bar{u}_1^2 - \bar{\rho} \bar{u}_1^2 + \bar{u}_1^2 d_1 - 2\bar{u}_1 d_2 \right)_{x_1} \psi_1 \\
&\quad - (p(\bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1} \psi_1 - (2\mu_1 + \lambda_1) \frac{\bar{u}_{1x_1 x_1} \phi \psi_1}{\bar{\rho}} \\
&\quad - \frac{1}{\bar{\rho}} (\bar{\rho} \bar{u}_1^2 - \bar{\rho} \bar{u}_1^2 + \bar{u}_1^2 d_1 - 2\bar{u}_1 d_2)_{x_1} \phi \psi_1 - \frac{1}{\bar{\rho}} (p(\bar{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1} \phi \psi_1. \quad (4.2)
\end{aligned}$$

Define potential energy by

$$\Phi(\rho, \bar{\rho}) := \int_{\bar{\rho}}^{\rho} \frac{p(s) - p(\bar{\rho})}{s^2} ds = \frac{1}{(\gamma-1)\rho} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi).$$

Direct computations yield

$$\begin{aligned}
&(\rho \Phi)_t + \operatorname{div}(\rho u \Phi + (p(\rho) - p(\bar{\rho})) \Psi) + \bar{u}_{1x_1} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi) \\
&\quad - p'(\rho) \nabla \phi \cdot \Psi - \left(p'(\rho) - \frac{\rho}{\bar{\rho}} p'(\bar{\rho}) \right) \bar{\rho}_{x_1} \phi_1 = 0. \quad (4.3)
\end{aligned}$$

Note that

$$\rho \psi_1^2 \bar{u}_{1x_1} = \rho \psi_1^2 \bar{u}_{1x_1} + \rho \psi_1^2 \left(\frac{-d_1 \bar{u}_1 + d_2}{\bar{\rho}} \right)_{x_1},$$

and due to $\|(d_1, d_2)\|_{L^\infty}, \|\phi\|_{L^\infty} \leq \nu/4$, we have $\bar{\rho} \sim \bar{\rho} \sim \rho$, then

$$\rho \Phi(\rho, \bar{\rho}) = \frac{1}{(\gamma-1)} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi) \geq C \bar{\rho}^{\gamma-2} \phi^2 \sim C \bar{\rho}^{\gamma-2} \phi^2,$$

and

$$\begin{aligned}
&\bar{u}_{1x_1} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi) \\
&= \bar{u}_{1x_1} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi) + \left(\frac{-d_1 \bar{u}_1 + d_2}{\bar{\rho}} \right)_{x_1} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi) \\
&\geq C \bar{\rho}^{\gamma-2} \bar{u}_{1x_1} \phi^2 + \left(\frac{-d_1 \bar{u}_1 + d_2}{\bar{\rho}} \right)_{x_1} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho})\phi).
\end{aligned}$$

Using the above relations and integrating (4.2) and (4.3) over $[0, t] \times \Omega$ yield that

$$\begin{aligned}
&\left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \phi, \bar{\rho}^{\frac{1}{2}} \Psi \right) (t, \cdot) \right\|^2 + \int_0^t \left(\left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\|^2 + \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \Psi \right\|^2 + \varepsilon \|\nabla \Psi\|^2 \right) dt \\
&\leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \phi_0, \bar{\rho}_0^{\frac{1}{2}} \Psi_0 \right) \right\|^2 + C \left| \int_0^t \iiint_{\Omega} \left[(2\mu_1 + \lambda_1) \left(\frac{-d_1 \bar{u}_1 + d_2}{\bar{\rho}} \right)_{x_1 x_1} \psi_1 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& - \left(\tilde{\rho} \tilde{u}_1^2 - \bar{\rho} \bar{u}_1^2 + \tilde{u}_1^2 d_1 - 2 \tilde{u}_1 d_2 \right)_{x_1} \psi_1 - (p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1} \psi_1 \\
& - (2\mu_1 + \lambda_1) \frac{\tilde{u}_{1x_1 x_1} \phi \psi_1}{\tilde{\rho}} - \frac{(\tilde{\rho} \tilde{u}_1^2 - \bar{\rho} \bar{u}_1^2 + \tilde{u}_1^2 d_1 - 2 \tilde{u}_1 d_2)_{x_1} \phi \psi_1}{\tilde{\rho}} \\
& - \frac{(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1} \phi \psi_1}{\tilde{\rho}} - \rho \psi_1^2 \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1} \\
& - \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1} (p(\rho) - p(\bar{\rho}) - p'(\bar{\rho}) \phi) \Big] dx_1 dx_2 dx_3 dt \Big| \\
& := C \left\| \left(\tilde{\rho}_0^{\frac{\gamma-2}{2}} \phi_0, \tilde{\rho}_0^{\frac{1}{2}} \Psi_0 \right) \right\|^2 + \left| \sum_{i=1}^8 I_i \right|. \tag{4.4}
\end{aligned}$$

For I_1 , by Lemmas 2.3 and 2.5, it holds that

$$\begin{aligned}
I_1 & \leq C \left| \int_0^t \iiint_{\Omega} (2\mu_1 + \lambda_1) \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \psi_1 dx_1 dx_2 dx_3 dt \right| \\
& \leq \frac{C\varepsilon}{\nu^{\frac{1}{2}}} \int_0^t \left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \right\| \cdot \left\| \tilde{\rho}^{\frac{1}{2}} \psi_1 \right\| dt \\
& \leq \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{1}{2}} \psi_1 \right\|^2 + \frac{C_T \varepsilon^2}{\nu} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \right\|^2 \\
& \leq \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{1}{2}} \psi_1 \right\|^2 + \frac{C_T \varepsilon^4}{\nu^{3\alpha+3} \delta^6},
\end{aligned}$$

where we use the facts that

$$\begin{aligned}
\left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} & = \frac{(-d_1 \bar{u}_1 + d_2)_{x_1 x_1}}{\tilde{\rho}} - 2 \frac{(-d_1 \bar{u}_1 + d_2)_{x_1} \tilde{\rho}_{x_1}}{\tilde{\rho}^2} \\
& \quad + (-d_1 \bar{u}_1 + d_2) \left(-\frac{\tilde{\rho}_{x_1 x_1}}{\tilde{\rho}^2} + 2 \frac{\tilde{\rho}_{x_1}^2}{\tilde{\rho}^3} \right),
\end{aligned}$$

and

$$\begin{aligned}
\left\| \frac{d_{2x_1 x_1}}{\tilde{\rho}} \right\| & \leq \frac{C}{\nu} \|d_{2x_1 x_1}\| \leq C_T \frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1} \delta^3}, \\
\left\| \frac{d_{1x_1} \bar{u}_{1x_1}}{\tilde{\rho}} \right\| & \leq \frac{C}{\nu} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_{1x_1}\| \leq C_T \frac{\varepsilon}{\nu^{\alpha+1} \delta^3},
\end{aligned}$$

$$\begin{aligned}
\left\| \frac{d_1 \bar{u}_{1x_1 x_1}}{\tilde{\rho}} \right\| &\leq \frac{C}{\nu} \|\bar{u}_{1x_1 x_1}\|_{L^\infty} \|d_1\| \leq C_T \frac{\varepsilon}{\nu^{\frac{\alpha}{2}+1} \delta^3}, \\
\left\| \frac{d_{2x_1} \tilde{\rho}_{x_1}}{\tilde{\rho}^2} \right\| &\leq \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_{2x_1}\| + \frac{C}{\nu^2} \|d_{1x_1}\|_{L^\infty} \|d_{2x_1}\| \leq C_T \frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1} \delta^3}, \\
\left\| \frac{d_1 \bar{u}_{1x_1} \tilde{\rho}_{x_1}}{\tilde{\rho}^2} \right\| &\leq \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|u_{1x_1}\|_{L^\infty}^2 \|d_1\| + \frac{C}{\nu^2} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_{1x_1}\|_{L^\infty} \|d_1\| \leq C_T \frac{\varepsilon}{\nu^{\alpha+1} \delta^3}, \\
\left\| \frac{d_2 \tilde{\rho}_{x_1}^2}{\tilde{\rho}^3} \right\| &\leq \frac{C}{\nu^{\alpha+1}} \|\bar{u}_{1x_1}\|_{L^\infty}^2 \|d_2\| + \frac{C}{\nu^3} \|d_{1x_1}\|_{L^\infty}^2 \|d_2\| \leq C_T \frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1} \delta^3}, \\
\left\| \frac{d_2 \tilde{\rho}_{x_1 x_1}}{\tilde{\rho}^2} \right\| &\leq \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1 x_1}\| \|d_1\| + \frac{C}{\nu^{\alpha+1}} \|\bar{u}_{1x_1}\|_{L^\infty}^2 \|d_2\| + \frac{C}{\nu^2} \|d_2\|_{L^\infty} \|d_{1x_1 x_1}\| \\
&\leq C_T \frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1} \delta^3},
\end{aligned}$$

therefore,

$$\left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \right\| \leq C_T \frac{\varepsilon}{\nu^{\frac{3}{2}\alpha+1} \delta^3}.$$

Similarly, we estimate the second term I_2 as

$$\begin{aligned}
I_2 &\leq C \left| \int_0^t \iiint_{\Omega} \left(\tilde{\rho} \bar{u}_1^2 - \tilde{\rho} \bar{u}_1^2 + \bar{u}_1^2 d_1 - 2 \bar{u}_1 d_2 \right)_{x_1} \psi_1 dx_1 dx_2 dx_3 dt \right| \\
&= C \left| \int_0^t \iiint_{\Omega} \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \psi_1 dx_1 dx_2 dx_3 dt \right| \\
&\leq \frac{C}{\nu^{\frac{1}{2}}} \int_0^t \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\| \cdot \|\tilde{\rho}^{\frac{1}{2}} \psi_1\| dt \\
&\leq \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \|\tilde{\rho}^{\frac{1}{2}} \psi_1\|^2 + \frac{C_T}{\nu} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\|^2 \\
&\leq \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \|\tilde{\rho}^{\frac{1}{2}} \psi_1\|^2 + \frac{C_T \varepsilon^4}{\nu^{\frac{7}{2}\alpha+3} \delta^7},
\end{aligned}$$

where we use the following facts that:

$$\left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} = \frac{(-d_1 \bar{u}_1 + d_2)_{x_1}^2}{\tilde{\rho}} - \frac{(-d_1 \bar{u}_1 + d_2)^2 \tilde{\rho}_{x_1}}{\tilde{\rho}^2},$$

$$\begin{aligned} \left\| \frac{d_2 d_{2x_1}}{\tilde{\rho}} \right\| &\leq \frac{C}{\nu} \|d_2\|_{L^\infty} \|d_{2x_1}\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\ \left\| \frac{d_1^2 \bar{u}_1 \bar{u}_{1x_1}}{\tilde{\rho}} \right\| &\leq \frac{C}{\nu} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty} \|d_1\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{5}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\ \left\| \frac{d_2^2 \tilde{\rho}_{x_1}}{\tilde{\rho}^2} \right\| &\leq \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_2\|_{L^\infty} \|d_2\| + \frac{C}{\nu^2} \|d_2\|_{L^\infty}^2 \|d_{2x_1}\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}, \end{aligned}$$

and so,

$$\left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}.$$

To estimate I_3 , it follows from Lemmas 2.3 and 2.5 that

$$\begin{aligned} I_3 &\leq C \left| \int_0^t \iiint_{\Omega} (p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1} \psi_1 dx_1 dx_2 dx_3 dt \right| \\ &\leq \frac{C}{\nu^{\frac{1}{2}}} \int_0^t \|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\| \cdot \|\rho^{\frac{1}{2}} \psi_1\| dt \\ &\leq \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \|\tilde{\rho}^{\frac{1}{2}} \psi_1\|^2 + \frac{C_T}{\nu} \sup_{0 \leq t \leq t_1(\varepsilon)} \|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\|^2 \\ &\leq \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \|\tilde{\rho}^{\frac{1}{2}} \psi_1\|^2 + \frac{C_T \varepsilon^4}{\nu^{\frac{7}{2}\alpha+3} \delta^7}, \end{aligned}$$

where we use the facts that

$$\begin{aligned} (p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1} &= (p'(\tilde{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})d_1)\tilde{\rho}_{x_1} + p''(\bar{\rho})d_1 d_{1x_1}, \\ \|(p'(\tilde{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})d_1)\tilde{\rho}_{x_1}\| \\ &\leq \frac{C}{\nu} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty} \|d_1\| + \|d_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty} \|d_1\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{5}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\ \|p''(\bar{\rho})d_1 d_{1x_1}\| &\leq \frac{C}{\nu} \|d_{1x_1}\|_{L^\infty} \|d_1\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}, \end{aligned}$$

and thus,

$$\|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\| \leq C_T \frac{\varepsilon^2}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}.$$

For I_4 , it holds from the Young's inequality that

$$I_4 \leq C \left| \int_0^t \iiint_{\Omega} (2\mu_1 + \lambda_1) \frac{\bar{u}_{1x_1 x_1} \phi \psi_1}{\tilde{\rho}} dx_1 dx_2 dx_3 dt \right|$$

$$\begin{aligned}
&\leq \frac{C\varepsilon}{\nu^{\frac{\alpha}{2}+1}\delta} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_1 \right\| dt \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\|^2 dt + \frac{\varepsilon^2}{\nu^{\alpha+2}\delta^2} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_1 \right\|^2 dt.
\end{aligned}$$

Due to $\|\phi\|_{L^\infty} \leq \nu/4 \leq \tilde{\rho}$ and the estimate of I_2, I_3 , then

$$|I_5| + |I_6| \leq C_T \frac{\varepsilon^4}{\nu^{\frac{7}{2}\alpha+3}\delta^7}.$$

For I_7, I_8 , it follows from Lemmas 2.3 and 2.5 that

$$\begin{aligned}
|I_7| &= C \left| \int_0^t \iiint_{\Omega} \rho \psi_1^2 \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1} dx_1 dx_2 dx_3 dt \right| \\
&\leq C_T \left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1} \right\|_{L^\infty} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \psi_1 \right\|^2 \\
&\leq \frac{C_T \varepsilon}{\nu^{\frac{5}{4}\alpha+1}\delta^{\frac{5}{2}}} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \psi_1 \right\|^2, \\
|I_8| &= C \left| \int_0^t \iiint_{\Omega} \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1} (p(\rho) - p(\tilde{\rho}) - p'(\tilde{\rho})\phi) dx_1 dx_2 dx_3 dt \right| \\
&\leq C_T \left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1} \right\|_{L^\infty} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\|^2 \\
&\leq \frac{C_T \varepsilon}{\nu^{\frac{5}{4}\alpha+1}\delta^{\frac{5}{2}}} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\|^2,
\end{aligned}$$

where we use the facts that

$$\left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1} = \frac{(-d_1 \bar{u}_1 + d_2)_{x_1}}{\tilde{\rho}} - \frac{(-d_1 \bar{u}_1 + d_2) \tilde{\rho}_{x_1}}{\tilde{\rho}^2},$$

and

$$\begin{aligned}
\left\| \frac{d_{2x_1}}{\tilde{\rho}} \right\|_{L^\infty} &\leq \frac{C}{\nu} \|d_{2x_1}\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{5}{4}\alpha+1}\delta^{\frac{5}{2}}}, \\
\left\| \frac{d_1 \bar{u}_{1x_1}}{\tilde{\rho}} \right\|_{L^\infty} &\leq \frac{C}{\nu} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{3}{4}\alpha+1}\delta^{\frac{5}{2}}}, \\
\left\| \frac{d_2 \tilde{\rho}_{x_1}}{\tilde{\rho}^2} \right\|_{L^\infty} &\leq \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_2\|_{L^\infty} + \frac{C}{\nu^2} \|d_{1x_1}\|_{L^\infty} \|d_2\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{5}{4}\alpha+1}\delta^{\frac{5}{2}}},
\end{aligned}$$

thus,

$$\left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\bar{\rho}} \right)_{x_1} \right\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{5}{4}\alpha+1} \delta^{\frac{5}{2}}}.$$

Substituting these estimates into (4.4) and using the inequality (3.9), then all the nonlinear terms can be controlled, thus the current lemma is proved. Note that the inviscid decay rate depends on the estimate of the nonlinear advection flux error terms I_2 and I_3 . $\square$

4.2 The first-order estimates

Lemma 4.2. *Under the assumption of Proposition 3.2, there exists a positive constant C_T such that $\forall 0 \leq t \leq t_1(\varepsilon)$,*

$$\begin{aligned} & \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi, \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right) (t, \cdot) \right\|^2 \\ & + \int_0^{t_1(\varepsilon)} \left[\left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1} \nabla \phi \right\|^2 + \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1} (\Psi_{x_1}, \nabla \psi_1) \right\|^2 + \varepsilon \|\nabla^2 \Psi\|^2 \right] dt \\ & \leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \nabla \phi_0, \bar{\rho}_0^{\frac{1}{2}} \nabla \Psi_0 \right) \right\|^2 + \frac{C_T \varepsilon^3}{\nu^{\frac{7}{2}\alpha+3} \delta^8}. \end{aligned} \quad (4.5)$$

Proof. Applying the operator ∇ to the Eq. (3.4a) and multiplying the resulting equation by $p'(\rho)/\rho \nabla \phi$ lead to

$$\begin{aligned} & \left(\rho^{\gamma-2} \frac{|\nabla \phi|^2}{2} \right)_t + \operatorname{div} \left(\rho^{\gamma-2} u \frac{|\nabla \phi|^2}{2} - \rho^{\gamma-1} \nabla \psi_i \phi_{x_i} \right) + (p'(\rho) \nabla \psi_i \nabla \phi)_{x_i} \\ & + \frac{\gamma-1}{2} \rho^{\gamma-2} \tilde{u}_{1x_1} |\nabla \phi|^2 + \rho^{\gamma-2} \tilde{u}_{1x_1} |\phi_{x_1}|^2 + p'(\rho) \nabla \phi \Delta \Psi \\ & = -\frac{\gamma-1}{2} \rho^{\gamma-2} |\nabla \phi|^2 \operatorname{div} \Psi - \rho^{\gamma-2} \tilde{\rho}_{x_1} \phi_{x_1} \operatorname{div} \Psi + (\gamma-2) \rho^{\gamma-2} \tilde{\rho}_{x_1} \nabla \phi \nabla \psi_1 \\ & - (\gamma-1) \rho^{\gamma-2} \tilde{\rho}_{x_1} \Psi_{x_1} \nabla \phi - \rho^{\gamma-2} \nabla \phi \nabla \psi_i \phi_{x_i} \\ & - \rho^{\gamma-2} \tilde{u}_{1x_1 x_1} \phi_{x_1} \phi - \rho^{\gamma-2} \tilde{\rho}_{x_1 x_1} \phi_{x_1} \psi_1 \\ & := J(t, x_1, x_2, x_3). \end{aligned} \quad (4.6)$$

Multiplying the Eq (3.4b) by $-\Delta \Psi$ yields

$$\begin{aligned} & \left(\rho \frac{|\nabla \Psi|^2}{2} \right)_t + \operatorname{div} \left(\rho u \frac{|\nabla \Psi|^2}{2} - \rho \psi_{jt} \nabla \psi_j - \rho u_i \psi_{jx_i} \nabla \psi_j + (\mu_1 + \lambda_1) \operatorname{div} \Psi \Delta \Psi \right. \\ & \quad \left. - (\mu_1 + \lambda_1) \operatorname{div} \Psi \nabla \operatorname{div} \Psi - \rho \tilde{u}_{1x_1} \psi_1 \nabla \psi_1 \right) \end{aligned}$$

$$\begin{aligned}
& + \mu_1 |\Delta \Psi|^2 + (\mu_1 + \lambda_1) |\nabla \operatorname{div} \Psi|^2 \rho \tilde{u}_{1x_1} |\Psi_{x_1}|^2 + \rho \tilde{u}_{1x_1} |\nabla \psi_1|^2 - p'(\rho) \nabla \phi \Delta \Psi \\
& = -\nabla \rho \nabla \psi_j \left(\psi_{jt} + u \nabla \psi_j + (\psi_1 \tilde{u}_{1x_1}, 0, 0)_j^T \right) - \rho \tilde{u}_{1x_1 x_1} \psi_{1x_1} \psi_1 - \rho \nabla \psi_j \nabla \psi_i \psi_{jx_i} \\
& \quad + \left(p'(\rho) - \frac{\rho}{\tilde{\rho}} p'(\tilde{\rho}) \right) \tilde{\rho}_{x_1} \Delta \psi_1 - (2\mu_1 + \lambda_1) \frac{\tilde{u}_{1x_1 x_1} \phi}{\tilde{\rho}} \Delta \psi_1 \\
& \quad + \frac{(\tilde{\rho} \tilde{u}_1^2 - \tilde{\rho} \tilde{u}_1^2 + \tilde{u}_1^2 d_1 - 2\tilde{u}_1 d_2)_{x_1}}{\tilde{\rho}} \rho \Delta \psi_1 + \frac{(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\tilde{\rho}) d_1)_{x_1}}{\tilde{\rho}} \rho \Delta \psi_1 \\
& \quad - (2\mu_1 + \lambda_1) \left(\frac{-d_1 \tilde{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \Delta \psi_1 \\
& := K(t, x_1, x_2, x_3),
\end{aligned} \tag{4.7}$$

where we have used

$$\begin{aligned}
& \nabla \rho \nabla \psi_1 \psi_1 \tilde{u}_{1x_1} + \rho \Delta \psi_1 \psi_1 \tilde{u}_{1x_1} \\
& = \operatorname{div}(\rho \nabla \psi_1) \psi_1 \tilde{u}_{1x_1} \\
& = \operatorname{div}(\rho \nabla \psi_1 \psi_1 \tilde{u}_{1x_1}) - \rho \tilde{u}_{1x_1} |\nabla \psi_1|^2 - \rho \tilde{u}_{1x_1 x_1} \psi_{1x_1} \psi_1.
\end{aligned}$$

Adding (4.6) and (4.7) together and then integrating the resulted equation over $[0, t] \times \Omega$ and using the elliptic estimate $\|\Delta \Psi\| \sim \|\nabla^2 \Psi\|$, $\|\nabla \operatorname{div} \Psi\| \sim \|\nabla^2 \Psi\|$, we can get

$$\begin{aligned}
& \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi, \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right) (t, \cdot) \right\|^2 \\
& \quad + \int_0^t \left[\left\| \bar{\rho}^{\frac{\gamma-2}{2}} \tilde{u}_{1x_1} \nabla \phi \right\|^2 + \left\| \bar{\rho}^{\frac{1}{2}} \tilde{u}_{1x_1} (\Psi_{x_1}, \nabla \psi_1) \right\|^2 + \varepsilon \|\nabla^2 \Psi\|^2 \right] dt \\
& \leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \nabla \phi_0, \bar{\rho}_0^{\frac{1}{2}} \nabla \Psi_0 \right) \right\|^2 \\
& \quad + C \left| \int_0^t \iiint_{\Omega} [J(t, x_1, x_2, x_3) + K(t, x_1, x_2, x_3)] dx_1 dx_2 dx_3 dt \right|.
\end{aligned} \tag{4.8}$$

By Lemmas 2.3, 2.5 and a prior assumptions (3.6), we first estimate $J(t, x_1, x_2, x_3)$ as follows:

If $1 < \gamma \leq 2$,

$$\begin{aligned}
& \left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} |\nabla \phi|^2 \operatorname{div} \Psi dx_1 dx_2 dx_3 dt \right| \\
& \leq \frac{C}{\nu^{\frac{2-\gamma}{2}}} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\| \cdot \|\nabla \phi\|_{L^4} \|\nabla \Psi\|_{L^4} dt
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{C}{\nu^{\frac{2-\gamma}{2}}} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\| \cdot \left\| \nabla \phi \right\|^{\frac{1}{4}} \left\| \nabla \phi \right\|^{\frac{3}{4}} \left\| \nabla \Psi \right\|^{\frac{1}{4}} \left\| \nabla \Psi \right\|^{\frac{3}{4}} dt \\
&\leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla \Psi \right\|_1^2 dt + \varepsilon^{-\frac{3}{5}} \nu^{-\frac{4}{5}(2-\gamma)} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2 \left\| \nabla \phi \right\|^{\frac{6}{5}} \left\| \nabla \Psi \right\|^{\frac{2}{5}} dt \\
&\leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla \Psi \right\|_1^2 dt + C_T \varepsilon^{-\frac{3}{5} + \frac{2}{5}b_1 + \frac{6}{5}b_2} \nu^{-\frac{4}{5}(2-\gamma)} |\ln \varepsilon|^{-\frac{8}{5}} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2. \quad (4.9a)
\end{aligned}$$

If $\gamma > 2$,

$$\begin{aligned}
&\left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} |\nabla \phi|^2 \operatorname{div} \Psi dx_1 dx_2 dx_3 dt \right| \\
&\leq C \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\| \cdot \left\| \nabla \phi \right\|_{L^4} \left\| \nabla \Psi \right\|_{L^4} dt \\
&\leq C \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\| \cdot \left\| \nabla \phi \right\|^{\frac{1}{4}} \left\| \nabla \phi \right\|^{\frac{3}{4}} \left\| \nabla \Psi \right\|^{\frac{1}{4}} \left\| \nabla \Psi \right\|^{\frac{3}{4}} dt \\
&\leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla \Psi \right\|_1^2 dt + \varepsilon^{-\frac{3}{5}} \nu^{-\frac{1}{5}(\gamma-2)} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2 \left\| \nabla \phi \right\|^{\frac{6}{5}} \left\| \nabla \Psi \right\|^{\frac{2}{5}} dt \\
&\leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla \Psi \right\|_1^2 dt + C_T \varepsilon^{-\frac{3}{5} + \frac{2}{5}b_1 + \frac{6}{5}b_2} \nu^{-\frac{1}{5}(\gamma-2)} |\ln \varepsilon|^{-\frac{8}{5}} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2. \quad (4.9b)
\end{aligned}$$

Using the Hölder's inequality, the Young's inequality and Lemma 4.1, we have

$$\begin{aligned}
&\left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \tilde{\rho}_{x_1} \phi_{x_1} \operatorname{div} \Psi dx_1 dx_2 dx_3 dt \right| \\
&\leq \int_0^t \left\| \bar{u}_{1x_1} \right\|_{L^\infty}^{\frac{1}{2}} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1} \right\| \cdot \left\| \nabla \Psi \right\| dt + \int_0^t \left\| d_{1x_1} \right\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1} \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right\| dt \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1} \right\|^2 dt + \frac{1}{\delta} \int_0^t \left\| \nabla \Psi \right\|^2 dt + \frac{C_T \varepsilon^2}{\nu^{\frac{5}{2}\alpha} \delta^5} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right\|^2 \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1} \right\|^2 dt + \frac{C_T \varepsilon^2}{\nu^{\frac{5}{2}\alpha} \delta^5} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right\|^2 + C_T \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+3} \delta^8}
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \tilde{u}_{1x_1 x_1} \phi_{x_1} \phi dx_1 dx_2 dx_3 dt \right| \\
&\leq \frac{C}{\delta} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1} \right\| \cdot \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_1 \right\| dt
\end{aligned}$$

$$\begin{aligned}
& + \int_0^t \left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \right\|_{L^\infty} \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \phi_{x_1} \right\| \cdot \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \phi \right\| dt \\
& \leq \frac{1}{160} \int_0^t \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\|^2 dt + \frac{C}{\delta^2} \int_0^t \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2 \\
& \quad + C_T \left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \right\|_{L^\infty}^2 \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \phi \right\|^2 \\
& \leq \frac{1}{160} \int_0^t \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2 + C_T \frac{\varepsilon}{\delta} \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+3} \delta^8} \\
& \quad + C_T \frac{\varepsilon^2}{\nu^{\frac{7}{2}\alpha+2} \delta^7} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \phi \right\|^2,
\end{aligned}$$

where we employ the facts that

$$\begin{aligned}
\left\| \frac{d_{2x_1 x_1}}{\tilde{\rho}} \right\|_{L^\infty} & \leq \frac{C}{\nu} \|d_{2x_1 x_1}\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\
\left\| \frac{d_{1x_1} \bar{u}_{1x_1}}{\tilde{\rho}} \right\|_{L^\infty} & \leq \frac{C}{\nu} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_{1x_1}\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{5}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\
\left\| \frac{d_1 \bar{u}_{1x_1 x_1}}{\tilde{\rho}} \right\|_{L^\infty} & \leq \frac{C}{\nu} \|\bar{u}_{1x_1 x_1}\|_{L^\infty} \|d_1\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{3}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\
\left\| \frac{d_{2x_1} \tilde{\rho}^{x_1}}{\tilde{\rho}^2} \right\|_{L^\infty} & \leq \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_{2x_1}\|_{L^\infty} + \frac{C}{\nu^2} \|d_{1x_1}\|_{L^\infty} \|d_{2x_1}\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\
\left\| \frac{d_1 \bar{u}_{1x_1} \tilde{\rho}_{x_1}}{\tilde{\rho}^2} \right\|_{L^\infty} & \leq C_T \frac{1}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1}\|_{L^\infty}^2 \|d_1\|_{L^\infty} + \frac{C}{\nu^2} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty} \\
& \leq C_T \frac{\varepsilon}{\nu^{\frac{5}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\
\left\| \frac{d_2 \tilde{\rho}_{x_1 x_1}}{\tilde{\rho}^2} \right\|_{L^\infty} & \leq C_T \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1 x_1}\|_{L^\infty} \|d_2\|_{L^\infty} + \frac{C}{\nu^{\alpha+1}} \|\bar{u}_{1x_1}\|_{L^\infty}^2 \|d_2\|_{L^\infty} \\
& \quad + \frac{C}{\nu^2} \|d_{1x_1 x_1}\|_{L^\infty} \|d_2\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}, \\
\left\| \frac{d_2 \tilde{\rho}_{x_1}^2}{\tilde{\rho}^3} \right\|_{L^\infty} & \leq C_T \frac{2}{\nu^{\alpha+1}} \|\bar{u}_{1x_1}\|_{L^\infty}^2 \|d_2\|_{L^\infty} + \frac{C}{\nu^3} \|d_{1x_1}\|_{L^\infty}^2 \|d_2\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}},
\end{aligned}$$

therefore,

$$\left\| \left(\frac{-d_1 \bar{u}_1 + d_2}{\tilde{\rho}} \right)_{x_1 x_1} \right\|_{L^\infty} \leq C_T \frac{\varepsilon}{\nu^{\frac{7}{4}\alpha+1} \delta^{\frac{7}{2}}}.$$

The other terms of $J(t, x_1, x_2, x_3)$ can be estimated similarly and the details will be omitted for brevity. Now we estimate $K(t, x_1, x_2, x_3)$. By Lemma 2.5 and a priori assumptions (3.6), we have

$$\begin{aligned}
 & \left| \int_0^t \iiint_{\Omega} \rho \nabla \psi_j \nabla \psi_i \psi_{j x_i} dx_1 dx_2 dx_3 dt \right| \\
 & \leq C \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \nabla \psi \right\| \cdot \left\| \nabla \psi \right\|_{L^4}^2 dt \\
 & \leq C \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \nabla \psi \right\| \cdot \left\| \nabla \psi \right\|^{\frac{1}{2}} \cdot \left\| \nabla \psi \right\|_1^{\frac{3}{2}} dt \\
 & \leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla \psi \right\|_1^2 dt + C_T \varepsilon^{-3} \left\| \nabla \psi \right\|^4 \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \psi \right\|^2 \\
 & \leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla \psi \right\|_1^2 dt + C_T \varepsilon^{-3+4b_1} |\ln \varepsilon|^{-4} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \psi \right\|^2. \quad (4.10)
 \end{aligned}$$

Using the Hölder's inequality, the Young's inequality and Lemma 4.1, we can obtain

$$\begin{aligned}
 & \left| \int_0^t \iiint_{\Omega} \left(p'(\rho) - \frac{\rho}{\bar{\rho}} p'(\bar{\rho}) \right) \bar{\rho}_{x_1} \Delta \psi_1 dx_1 dx_2 dx_3 dt \right| \\
 & \leq \frac{C}{\delta^{\frac{1}{2}}} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\| \cdot \left\| \nabla^2 \psi_1 \right\| dt + \frac{C}{\nu^{\frac{1}{2}}} \int_0^t \|d_{1x_1}\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\| \cdot \left\| \nabla^2 \psi_1 \right\| dt \\
 & \leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla^2 \psi_1 \right\|^2 dt + \frac{1}{\varepsilon \delta} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\|^2 dt + \frac{C_T}{\varepsilon \nu} \|d_{1x_1}\|_{L^\infty}^2 \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\|^2 \\
 & \leq \frac{\varepsilon}{160} \int_0^t \left\| \nabla^2 \psi_1 \right\|^2 dt + C_T \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+3} \delta^8} + C_T \frac{\varepsilon^2}{\nu^{\frac{5}{2}\alpha+1} \delta^4} \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+3} \delta^8}.
 \end{aligned}$$

Using the Young's inequality and the estimates similar as in the proof of Lemma 4.1, one has

$$\begin{aligned}
 & \left| \int_0^t \iiint_{\Omega} \left(\frac{p'(\rho)}{\rho} - \frac{p'(\bar{\rho})}{\bar{\rho}} \right) \bar{\rho}_{x_1} \nabla \rho \nabla \psi_1 dx_1 dx_2 dx_3 dt \right| \\
 & \leq \left| \int_0^t \iiint_{\Omega} (\rho^{\gamma-2} - \bar{\rho}^{\gamma-2}) \bar{\rho}_{x_1}^2 \psi_{1x_1} dx_1 dx_2 dx_3 dt \right| \\
 & \quad + \left| \int_0^t \iiint_{\Omega} (\rho^{\gamma-2} - \bar{\rho}^{\gamma-2}) \bar{\rho}_{x_1} \nabla \phi \nabla \psi_1 dx_1 dx_2 dx_3 dt \right| \\
 & \leq \frac{C}{\nu^{\frac{\alpha}{2}} \delta} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1} \right\| dt
 \end{aligned}$$

$$\begin{aligned}
& + \frac{C}{\nu^2} \int_0^t \|d_{1x_1}\|_{L^\infty}^2 \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \psi_{1x_1} \right\| dt \\
& + \frac{1}{\nu} \int_0^t \|\phi\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \psi_1 \right\| dt \\
& + \frac{1}{\nu^2} \int_0^t \|\phi\|_{L^\infty} \|d_{1x_1}\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \nabla \psi_1 \right\| dt \\
& \leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1} \right\|^2 dt + \frac{1}{\nu^\alpha \delta^2} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\|^2 dt \\
& \quad + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \psi_{1x_1} \right\|^2 + C_T \frac{\|d_{1x_1}\|_{L^\infty}^4}{\nu^4} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\| \\
& \quad + \frac{\|\phi\|_{L^\infty}^2}{\nu^2} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\|^2 dt + C_T \frac{\|\phi\|_{L^\infty}^2 \|d_{1x_1}\|_{L^\infty}^2}{\nu^4} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \psi_1 \right\|^2 \\
& \leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1} \right\|^2 dt + \frac{\varepsilon}{\nu^\alpha \delta} \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+3} \delta^8} + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \psi_{1x_1} \right\|^2 \\
& \quad + C_T \frac{\varepsilon^4}{\nu^{5\alpha+4} \delta^{10}} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi \right\|^2 + \frac{\|\phi\|_{L^\infty}^2}{\nu^2} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\|^2 dt \\
& \quad + C_T \frac{\|\phi\|_{L^\infty}^2}{\nu^2} \frac{\varepsilon^2}{\nu^{\frac{5}{2}\alpha+2} \delta^5} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \psi_1 \right\|^2.
\end{aligned}$$

By Hölder's inequality, Young's inequality and Lemma 4.1, then

$$\begin{aligned}
& \left| \int_0^t \iiint_{\Omega} \frac{(\bar{\rho} \bar{u}_1^2 - \bar{\rho} \bar{u}_1^2 + \bar{u}_1^2 d_1 - 2 \bar{u}_1 d_2)_{x_1}}{\bar{\rho}} \nabla \rho \nabla \psi_1 dx_1 dx_2 dx_3 dt \right| \\
& = \left| \int_0^t \iiint_{\Omega} \frac{1}{\bar{\rho}} \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\bar{\rho}} \right)_{x_1} \nabla \rho \nabla \psi_1 dx_1 dx_2 dx_3 dt \right| \\
& \leq \left| \int_0^t \iiint_{\Omega} \frac{1}{\bar{\rho}} \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\bar{\rho}} \right)_{x_1} \bar{\rho}_{x_1} \psi_{1x_1} dx_1 dx_2 dx_3 dt \right| \\
& \quad + \left| \int_0^t \iiint_{\Omega} \frac{1}{\bar{\rho}} \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\bar{\rho}} \right)_{x_1} \nabla \phi \nabla \psi_1 dx_1 dx_2 dx_3 dt \right| \\
& \leq \frac{C}{\nu^{\frac{\alpha+1}{2}} \delta^{\frac{1}{2}}} \int_0^t \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\bar{\rho}} \right)_{x_1} \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1} \right\| dt \\
& \quad + \frac{\|d_{1x_1}\|_{L^\infty}}{\nu^{\frac{3}{2}}} \int_0^t \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\bar{\rho}} \right)_{x_1} \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \psi_{1x_1} \right\| dt
\end{aligned}$$

$$\begin{aligned}
& + \frac{C}{\nu^{\frac{\alpha}{2}+1}} \int_0^t \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\|_{L^\infty} \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\| \cdot \left\| \tilde{\rho}^{\frac{1}{2}} \nabla \psi_1 \right\| dt \\
& \leq \frac{1}{160} \int_0^t \left\| \tilde{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1} \right\|^2 dt + \left(\frac{C_T}{\nu^{\alpha+1} \delta} + \frac{C_T \|d_{1x_1}\|_{L^\infty}^2}{\nu^3} \right) \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\|^2 \\
& \quad + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{1}{2}} \nabla \psi_1 \right\|^2 + \frac{C_T}{\nu^{\alpha+2}} \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\|_{L^\infty}^2 \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{\gamma-2}{2}} \nabla \psi_1 \right\|^2 \\
& \leq \frac{1}{160} \int_0^t \left\| \tilde{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1} \right\|^2 dt + C_T \left(\frac{\varepsilon}{\nu^\alpha} + \frac{\varepsilon^3}{\nu^{\frac{5}{2}\alpha+2} \delta^4} \right) \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+3} \delta^8} \\
& \quad + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \tilde{\rho}^{\frac{1}{2}} \nabla \psi_1 \right\|^2 + C_T \frac{\varepsilon^4}{\nu^{5\alpha+4} \delta^8} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\|^2,
\end{aligned}$$

where we utilize the facts that

$$\begin{aligned}
\left\| \frac{2d_2 d_{2x_1}}{\tilde{\rho}} \right\|_{L^\infty} & \leq \frac{C}{\nu} \|d_2\|_{L^\infty} \|d_{2x_1}\|_{L^\infty} \leq C_T \frac{\varepsilon^2}{\nu^{2\alpha+1} \delta^4}, \\
\left\| \frac{2d_1^2 \bar{u}_1 \bar{u}_{1x_1}}{\tilde{\rho}} \right\|_{L^\infty} & \leq \frac{C}{\nu} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty}^2 \leq C_T \frac{\varepsilon^2}{\nu^{\frac{3}{2}\alpha+1} \delta^4}, \\
\left\| \frac{d_2^2 \tilde{\rho}_{x_1}}{\tilde{\rho}^2} \right\|_{L^\infty} & \leq \frac{C}{\nu^{\frac{\alpha}{2}+1}} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_2\|_{L^\infty}^2 + \frac{C}{\nu^2} \|d_{1x_1}\|_{L^\infty} \|d_2\|_{L^\infty}^2 \leq C_T \frac{\varepsilon^2}{\nu^{2\alpha+1} \delta^4},
\end{aligned}$$

and therefore,

$$\left\| \left(\frac{(-d_1 \bar{u}_1 + d_2)^2}{\tilde{\rho}} \right)_{x_1} \right\|_{L^\infty}^2 \leq C_T \frac{\varepsilon^2}{\nu^{2\alpha+1} \delta^4}.$$

Similarly,

$$\begin{aligned}
& \left| \int_0^t \iiint_{\Omega} \frac{(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1}}{\tilde{\rho}} \nabla \rho \nabla \psi_1 dx_1 dx_2 dx_3 dt \right| \\
& \leq \left| \int_0^t \iiint_{\Omega} \frac{(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1}}{\tilde{\rho}} \tilde{\rho}_{x_1} \psi_{1x_1} dx_1 dx_2 dx_3 dt \right| \\
& \quad + \left| \int_0^t \iiint_{\Omega} \frac{(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1}}{\tilde{\rho}} \nabla \phi \nabla \psi_1 dx_1 dx_2 dx_3 dt \right| \\
& \leq \frac{C}{\nu^{\frac{\alpha+1}{2}} \delta^{\frac{1}{2}}} \int_0^t \left\| (p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho}) d_1)_{x_1} \right\| \cdot \left\| \tilde{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1} \right\| dt
\end{aligned}$$

$$\begin{aligned}
& + \frac{C}{\nu^{\frac{3}{2}}} \int_0^t \|d_{1x_1}\|_{L^\infty} \|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\| \cdot \|\bar{\rho}^{\frac{1}{2}}\psi_{1x_1}\| dt \\
& + \frac{C}{\nu^{\frac{\alpha}{2}+1}} \int_0^t \|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\|_{L^\infty} \|\bar{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\| \cdot \|\bar{\rho}^{\frac{1}{2}}\nabla\psi_1\| dt \\
& \leq \frac{1}{160} \int_0^t \|\bar{\rho}^{\frac{1}{2}}\bar{u}_{1x_1}^{\frac{1}{2}}\psi_{1x_1}\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \|\bar{\rho}^{\frac{1}{2}}\nabla\psi_1\|^2 \\
& \quad + \left(\frac{C_T}{\nu^{\alpha+1}\delta} + \frac{C_T\|d_{1x_1}\|_{L^\infty}^2}{\nu^3} \right) \sup_{0 \leq t \leq t_1(\varepsilon)} \|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\|^2 \\
& \quad + \frac{C_T}{\nu^{\alpha+2}} \|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\|_{L^\infty}^2 \sup_{0 \leq t \leq t_1(\varepsilon)} \|\bar{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2 \\
& \leq \frac{1}{160} \int_0^t \|\bar{\rho}^{\frac{1}{2}}\bar{u}_{1x_1}^{\frac{1}{2}}\psi_{1x_1}\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \|\bar{\rho}^{\frac{1}{2}}\nabla\psi_1\|^2 \\
& \quad + C_T \left(\frac{\varepsilon}{\nu^\alpha} + \frac{\varepsilon^3}{\nu^{\frac{5}{2}\alpha+2}\delta^4} \right) \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+3}\delta^8} + C_T \frac{\varepsilon^4}{\nu^{5\alpha+4}\delta^8} \sup_{0 \leq t \leq t_1(\varepsilon)} \|\bar{\rho}^{\frac{\gamma-2}{2}}\nabla\phi\|^2,
\end{aligned}$$

where we use the facts that

$$\begin{aligned}
& \|(p'(\tilde{\rho}) - p'(\bar{\rho}) - p''(\bar{\rho})d_1)\tilde{\rho}_{x_1}\|_{L^\infty} \\
& \leq \frac{C}{\nu} \|\bar{u}_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty}^2 + \frac{C}{\nu^2} \|d_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty}^2 \leq C_T \frac{\varepsilon^2}{\nu^{\frac{3}{2}\alpha+1}\delta^4}, \\
& \|p''(\bar{\rho})d_{1x_1}d_1\|_{L^\infty} \leq \frac{C}{\nu} \|d_{1x_1}\|_{L^\infty} \|d_1\|_{L^\infty} \leq C_T \frac{\varepsilon^2}{\nu^{2\alpha+1}\delta^4},
\end{aligned}$$

then

$$\|(p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1}\|_{L^\infty} \leq C_T \frac{\varepsilon^2}{\nu^{2\alpha+1}\delta^4}.$$

The other terms can be estimated similarly and the details will be omitted for brevity. Then Lemma 4.2 is obtained if we substitute the above estimates for $J(t, x_1, x_2, x_3)$ and $K(t, x_1, x_2, x_3)$ into (4.8) and use the inequality (3.9) with a_1 defined in (1.15). $\square$

4.3 The second-order estimates

Lemma 4.3. *Under the assumption of Proposition 3.2, there exists a positive constant C_T such that $\forall 0 \leq t \leq t_1(\varepsilon)$,*

$$\sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \left(\bar{\rho}^{\frac{\gamma-2}{2}} \nabla^2 \phi, \bar{\rho}^{\frac{1}{2}} \nabla^2 \Psi \right) (t, \cdot) \right\|^2$$

$$\begin{aligned}
& + \int_0^{t_1(\varepsilon)} \left[\left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1} \nabla^2 \phi \right\|^2 + \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1} (\nabla \Psi_{x_1}, \nabla^2 \psi_1) \right\|^2 + \varepsilon \left\| \nabla^3 \Psi \right\|^2 \right] dt \\
& \leq C \left\| \left(\bar{\rho}_0^{\frac{\gamma-2}{2}} \nabla^2 \phi_0, \bar{\rho}_0^{\frac{1}{2}} \nabla^2 \Psi_0 \right) \right\|^2 + \frac{C_T \varepsilon^2}{\nu^{\frac{7}{2} \alpha + 3} \delta^9}.
\end{aligned} \tag{4.11}$$

Proof. Applying the operator ∂_{ij} to the Eq. (3.4a) and multiplying the resulting equation by $\partial_{ij}\phi(p'(\rho)/\rho)$, we can obtain

$$\begin{aligned}
& \left(\bar{\rho}^{\gamma-2} \frac{|\nabla^2 \phi|^2}{2} \right)_t + \operatorname{div} \left(\rho^{\gamma-2} u \frac{|\nabla^2 \phi|^2}{2} + p'(\rho) \nabla^2 \phi \nabla^2 \Psi \right) - (p'(\rho) \partial_{ki} \phi \partial_{ij} \psi_k)_{x_j} \\
& + \frac{\gamma-1}{2} \rho^{\gamma-2} \bar{u}_{1x_1} |\nabla^2 \phi|^2 + 2\rho^{\gamma-2} \bar{u}_{1x_1} |\nabla \phi_{x_1}|^2 + p'(\rho) \nabla^2 \phi \nabla \Delta \Psi \\
& = -\frac{\gamma-1}{2} \rho^{\gamma-2} |\nabla^2 \phi|^2 \operatorname{div} \Psi - 2\rho^{\gamma-2} \partial_{ij} \phi \partial_i \phi \partial_j \operatorname{div} \Psi - 2\rho^{\gamma-2} \bar{\rho}_{x_1} \nabla \phi_{x_1} \nabla \operatorname{div} \Psi \\
& - \rho^{\gamma-2} \bar{\rho}_{x_1 x_1} \phi_{x_1 x_1} \operatorname{div} \Psi - 2\rho^{\gamma-2} \bar{u}_{x_1 x_1} \nabla \phi_{x_1} \nabla \phi - \rho^{\gamma-2} \bar{u}_{1x_1 x_1 x_1} \phi_{x_1 x_1} \phi \\
& - \rho^{\gamma-2} \partial_{ij} \phi \partial_{ij} \Psi \nabla \phi - 2\rho^{\gamma-2} \partial_{ij} \phi \partial_i \Psi \nabla \partial_j \phi - \rho^{\gamma-2} \bar{u}_{x_1 x_1} \phi_{x_1 x_1} \phi_{x_1} \\
& - \rho^{\gamma-2} (\bar{\rho}_{x_1} \partial_{ij} \phi \partial_{ij} \psi_1 + 2\bar{\rho}_{x_1 x_1} \nabla \phi_{x_1} \nabla \psi_1 + \bar{\rho}_{x_1 x_1 x_1} \phi_{x_1 x_1} \psi_1) \\
& + (\gamma-1) \rho^{\gamma-2} \partial_k \partial_{ij} \phi \partial_{ij} \psi_k + (\gamma-1) \rho^{\gamma-2} \bar{\rho}_{x_1} \partial_{ij} \phi \partial_{ij} \psi_1 \\
& - (\gamma-1) \rho^{\gamma-2} \partial_j \phi \partial_{ki} \phi \partial_{ij} \psi_k - (\gamma-1) \rho^{\gamma-2} \bar{\rho}_{x_1} \partial_{ki} \phi \partial_i \psi_{kx_1} \\
& := N(t, x_1, x_2, x_3).
\end{aligned} \tag{4.12}$$

Applying ∂_{ij} to (3.4b) and multiplying the resulted equation by $\rho \partial_{ij} \Psi$ yield

$$\begin{aligned}
& \left(\rho \frac{|\nabla^2 \Psi|^2}{2} \right)_t + \operatorname{div} \left(\rho u \frac{|\nabla^2 \Psi|^2}{2} + (\mu_1 + \lambda_1) \nabla \operatorname{div} \Psi \nabla \Delta \Psi \right. \\
& \quad \left. - (\mu_1 + \lambda_1) \nabla \operatorname{div} \Psi \nabla^2 \operatorname{div} \Psi \right) \\
& - \partial_j (\partial_i (\psi_1 \bar{u}_{1x_1}) \rho \partial_{ij} \psi_1) + \rho \bar{u}_{1x_1} |\nabla^2 \psi_1|^2 + 2\rho \bar{u}_{1x_1} |\nabla \Psi_{x_1}|^2 \\
& + \mu_1 |\nabla \Delta \Psi|^2 + (\mu_1 + \lambda_1) |\nabla^2 \operatorname{div} \Psi|^2 - p'(\rho) \nabla^2 \phi \nabla \Delta \Psi \\
& = -2\rho \partial_i \Psi \nabla \partial_j \Psi \partial_{ij} \Psi - \rho \partial_{ij} \Psi \nabla \Psi \partial_{ij} \Psi - \rho \bar{u}_{x_1 x_1} \Psi_{x_1} \Psi_{x_1 x_1} - 2\rho \bar{u}_{x_1 x_1} \partial_i \psi_1 \partial_i \psi_{1x_1} \\
& - \rho \bar{u}_{1x_1 x_1 x_1} \psi_{1x_1 x_1} \psi_1 + (\gamma-2) \rho^{\gamma-2} \partial_i \rho \nabla \phi \partial_i \Delta \Psi + \rho^{\gamma-2} \partial_i \nabla \phi \partial_j \rho \partial_{ij} \Psi \\
& + (\gamma-2) \rho^{\gamma-3} \partial_i \rho \partial_j \rho \nabla \phi \partial_{ij} \Psi - \left(\frac{\mu_1}{\rho} \partial_i \Delta \Psi + \frac{\mu_1 + \lambda_1}{\rho} \partial_i \nabla \operatorname{div} \Psi \right) \partial_j \rho \partial_{ij} \Psi \\
& + \left(\frac{\mu_1}{\rho^2} \partial_i \rho \Delta \Psi + \frac{\mu_1 + \lambda_1}{\rho^2} \partial_i \rho \nabla \operatorname{div} \Psi \right) (\partial_j \rho \partial_{ij} \Psi + \rho \partial_i \Delta \Psi)
\end{aligned}$$

$$\begin{aligned}
& +(\gamma-2)\rho^{\gamma-3}\partial_i\phi\tilde{\rho}_{x_1}(\partial_j\rho\partial_{ij}\psi_1+\rho\partial_i\Delta\psi_1)+(\gamma-2)(\rho^{\gamma-3}-\tilde{\rho}^{\gamma-3})\tilde{\rho}_{x_1}^2 \\
& \times(\partial_j\rho\partial_j\psi_{1x_1}+\rho\Delta\psi_{1x_1})+(\rho^{\gamma-2}-\tilde{\rho}^{\gamma-2})\tilde{\rho}_{x_1x_1}(\partial_j\rho\partial_j\psi_{1x_1}+\rho\Delta\psi_{1x_1}) \\
& +\left(\frac{(\tilde{\rho}\tilde{u}_1^2-\tilde{\rho}\tilde{u}_1^2+\tilde{u}_1^2d_1-2\tilde{u}_1d_2)_{x_1}}{\tilde{\rho}}\right)_{x_1}(\partial_j\rho\partial_j\psi_{1x_1}+\rho\Delta\psi_{1x_1}) \\
& +\left(\frac{(p(\tilde{\rho})-p(\bar{\rho})-p'(\bar{\rho})d_1)_{x_1}}{\tilde{\rho}}\right)_{x_1}(\partial_j\rho\partial_j\psi_{1x_1}+\rho\Delta\psi_{1x_1}) \\
& +(2\mu_1+\lambda_1)\partial_i\left(\frac{\tilde{u}_{x_1x_1}\phi}{\tilde{\rho}\rho}\right)(\partial_j\rho\partial_{ij}\psi_1+\rho\partial_i\Delta\psi_1) \\
& +(2\mu_1+\lambda_1)\partial_i\left(\frac{1}{\rho}\left(\frac{-d_1\tilde{u}_1+d_2}{\tilde{\rho}}\right)_{x_1x_1}\right)(\partial_j\rho\partial_{ij}\psi_1+\rho\partial_i\Delta\psi_1) \\
& :=M(t,x_1,x_2,x_3). \tag{4.13}
\end{aligned}$$

Integrating the above two equations over $[0,t] \times \Omega$ together and using the elliptic estimates $\|\nabla\Delta\Psi\| \sim \|\nabla^3\Psi\|$, $\|\nabla^2\operatorname{div}\Psi\| \sim \|\nabla^3\Psi\|$, we can get

$$\begin{aligned}
& \left\|\left(\bar{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi,\bar{\rho}^{\frac{1}{2}}\nabla^2\Psi\right)(t,\cdot)\right\|^2 \\
& +\int_0^t\left[\left\|\bar{\rho}^{\frac{\gamma-2}{2}}\bar{u}_{1x_1}^{\frac{1}{2}}\nabla^2\phi\right\|^2+\left\|\bar{\rho}^{\frac{1}{2}}\bar{u}_{1x_1}^{\frac{1}{2}}(\nabla\Psi_{x_1},\nabla^2\psi_1)\right\|^2+\varepsilon\|\nabla^3\Psi\|^2\right]dt \\
& \leq C\left\|\left(\bar{\rho}_0^{\frac{\gamma-2}{2}}\nabla^2\phi_0,\bar{\rho}_0^{\frac{1}{2}}\nabla^2\Psi_0\right)\right\|^2 \\
& +C\left|\int_0^t\iiint_{\Omega}[N(t,x_1,x_2,x_3)+M(t,x_1,x_2,x_3)]dx_1dx_2dx_3dt\right|. \tag{4.14}
\end{aligned}$$

For $N(t,x_1,x_2,x_3)$, first we have

$$\begin{aligned}
& \left|\int_0^t\iiint_{\Omega}\rho^{\gamma-2}|\nabla^2\phi|^2\operatorname{div}\Psi dx_1dx_2dx_3dt\right| \\
& \leq\int_0^t\left\|\bar{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\right\|^2\cdot\|\nabla\Psi\|_{L^\infty}dt \\
& \leq\int_0^t\left\|\bar{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\right\|^2\left(\|\nabla\Psi\|^{\frac{1}{2}}\|\nabla^2\Psi\|^{\frac{1}{2}}+\|\nabla^2\Psi\|^{\frac{1}{2}}\|\nabla^3\Psi\|^{\frac{1}{2}}\right)dt \\
& \leq C_T\varepsilon^{\frac{1}{2}(b_1+b_2)}|\ln\varepsilon|^{-1}\sup_{0\leq t\leq t_1(\varepsilon)}\left\|\bar{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\right\|^2+\frac{\varepsilon}{160}\int_0^t\|\nabla^3\Psi\|^2dt \\
& \quad +\varepsilon^{-\frac{1}{3}}\int_0^t\left\|\bar{\rho}^{\frac{\gamma-2}{2}}\nabla^2\phi\right\|^{\frac{8}{3}}\|\nabla^2\Psi\|^{\frac{2}{3}}dt
\end{aligned}$$

$$\begin{aligned}
&\leq C_T \varepsilon^{\frac{1}{2}(b_1+b_2)} |\ln \varepsilon|^{-1} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla^2 \phi \right\|^2 + \frac{\varepsilon}{160} \int_0^t \|\nabla^3 \Psi\|^2 dt \\
&\quad + C_T \varepsilon^{-\frac{1}{3} + \frac{4}{3}b_2} \nu^{-\frac{s_1}{3}} |\ln \varepsilon|^{-\frac{4}{3}} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla^2 \phi \right\|^2,
\end{aligned} \tag{4.15}$$

where

$$s_1 = \begin{cases} 2-\gamma, & 1 < \gamma \leq 2, \\ 0, & \gamma > 2. \end{cases}$$

By Hölder's inequality, Young's inequality and Lemma 4.2, it follows that

$$\begin{aligned}
&\left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \tilde{\rho}_{x_1} \nabla \phi_{x_1} \nabla \operatorname{div} \Psi dx_1 dx_2 dx_3 dt \right| \\
&\leq \frac{C}{\delta^{\frac{1}{2}}} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^2 \phi \right\| \cdot \|\nabla^2 \Psi\| dt + \frac{C}{\nu^{\frac{1}{2}}} \int_0^t \|d_{1x_1}\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla^2 \phi \right\| \|\nabla^2 \Psi\| dt \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^2 \phi \right\|^2 dt + \frac{C}{\delta} \int_0^t \|\nabla^2 \Psi\|^2 dt \\
&\quad + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla^2 \phi \right\|^2 + C_T \frac{\|d_{1x_1}\|_{L^\infty}^2}{\nu} \int_0^t \|\nabla^2 \Psi\|^2 dt \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^2 \phi \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla^2 \phi \right\|^2 \\
&\quad + C_T \left(1 + \frac{\varepsilon^2}{\nu^{\frac{5}{2}\alpha+1} \delta^4} \right) \frac{\varepsilon^2}{\nu^{\frac{7}{2}\alpha+3} \delta^9}.
\end{aligned} \tag{4.16}$$

Similarly, one has

$$\begin{aligned}
&\left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \tilde{\rho}_{x_1 x_1} \phi_{x_1 x_1} \operatorname{div} \Psi dx_1 dx_2 dx_3 dt \right| \\
&\leq C \left(\frac{1}{\delta^{\frac{3}{2}}} + \frac{C}{\nu^{\frac{\alpha}{2}} \delta^{\frac{3}{2}}} \right) \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^2 \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right\| dt \\
&\quad + \frac{C}{\nu} \int_0^t \|d_{1x_1 x_1}\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi_{x_1 x_1} \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right\| dt \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^2 \phi \right\|^2 dt + \frac{C_T}{\nu^\alpha \delta^3} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right\|^2 \\
&\quad + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi_{x_1 x_1} \right\|^2 + C_T \frac{\|d_{1x_1 x_1}\|_{L^\infty}^2}{\nu^2} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla \Psi \right\|^2
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla^2 \phi \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi_{x_1 x_1} \right\|^2 \\ &\quad + C_T \left(\frac{\varepsilon}{\nu^\alpha \delta^2} + \frac{\varepsilon^3}{\nu^{\frac{7}{2}\alpha+2} \delta^6} \right) \frac{\varepsilon^2}{\nu^{\frac{7}{2}\alpha+3} \delta^9}. \end{aligned}$$

Also, it holds that

$$\begin{aligned} &\left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \tilde{\rho}_{x_1 x_1 x_1} \phi_{x_1 x_1} \psi_1 dx_1 dx_2 dx_3 dt \right| \\ &\leq C \left(\frac{1}{\nu^{\frac{\alpha}{2}} \delta^2} + \frac{1}{\delta^2} + \frac{1}{\nu^\alpha \delta^2} \right) \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1 x_1} \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_1 \right\| dt \\ &\quad + \frac{C}{\nu} \int_0^t \|d_{1x_1 x_1 x_1}\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi_{x_1 x_1} \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \psi_1 \right\| dt \\ &\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1 x_1} \right\|^2 dt + \frac{C}{\nu^{2\alpha} \delta^4} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_1 \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi_{x_1 x_1} \right\|^2 \\ &\quad + C_T \frac{1}{\nu^2} \|d_{1x_1 x_1 x_1}\|_{L^\infty}^2 \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \psi_1 \right\|^2 \\ &\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi_{x_1 x_1} \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \phi_{x_1 x_1} \right\|^2 \\ &\quad + C_T \left(\frac{\varepsilon^2}{\nu^{2\alpha} \delta^2} + \frac{\varepsilon^4}{\nu^{\frac{9}{2}\alpha+2} \delta^7} \right) \frac{\varepsilon^2}{\nu^{\frac{7}{2}\alpha+3} \delta^9}. \end{aligned}$$

The other terms of $N(t, x_1, x_2, x_3)$ can be estimated similarly. Next, we estimate $M(t, x_1, x_2, x_3)$ as follows:

If $1 < \gamma \leq 2$,

$$\begin{aligned} &\left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \partial_i \phi \nabla \phi \partial_i \Delta \Psi dx_1 dx_2 dx_3 dt \right| \\ &\leq \frac{C}{\nu^{2-\gamma}} \int_0^t \|\nabla \Delta \Psi\| \cdot \|\nabla \phi\|_{L^4}^2 dt \\ &\leq \frac{\varepsilon}{160} \int_0^t \|\nabla \Delta \Psi\|^2 dt + \frac{C}{\varepsilon \nu^{2(2-\gamma)}} \int_0^t \|\nabla \phi\| \cdot \|\nabla \phi\|_1^3 dt \\ &\leq \frac{\varepsilon}{160} \int_0^t \|\nabla \Delta \Psi\|^2 dt + C_T \varepsilon^{b_1+b_2-1} \nu^{-2(2-\gamma)} |\ln \varepsilon|^{-2} \\ &\quad \times \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} (\nabla \phi, \nabla^2 \phi) \right\|^2. \end{aligned} \tag{4.17a}$$

If $\gamma > 2$,

$$\begin{aligned}
 & \left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \partial_i \phi \nabla \phi \partial_i \Delta \Psi dx_1 dx_2 dx_3 dt \right| \\
 & \leq C \int_0^t \|\nabla \Delta \Psi\| \cdot \|\nabla \phi\|_{L^4}^2 dt \\
 & \leq \frac{\varepsilon}{160} \int_0^t \|\nabla \Delta \Psi\|^2 dt + \frac{C}{\varepsilon} \int_0^t \|\nabla \phi\| \cdot \|\nabla \phi\|_1^3 dt \\
 & \leq \frac{\varepsilon}{160} \int_0^t \|\nabla \Delta \Psi\|^2 dt + C_T \varepsilon^{b_1+b_2-1} \nu^{-(\gamma-2)} |\ln \varepsilon|^{-2} \\
 & \quad \times \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} (\nabla \phi, \nabla^2 \phi) \right\|^2.
 \end{aligned} \tag{4.17b}$$

By Lemma 4.2, it holds that

$$\begin{aligned}
 & \left| \int_0^t \iiint_{\Omega} \rho^{\gamma-2} \tilde{\rho}_{x_1} \nabla \phi \Delta \Psi_{x_1} dx_1 dx_2 dx_3 dt \right| \\
 & \leq \frac{C}{\delta^{\frac{1}{2}}} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\| \cdot \|\nabla \Delta \Psi\| dt \\
 & \quad + \frac{C}{\nu^{\frac{1}{2}}} \int_0^t \|d_{1x_1}\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\| \cdot \|\nabla \Delta \Psi\| dt \\
 & \leq \frac{\varepsilon}{160} \int_0^t \|\nabla \Delta \Psi\|^2 dt + \frac{1}{\varepsilon \delta} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\|^2 dt \\
 & \quad + C_T \frac{\|d_{1x_1}\|_{L^\infty}^2}{\nu} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2 \\
 & \leq \frac{\varepsilon}{160} \int_0^t \|\nabla \Delta \Psi\|^2 dt + C_T \frac{\varepsilon^2}{\nu^{\frac{7}{2}+3} \delta^9} + C_T \frac{\varepsilon^3}{\nu^{\frac{5}{2}\alpha+1} \delta^4} \frac{\varepsilon^2}{\nu^{\frac{7}{2}+3} \delta^9}.
 \end{aligned}$$

Using 3D Sobolev inequality $\|f\|_{L^6} \leq C\|f\|_1$, we obtain

$$\begin{aligned}
 & \left| \int_0^t \iiint_{\Omega} \rho^{\gamma-3} \partial_i \phi \partial_j \phi \nabla \phi \partial_{ij} \Psi dx_1 dx_2 dx_3 dt \right| \\
 & \leq \frac{C}{\nu^2} \int_0^t \|\nabla \phi\|_{L^6}^3 \|\nabla^2 \Psi\| dt \leq \frac{C}{\nu^2} \int_0^t \|\nabla \phi\|_1^3 \cdot \|\nabla^2 \Psi\| dt \\
 & \leq \frac{C}{\delta} \int_0^t \|\nabla^2 \Psi\|^2 dt + \frac{\delta}{\nu^4} \int_0^t \|\nabla \phi\|_1^6 dt \\
 & \leq C_T \frac{\varepsilon^2}{\nu^{\frac{7}{2}+3} \delta^9} + C_T \frac{\varepsilon^{4b_2} \delta}{\nu^{4+\alpha}} |\ln \varepsilon|^{-4} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} (\nabla \phi, \nabla^2 \phi) \right\|^2.
 \end{aligned}$$

By 3D Sobolev inequality $\|f\|_{L^4} \leq C\|f\|^{\frac{1}{4}}\|\nabla f\|_1^{\frac{3}{4}}$, we have

$$\begin{aligned}
& \left| \int_0^t \iiint_{\Omega} \rho^{\gamma-3} \partial_i \phi \tilde{\rho}_{x_1} \nabla \phi \partial_i \Psi_{x_1} dx_1 dx_2 dx_3 dt \right| \\
& \leq \frac{C}{\nu^{\frac{3}{2}} \delta^{\frac{1}{2}}} \int_0^t \|\nabla \phi\|_{L^4}^2 \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{x_1}^{\frac{1}{2}} \nabla \Psi_{x_1} \right\| dt + \frac{C}{\nu^{\frac{5}{2}}} \int_0^t \|d_{1x_1}\|_{L^\infty} \|\nabla \phi\|_{L^4}^2 \left\| \bar{\rho}^{\frac{1}{2}} \nabla^2 \Psi \right\| dt \\
& \leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{x_1}^{\frac{1}{2}} \nabla \Psi_{x_1} \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla^2 \Psi \right\|^2 \\
& \quad + C \left(\frac{1}{\nu^3 \delta} + \frac{\|d_{1x_1}\|_{L^\infty}^2}{\nu^5} \right) \int_0^t \|\nabla \phi\| \cdot \|\nabla \phi\|_1^3 dt \\
& \leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{x_1}^{\frac{1}{2}} \nabla \Psi_{x_1} \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla^2 \Psi \right\|^2 \\
& \quad + C_T \left(\frac{\varepsilon^{b_1+b_2}}{\nu^{\alpha+3} \delta} + \frac{\varepsilon^{2+b_1+b_2}}{\nu^{\frac{7}{2}\alpha+5} \delta^5} \right) \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \nabla^{\frac{\gamma-2}{2}} \phi \right\|^2.
\end{aligned}$$

By Lemma 4.2, we have

$$\begin{aligned}
& \left| \int_0^t \iiint_{\Omega} \rho^{\gamma-3} \tilde{\rho}_{x_1}^2 \nabla \phi \Psi_{x_1 x_1} dx_1 dx_2 dx_3 dt \right| \\
& \leq \frac{C}{\nu^{\frac{\alpha}{2}} \delta} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1} \Psi_{x_1 x_1} \right\| dt \\
& \quad + \frac{1}{\nu^{\frac{\alpha}{2}+2}} \int_0^t \|d_{1x_1}\|_{L^\infty} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \Psi_{x_1 x_1} \right\| dt \\
& \leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \Psi_{x_1 x_1} \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \Psi_{x_1 x_1} \right\|^2 \\
& \quad + \frac{C}{\nu^\alpha \delta^2} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \nabla \phi \right\|^2 dt + C_T \frac{1}{\nu^{\alpha+4}} \|d_{1x_1}\|_{L^\infty}^2 \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \nabla \phi \right\|^2 \\
& \leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \Psi_{x_1 x_1} \right\|^2 dt + \frac{1}{160} \sup_{0 \leq t \leq t_1(\varepsilon)} \left\| \bar{\rho}^{\frac{1}{2}} \Psi_{x_1 x_1} \right\|^2 \\
& \quad + C_T \left(\frac{\varepsilon}{\nu^\alpha \delta} + \frac{\varepsilon^5}{\nu^{6\alpha+4} \delta^9} \right) \frac{\varepsilon^2}{\nu^{\frac{7}{2}\alpha+3} \delta^9}.
\end{aligned}$$

By the Hölder's inequality and Lemma 4.1, we have

$$\left| \int_0^t \iiint_{\Omega} \tilde{\rho}^{\gamma-4} |\bar{\rho}_{x_1}|^2 \phi \partial_j \phi \partial_j \psi_{1x_1} dx_1 dx_2 dx_3 dt \right|$$

$$\begin{aligned}
&\leq \frac{C}{\nu^\alpha \delta^2} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\| \cdot \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1 x_1} \right\| dt \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1 x_1} \right\|^2 dt + \frac{C}{\nu^{2\alpha} \delta^4} \int_0^t \left\| \bar{\rho}^{\frac{\gamma-2}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \phi \right\|^2 dt \\
&\leq \frac{1}{160} \int_0^t \left\| \bar{\rho}^{\frac{1}{2}} \bar{u}_{1x_1}^{\frac{1}{2}} \psi_{1x_1 x_1} \right\|^2 dt + C_T \frac{\varepsilon^2}{\nu^{2\alpha} \delta^2} \frac{\varepsilon^2}{\nu^{\frac{7}{2}\alpha+3} \delta^9}.
\end{aligned}$$

Substituting the above estimates for $N(t, x_1, x_2, x_3), M(t, x_1, x_2, x_3)$ into (4.14), we proved Lemma 4.3. $\square$

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