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ALIKHANOV LINEARIZED GALERKIN FINITE ELEMENT METHODS FOR NONLINEAR TIME-FRACTIONAL SCHRÖDINGER EQUATIONS*

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Abstract

We present Alikhanov linearized Galerkin methods for solving the nonlinear time fractional Schrödinger equations. Unconditionally optimal estimates of the fully-discrete scheme are obtained by using the fractional time-spatial splitting argument. The convergence results indicate that the error estimates hold without any spatial-temporal stepsize restrictions. Numerical experiments are done to verify the theoretical results.

Mathematics subject classification: 65M60, 65M12, 65M15.
Key words: Fractional Grönwall type inequality, Nonlinear time-fractional Schrödinger equation, Error analysis.

1. Introduction

We consider the Alikhanov finite element method (FEM) for solving the following nonlinear time fractional Schrödinger equations (TFSEs) [29]:

$$\begin{cases} i^C D_t^\alpha u + \Delta u + f\big(|u|^2\big)u = 0, & (\mathbf{x},t) \in \Omega \times (0,T], \\ u(\mathbf{x},0) = u_0(\mathbf{x},t), & \mathbf{x} \in \Omega, \\ u(\mathbf{x},t) = 0, & (\mathbf{x},t) \in \partial\Omega \times [0,T], \end{cases} \tag{1.1}$$

where $i=\sqrt{-1}, \Omega \in \mathcal{R}^d$, $d=1,2,3$, and $f \in C^3(\mathcal{R})$ is a nonlinear function, $u(\mathbf{x},t)$ is a complex-valued function. Here ${}^C D_t^\alpha u$ denotes the Captuo fractional derivative, which is defined as

$$^C D_t^\alpha u(x,t) = \int_0^t w_{1-\alpha}(t-s)\frac{\partial u(x,s)}{\partial s} ds,$$

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where $w_\beta(t) = t^{\beta-1}/\Gamma(\beta)$ and $\Gamma(\cdot)$ is the common Gamma function. Fractional Schrödinger equations were investigated extensively. For example, Laskin [16, 17] proposed the fractional Schrödinger equations by using the Feynman path integrals instead of Lévy ones. In 2004, Naber [29] pointed out that one could obtain a time fractional Schrödinger equation when non-Markovian evolution was considered. In 2006, Xu and Guo [7] studied some physical evolutions of fractional Schrödinger equation. Tofighi [32] considered the probability structure of TFSEs. More details can be found in [1, 9, 18, 28, 30].

In the past several years, TFSEs were numerically investigated by using different algorithms, including finite difference methods [5, 11, 14, 15, 26, 33], finite element methods [24], spectral methods [3, 8, 35], local discontinuous Galerkin methods [34] and Krylov projection method [6] and so on [2, 10, 12]. Most convergence results were obtained with certain time-step restrictions dependent on the spatial mesh sizes. In order to remove such restrictions, Li *et al.* [21] introduced the fractional temporal-spatial splitting argument and obtained unconditionally optimal L^2-error estimates for problem (1.1). The time-discretization in the paper is done by L1 scheme. And the convergence order of the scheme is $2-\alpha$ in temporal direction if the exact solutions satisfy $u \in C^2([0, T]; L^2(\Omega))$. The regularity of the problems is not considered in the paper.

In this paper, we propose a linearized fully-discrete numerical scheme for solving problem (1.1), taking the initial singularities into account. The temporal discretization is done by applying the Alikhanov scheme on graded meshes and the extrapolation method. The spatial discretization is done by using the r-degree Galerkin FEM. It is shown that the convergence order in L^2-norm of the fully-discrete scheme can be of 2 in the temporal direction and of $r+1$ in the spatial direction. Such error estimates hold without any spatial-temporal stepsize restrictions. The key to the proof is the so-called temporal-spatial splitting argument, which is firstly proposed by Li and Sun [19, 20]. This technique has a successful application in the time-dependent problems [21–23, 37]. We introduce the approach in our proof and obtain the unconditionally convergent results for the time fractional problems in the complex spaces.

The rest of this paper is organized as follows. In Section 2, Alikhanov linearized Galerkin FEM is established for solving problem (1.1), and our main results are also presented. In Section 3, a rigorous analysis of our results is obtained by applying discrete fractional Grönwall type inequality. In Section 4, numerical examples are given to confirm our theoretical results. Finally, some conclusions are drawn in Section 5.

Throughout this paper, C_ν and C_f denote two positive constants, not always the same in different occasions, which dependent on the given information but independent of temporal and spatial stepsizes.

2. The Alikhanov Galerkin FEM

Following the standard FEM discretization [31], let $\mathcal{T}_h$ be a subdivision of Ω into triangles T_k in $\mathcal{R}^1, \mathcal{R}^2$ or tetrahedra in $\mathcal{R}^3$ and $h = \max_{T_K \in \mathcal{T}_h}\{\text{diam } T_K\}$ be the mesh size. The finite-dimensional subspace of $H_0^1(\Omega)$ is named V_h. It is comprised by continuous piecewise polynomial $\{\phi_j\}_{j=1}^M$ whose order is r $(r \geq 1)$ on $\mathcal{T}_h$. Let $\tau_n = t_{n+1} - t_n$ be time step. Denote $t_n = T(n/N)^\delta, 0 \leq n \leq N, \delta \geq 1$, $t_{n-\alpha/2} = (1-\alpha/2)t_n + (\alpha/2)t_{n-1}$, where N is a given integer and $u^m = u(x, t_m)$. For a set of functions $\{\omega^n\}$, we define

$$\begin{aligned} \omega^{n,\alpha} &= \left(1 - \frac{\alpha}{2}\right)\omega^n + \frac{\alpha}{2}\omega^{n-1}, && 1 \leq n \leq N, \\ \hat{\omega}^n &= \left(2 - \frac{\alpha}{2}\right)\omega^{n-1} - \left(1 - \frac{\alpha}{2}\right)\omega^{n-2}, && n \geq 2. \end{aligned} \tag{2.1}$$

With the above notations, the Caputo's fractional derivative at $t_{n-\alpha/2}$ is approximated by

$$\begin{aligned}(\partial_t^\alpha\phi)^{n-\frac{\alpha}{2}} &= \int_0^{t_{n-\frac{\alpha}{2}}} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)\phi'(s)ds\\ &= \sum_{l=1}^{n-1}\int_{t_{l-1}}^{t_l} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)\phi'(s)ds + \int_{t_{n-1}}^{t_{n-\frac{\alpha}{2}}} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)\phi'(s)ds\\ &\approx \sum_{l=1}^{n-1}\int_{t_{l-1}}^{t_l} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)(\Pi_{2,l}\phi)'(s)ds + \int_{t_{n-1}}^{t_{n-\frac{\alpha}{2}}} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)(\Pi_{1,n}\phi)'(s)ds,\end{aligned}$$

where $\Pi_{2,l}\phi$ signifies the quadratic interpolation at t_{l-1}, t_l and t_{l+1}, and $\Pi_{1,l}\phi$ means the linear interpolation with t_{l-1}, t_l. The Alikhanov formula is obtained by removing the truncation errors, i.e.,

$$\begin{aligned}(D_{\Delta t}^\alpha\phi)^{n-\frac{\alpha}{2}} &:= \sum_{l=1}^{n-1}\int_{t_{l-1}}^{t_l} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)\left[\frac{\nabla_\tau\phi^l}{\tau_l} + \frac{2(s-t_{l-\frac{1}{2}})}{\tau_l(\tau_l+\tau_{l+1})}\big(\rho_l\nabla_\tau\phi^{l+1}-\nabla_\tau\phi^l\big)\right]ds\\ &\quad + \int_{t_{n-1}}^{t_{n-\frac{\alpha}{2}}} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)\frac{\nabla_\tau\phi^n}{\tau_n}ds\\ &= \tilde{z}_0^{(n)}\nabla_\tau\phi^n + \sum_{l=1}^{n-1}\Big(\tilde{z}_{n-l}^{(n)}\nabla_\tau\phi^l + \rho_l\tilde{y}_{n-l}^{(n)}\nabla_\tau\phi^{l+1} - \tilde{y}_{n-l}^{(n)}\nabla_\tau\phi^l\Big)\\ &= B_0^{(n)}\nabla_\tau\phi^n + \sum_{l=1}^{n-1} B_{n-l}^{(n)}\nabla_\tau\phi^l,\end{aligned} \tag{2.2}$$

where the discrete coefficients $\tilde{z}_0^{(n)}$, $\tilde{z}_{n-l}^{(n)}$ and $\tilde{y}_{n-l}^{(n)}$ are

$$\begin{aligned}\tilde{z}_0^{(n)} &= \frac{1}{\tau_n}\int_{t_{n-1}}^{t_{n-\frac{\alpha}{2}}} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)ds,\\ \tilde{z}_{n-l}^{(n)} &= \frac{1}{\tau_l}\int_{t_{l-1}}^{t_l} w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)ds,\\ \tilde{y}_{n-l}^{(n)} &= \frac{2}{\tau_l(\tau_l+\tau_{l+1})}\int_{t_{l-1}}^{t_l}(s-t_{l-\frac{1}{2}})w_{1-\alpha}\big(t_{n-\frac{\alpha}{2}}-s\big)ds,\end{aligned}$$

and

$$B_{n-l}^{(n)} = \begin{cases}\tilde{z}_0^{(n)} + \rho_{n-1}\tilde{y}_1^{(n)}, & l=n,\\ \tilde{z}_{n-l}^{(n)} + \rho_{l-1}\tilde{y}_{n-l+1}^{(n)} - \tilde{y}_{n-l}^{(n)}, & 2\le l\le n-1,\\ \tilde{z}_{n-1}^{(n)} - \tilde{y}_{n-1}^{(n)}, & l=1.\end{cases}$$

Let U_h^n approximate u^n. Then Alikhanov linearized Galerkin FEM scheme is to find $U_h^n \in V_h$ such that, for all $v\in V_h$,

$$i\Big(D_{\Delta t}^\alpha U_h^{n-\frac{\alpha}{2}}, v\Big) - \big(\nabla U_h^{n,\alpha},\nabla v\big) + \Big(f(|\hat{U}_h^n|^2)U_h^{n,\alpha}, v\Big) = 0, \quad n=2,\ldots,N. \tag{2.3}$$

We take U_h^1 to be the solution of the following equation:

$$i\left(\frac{U_h^1-U_h^0}{\mu\tau_1^\alpha}, v\right) - \Big(\nabla U_h^{1,\alpha},\nabla v\Big) + \Big(f\big(|U_h^0|^2\big)U_h^{1,\alpha}, v\Big) = 0 \tag{2.4}$$

with $\mu = (1-\alpha/2)/(\Gamma(2-\alpha))$. As for the initial time step, let $U_h^0 = \Pi_h u^0$, where Π_h is the interpolation operator. From the property of Eq. (2.4), we can get that it satisfies $\|U_h^1\|_{L^2} \le \|U_h^0\|_{L^2}$.

The typical solution of problem (1.1) satisfies (e.g., see [13])

$$\big\|u_t^{(j)}\big\|_{L^\infty(0,T;H^{r+1})} \le C(1+t^{\alpha-j}), \quad j=0,1,2,3, \quad r=1,2, \tag{2.5}$$

where C is a constant.

Now, the optimal convergent result is presented here and it is proof is left in the next section.

Theorem 2.1. *Suppose that $u_0 \in H^{r+1}(\Omega)\bigcap H_0^1(\Omega)$ and problem (1.1) has a unique solution, satisfying (2.5). Then there exist positive constants N_0 and h_0, such that when $N \ge N_0$ and $h < h_0$, system (2.3), (2.4) has a unique solution $U_h^m, m=1,2,3,\ldots,N$, satisfying*

$$\big\|u^m - U_h^m\big\|_{L^2} \le C_0\big(N^{-\delta\alpha} + h^{r+1}\big), \tag{2.6}$$

where $1 \le \delta \le 2/\alpha, u^m = u(\cdot, t_m)$ and C_0 is a positive constant independent of τ and h.

3. Proof of the Main Results

In this section, we focus on proving our main results.

3.1. Preliminaries

Some features of $B_{n-k}^{(n)}$ are listed here, which are proved in [25, 26].

B1. The discrete kernels are non-increasing with respect to k becoming larger, i.e.,

$$0 < B_{k-1}^{(n)} \le B_{k-2}^{(n)}, \quad 2 \le k \le n \le N.$$

B2. There exists a constant $\pi_A = 11/4$, such that

$$\frac{1}{\pi_A \tau_k}\int_{t_{k-1}}^{t_k} w_{1-\alpha}(t_n - s)ds \le B_{n-k}^{(n)}$$

holds for $1 \le k \le n \le N$.

B3. There is a positive constant ρ satisfying $\rho_k \le \rho, 1 \le k \le N-1$, where $\rho_k := \tau_k/\tau_{k+1}$.

The following lemmas could be obtained with the help of these properties.

Lemma 3.1 ([27]). *Let*

$$P_0^{(n)} := \frac{1}{B_0^{(n)}}, \quad P_{n-j}^{(n)} := \frac{1}{B_0^{(j)}}\sum_{k=j+1}^{n}\Big(B_{k-j-1}^{(k)} - B_{k-j}^{(k)}\Big)P_{n-k}^{(n)}, \quad 1 \le j \le n-1. \tag{3.1}$$

Then, we have

$$\sum_{j=1}^{n} P_{n-j}^{(n)} w_{1-\alpha}(t_j) \le \pi_A, \quad 1 \le n \le N,$$

and

$$\sum_{j=1}^{n} P_{n-j}^{(n)} \le t_n^\alpha \pi_A \Gamma(2-\alpha). \tag{3.2}$$

Lemma 3.2 ([27]). *For any sequence* $\{v^n\}_{n=0}^N$, *it holds*

$$\frac{1}{2}\sum_{k=1}^{n} B_{n-k}^{(n)}\nabla_\tau\big(\|v^k\|^2\big) \le Re\big\langle v^{n,\alpha}, (D_{\Delta t}^{\alpha}v)^{n-\frac{\alpha}{2}}\big\rangle \quad for \quad 1\le n\le N. \tag{3.3}$$

Lemma 3.3 ([27]). *Suppose the nonnegative sequences* $\{v^n,\xi^n\}_{n=0}^N$ *satisfy*

$$\sum_{k=1}^{n} B_{n-k}^{(n)}\nabla_\tau(v^k)^2 \le \lambda_1(v^n)^2+\lambda_2(v^{n-1})^2+\lambda_3(v^{n-2})^2+v^{n,\alpha}(\xi^n+\eta), \quad n\ge 2. \tag{3.4}$$

Then, it holds

$$v^n \le 2E_\alpha\big(2\pi_A\lambda t_n^\alpha\big)\left[v^0+\max_{1\le k\le n}\sum_{j=1}^{k}P_{k-j}^{(k)}\xi^j+\pi_A\Gamma(2-\alpha)t_n^\alpha\eta\right], \tag{3.5}$$

where τ_n *satisfies* $\max_{1\le n\le N}\tau_n \le (2\pi_A\Gamma(2-\alpha)\lambda)^{-1/\alpha}, \lambda=\lambda_1+\lambda_2+\lambda_3,$ *and*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}$$

is the Mittag-Leffler function.

Lemma 3.4 ([25]). *Suppose that* $v\in C^3((0,T])$ *and there exists a constant* $C_v>0$ *such that*

$$|v'''(t)|\le C_v(1+t^{\alpha-3}) \quad for \quad 0\le t\le T.$$

Then, it holds that

$$\sum_{j=1}^{n}P_{n-j}^{(n)}\big|\Upsilon_1^j\big| \le C_v\big(T^\alpha N^{-\delta\alpha}+T^\alpha\delta^{3-\alpha}8^{\delta-1}N^{-\min\{\delta\alpha,3-\alpha\}}\big),$$

where

$$\Upsilon_1^n=\frac{1}{\Gamma(1-\alpha)}\int_0^{t_{n-\frac{\alpha}{2}}}\frac{v'(s)}{(t-s)^\alpha}ds-\big(D_{\Delta t}^\alpha v\big)^{n-\frac{\alpha}{2}}.$$

Lemma 3.5 ([26]). *Suppose that* $v\in C^2(0,T]$ *with*

$$\|v''(t)\|_{L^\infty(0,T;H^2)}\le C_\nu(1+t^{\alpha-2}).$$

Then, it holds that

$$\sum_{j=1}^{n}P_{n-j}^{(n)}|\Upsilon_2^{j,\alpha}|\le C_v\big(T^{2\alpha}N^{-\min\{\delta\alpha,2\}}\big), \quad 1\le n\le N, \tag{3.6}$$

where

$$\Upsilon_2^{n,\alpha}=\Delta v(t_{n-\alpha/2})-\Delta v^{n,\alpha}, \quad 1\le n\le N.$$

Lemma 3.6. *Suppose that* $\nu\in C^2((0,T])$ *satisfies* $|\nu'(t)|\le C_\nu(1+t^{\alpha-1})$ *and the nonlinear function* $g\in C^2(\mathcal{R})$. *Denote*

$$\nu^n=\nu(t_n), \quad \tilde{\nu}^{1,\alpha}=\Big(1-\frac{\alpha}{2}\Big)\tilde{\nu}^1+\frac{\alpha}{2}\nu^0,$$

and

$$R_{\nu}^{1}=g\big(|\nu(t_{1-\frac{\alpha}{2}})|^2\big)\nu(t_{1-\frac{\alpha}{2}})-g\big(|\nu^0|^2\big)\tilde{\nu}^{1,\alpha},$$
$$R_{\nu}^{j}=g\big(|\nu^{j-\frac{\alpha}{2}}|^2\big)\nu^{j-\frac{\alpha}{2}}-g\big(|\hat{\nu}^j|^2\big)\nu^{j,\alpha},\quad j=2,3,\dots,N.$$

Then

$$\sum_{j=1}^{n}P_{n-j}^{(n)}|R_{\nu}^{j}|\le 2C_{\nu}T^{2\alpha}N^{-\min\{\delta\alpha,2\}},\quad 1\le n\le N.$$

Proof. Applying Taylor expansion which has integral reminder, we can arrive at

$$\nu^{j-1}=\nu^{j-\frac{\alpha}{2}}+\nu'\big(t_{j-\frac{\alpha}{2}}\big)\big(t_{j-1}-t_{j-\frac{\alpha}{2}}\big)+\int_{t_{j-\frac{\alpha}{2}}}^{t_{j-1}}\nu''(s)(s-t_{j-1})ds,$$
$$\nu^{j-2}=\nu^{j-\frac{\alpha}{2}}+\nu'\big(t_{j-\frac{\alpha}{2}}\big)\big(t_{j-2}-t_{j-\frac{\alpha}{2}}\big)+\int_{t_{j-\frac{\alpha}{2}}}^{t_{j-2}}\nu''(s)(s-t_{j-2})ds.$$

Recalling the definition of $\hat{\omega}^j$, we can get

$$\begin{aligned}\nu^{j-\frac{\alpha}{2}}-\hat{\nu}^j&=\Big(2-\frac{\alpha}{2}\Big)\big(\nu^{j-\frac{\alpha}{2}}-\nu^{j-1}\big)-\Big(1-\frac{\alpha}{2}\Big)\big(\nu^{j-\frac{\alpha}{2}}-\nu^{j-2}\big)\\&\le\Big(2-\frac{\alpha}{2}\Big)\int_{t_{j-1}}^{t_{j-\frac{\alpha}{2}}}(1+s^{\alpha-2})(s-t_{j-1})ds-\Big(1-\frac{\alpha}{2}\Big)\int_{t_{j-2}}^{t_{j-\frac{\alpha}{2}}}\nu''(s)\big(s-t_{j-\frac{\alpha}{2}}\big)ds\\&\le\Big(2-\frac{\alpha}{2}\Big)\Big(1-\frac{\alpha}{2}\Big)\big(\tau_j^2+\tau_jt_{j-2}^{\alpha-1}\big).\end{aligned}\tag{3.7}$$

Together with inequalities (3.6), (3.7) and the assumption $g\in C^2(\mathcal{R})$, we obtain there exists a positive constant C_g, such that the following inequalities hold:

$$\begin{aligned}|R_v^j|&=\big|g\big(|\nu^{j-\frac{\alpha}{2}}|^2\big)\nu^{j-\frac{\alpha}{2}}-g\big(|\hat{\nu}^j|^2\big)\nu^{j-\frac{\alpha}{2}}+g\big(|\hat{\nu}^j|^2\big)\nu^{j-\frac{\alpha}{2}}-g\big(|\hat{\nu}^j|^2\big)\nu^{j,\alpha}\big|\\&=\|\nu^{j-\frac{\alpha}{2}}\|_{L^\infty}|g'(\xi)|\big\||\nu^{j-\frac{\alpha}{2}}|^2-|\hat{\nu}^j|^2\big\|_{L^2}+\big|g(|\hat{\nu}|^2)\big|\big\|\nu^{j-\frac{\alpha}{2}}-\nu^{j,\alpha}\big\|_{L^2}\\&\le C_g\big\||\nu^{j-\frac{\alpha}{2}}|^2-|\hat{\nu}^j|^2\big\|_{L^2}+\|\nu^{j-\frac{\alpha}{2}}-\nu^{j,\alpha}\|_{L^2}\\&\le C_g\big(\|\nu^{j-\frac{\alpha}{2}}\|_{L^\infty}+\|\hat{\nu}^j\|_{L^\infty}\big)\big(\|\nu^{j-\frac{\alpha}{2}}-\hat{\nu}^j\|_{L^2}\big)+C_g\|\nu^{j-\frac{\alpha}{2}}-\nu^{j,\alpha}\|_{L^2}\\&\le C_\nu\big(\tau_j^2+\tau_jt_{j-2}^{\alpha-1}+t_{j-1}^{\alpha-2}\tau_j^2\big).\end{aligned}$$

Moreover

$$\big|R_v^1\big|\le\tilde{C}_3\big(\tau_1+\tau_1^\alpha\big),\tag{3.8}$$

which further implies

$$\begin{aligned}\sum_{j=1}^{n}P_{n-j}^{(n)}\big|R_v^j\big|&=P_{n-1}^{(n)}\big|R_v^1\big|+\sum_{j=2}^{n}P_{n-j}^{(n)}\big|R_v^j\big|\\&\le\Gamma(2-\alpha)\pi_A\tau_1^\alpha\big|R_v^1\big|+\max_{2\le k\le n}\big|R_v^k\big|\sum_{j=2}^{n}P_{n-j}^{(n)}\\&\le C_v\left(\tau_1^{2\alpha}+t_n^\alpha\max_{2\le k\le n}\tau_k^2t_{k-1}^{\alpha-2}\right)\\&\le C_v\left(T^{2\alpha}N^{-2\delta\alpha}+t_n^\alpha\max_{2\le k\le n}\big(T^\alpha\delta^2N^{-\delta\alpha}k^{2(\delta-1)}(k-1)^{\delta(\alpha-2)}\big)\right)\\&\le C_v\left[T^{2\alpha}N^{-2\delta\alpha}+t_n^\alpha\max_{2\le k\le n}4^\delta\left(\frac{k-1}{N}\right)^{\delta\alpha-\min\{\delta\alpha,2\}}N^{-\min\{\delta\alpha,2\}}\right]\\&\le C_v\big(T^{2\alpha}N^{-\min\{\delta\alpha,2\}}\big).\end{aligned}$$

Here the property of graded meshes $\tau_k \le \delta k^{\delta-1} T N^{-\delta}, 1 \le k \le n$, is used, which finishes the proof. $\square$

Let $R_h : H_0^1(\Omega) \to V_h^0$ be Ritz projection defined by

$$(\nabla \chi, \nabla \omega) = (\nabla R_h \chi, \nabla \omega), \quad \forall \omega \in V_h^0. \tag{3.9}$$

According to classical FEM theory [31], for any $v \in H^s(\Omega) \cap H_0^1(\Omega)$, we have

$$\|v - R_h v\|_{L^2} + h\|\nabla(v - R_h v)\|_{L^2} \le C_\Omega h^s \|v\|_{H^s}, \quad 1 \le s \le r+1. \tag{3.10}$$

The following inequality is important for our proof [4]:

$$\|v\|_{L^\infty} \le C_\Omega h^{-\frac{d}{2}} \|v\|_{L^2}, \quad \forall v \in V_h. \tag{3.11}$$

Here, we give the following time-discrete system:

$$iD_{\Delta t}^\alpha U^{n-\frac{\alpha}{2}} + \Delta U^{n,\alpha} + f\big(|\hat{U}^n|^2\big) U^{n,\alpha} = 0, \quad n = 2, \ldots, N \tag{3.12}$$

with initial and boundary conditions

$$U^n(x) = 0, \qquad x \in \partial\Omega, \quad n = 2, 3, \ldots, N, \tag{3.13}$$

$$U^0(x) = u_0(x), \quad x \in \Omega. \tag{3.14}$$

In the case of $n = 1$, we have the following scheme:

$$i\frac{U^1 - U^0}{\mu \tau_1^\alpha} + \Delta U^{1,\alpha} + f\big(|U^0|^2\big) U^{1,\alpha} = 0. \tag{3.15}$$

We divide the errors into two parts, i.e.,

$$\|u^n - U_h^n\| \le \|u^n - U^n\| + \|U^n - U_h^n\| := \|e^n\| + \|U^n - U_h^n\|. \tag{3.16}$$

Then, we will prove our main results by giving the error estimates of two terms in above inequality, respectively.

3.2. Analysis of $\|u^n - U^n\|$

We now focus on the regularity of U^n, followed with the error estimates.

Taking $t = t_{n-\alpha/2}$ in Eq. (3.12), we can find u^n satisfies the following equation:

$$iD_{\Delta t}^\alpha u^{n-\frac{\alpha}{2}} + \Delta u^{n,\alpha} + f\big(|\hat{u}^n|^2\big) u^{n,\alpha} = P^n, \quad n = 2, 3, \ldots, N, \tag{3.17}$$

where

$$\begin{aligned} P^n = {} & i\Big(D_{\Delta t}^\alpha u^{n-\frac{\alpha}{2}} - {}^C D_{t_{n-\frac{\alpha}{2}}} u\Big) + \Delta u^{n,\alpha} - \Delta u\big(t_{n-\frac{\alpha}{2}}\big) \\ & + f\big(|\hat{u}^n|^2\big) u^{n,\alpha} - f\big(|u(t_{n-\frac{\alpha}{2}})|^2\big) u\big(t_{n-\frac{\alpha}{2}}\big). \end{aligned} \tag{3.18}$$

Moreover, u^1 satisfies

$$i\frac{u^1 - u^0}{\mu \tau_1^\alpha} + \Delta u^{1,\alpha} + f\big(|u^0|^2\big) u^{1,\alpha} = P^1, \tag{3.19}$$

where

$$P^1 = i\left(\frac{u^1 - u^0}{\mu\tau^\alpha} -^C D_{t_{1-\frac{\alpha}{2}}} u\right) + \Delta u^{1,\alpha} - \Delta u\big(t_{1-\frac{\alpha}{2}}\big) + f\big(|u^0|^2\big)u^{1,\alpha} - f\big(|u(t_{1-\frac{\alpha}{2}})|^2\big)u\big(t_{1-\frac{\alpha}{2}}\big). \tag{3.20}$$

Applying (3.4), (3.20) and Taylor's theorem, there exists a positive constant C_K dependent on C such that

$$\|P^1\|_{H^2} \le C_K N^{-\delta\alpha}. \tag{3.21}$$

Let $e^n := u^n - U^n, n = 1, 2, \ldots, N$. Subtracting (3.12) from (3.17), we arrive at that

$$iD^\alpha_{\Delta t} e^{n-\frac{\alpha}{2}} + \Delta e^{n,\alpha} + R^n_1 = P^n, \quad n = 2, 3, \ldots, N, \tag{3.22}$$

where

$$R^n_1 = f\big(|\hat{u}^n|^2\big)u^{n,\alpha} - f\big(|\hat{U}^n|^2\big)U^{n,\alpha}.$$

Similarly subtracting (3.15) from (3.20) yields

$$i\frac{e^1}{\mu\tau^\alpha_1} + \Delta e^{1,\alpha} + f\big(|u^0|^2\big)e^{1,\alpha} = P^1. \tag{3.23}$$

Define

$$K_1 := \max_{1\le n\le N} \|u^n\|_{L^\infty} + \max_{1\le n\le N} \|u^n\|_{H^2} + 1. \tag{3.24}$$

Lemma 3.7. *The solution U^1 of semi-discrete system* (3.15) *is valid and unique, and there is a constant $N^*_1 > 0$ such that, when $N \ge N^*_1$,*

$$\|e^1\|_{H^2} \le C^*_1 N^{-\delta\alpha}, \tag{3.25}$$

$$\|U^1\|_{H^2} + \big\|\big(D^\alpha_{\Delta t}U\big)^{1-\frac{\alpha}{2}}\big\|_{H^2} \le C^{**}_1, \tag{3.26}$$

where C^*_1, C^{**}_1 are two positive numbers independent of N and h.

Proof. We leave the proof in the appendix for the reader's convenience. □

Theorem 3.1. *Suppose that the regularity condition* (2.5) *holds. Then system* (3.12)-(3.14) *has a unique solution U^n, and there is a constant $N^*_2 > 0$ such that, when $N \ge N^*_2$, for $n = 2, 3, \ldots, N$,*

$$\|e^n\|_{H^2} \le C^*_1 N^{-\delta\alpha}, \tag{3.27}$$

$$\|U^n\|_{H^2} + \|(D^\alpha_{\Delta t}U)^{n-\frac{\alpha}{2}}\|_{H^2} \le C^{**}_1, \tag{3.28}$$

where $\delta\alpha \le 2$ and C^*_1, C^{**}_1 are two positive numbers independent of N and h.

Proof. It is sufficient to gain the solution U^n is existent and unique, by the fact that system (3.12) becomes a linear elliptic equation at each time level. Next, the main results were proved by taking the mathematical induction into account. Firstly, one can check that the result holds for $n = 1$ by using Lemma 3.7. Now, we suppose that (3.27) holds for $n \le k - 1$. Based on inequality (3.24), we have, for $n \le k - 1$,

$$\begin{aligned}\|U^k\|_{L^\infty} &\le \|u^k\|_{L^\infty} + \|e^k\|_{L^\infty} \\ &\le \|u^k\|_{L^\infty} + C_\Omega\|e^k\|_{H^2} \\ &\le \|u^k\|_{L^\infty} + C_\Omega C^*_1 N^{-\delta\alpha} \le K_1,\end{aligned} \tag{3.29}$$

where $N \geq \tilde{N}_1 = (C_\Omega C_1^*)^{1/(\delta\alpha)}$. Together with $f \in C^3(\mathcal{R})$, the boundedness of U^n and the regularity of u, we arrive at

$$\begin{aligned}
\big\|R_1^n\big\|_{L^2} &= \big\|f\big(|\hat{u}^n|^2\big)u^{n,\alpha} - f\big(|\hat{U}^n|^2\big)U^{n,\alpha}\big\|_{L^2} \\
&\leq \big\|f\big(|\hat{u}^n|^2\big)u^{n,\alpha} - f\big(|\hat{U}^n|^2\big)u^{n,\alpha}\big\|_{L^2} + \big\|f\big(|\hat{U}^n|^2\big)u^{n,\alpha} - f\big(|\hat{U}^n|^2\big)U^{n,\alpha}\big\|_{L^2} \\
&\leq \big\|f'(\xi_1)\big(|\hat{u}^n|^2 - |\hat{U}^n|^2\big)\big\|_{L^2}\|u^{n,\alpha}\|_{L^\infty} + \big\|f\big(|\hat{U}^n|^2\big)\big\|_{L^\infty}\|e^{n,\alpha}\|_{L^2} \\
&\leq C_f K_1^2\|\hat{e}^n\|_{L^2} + C_f\|e^{n,\alpha}\|_{L^2}.
\end{aligned} \tag{3.30}$$

Now we consider the case of $k = n$. Taking inner product of each side of (3.22) with $e^{n,\alpha}$ and the imaginary part of the resulting equation to obtain

$$\begin{aligned}
\mathrm{Re}\big(D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}, e^{n,\alpha}\big) &= -\mathrm{Im}\big(R_1^n, e^{n,\alpha}\big) + \mathrm{Im}(P^n, e^{n,\alpha}) \\
&\leq \big\|R_1^n\big\|_{L^2}\|e^{n,\alpha}\|_{L^2} + \|P^n\|_{L^2}\|e^{n,\alpha}\|_{L^2}.
\end{aligned}$$

Substituting (3.30) into the above equation and recalling Lemma 3.2, we have

$$\begin{aligned}
\frac{1}{2}\sum_{k=1}^n B_{n-k}^{(n)}\nabla_\tau\|e^k\|_{L^2}^2 &\leq \big(C_f K_1^2\|\hat{e}^n\|_{L^2} + C_f\|e^{n,\alpha}\|_{L^2}\big)\|e^{n,\alpha}\|_{L^2} + \|P^n\|_{L^2}\|e^{n,\alpha}\|_{L^2} \\
&\leq \big(C_f K_1^2\big)^2\|\hat{e}^n\|_{L^2}^2 + \frac{1}{4}\|e^{n,\alpha}\|_{L^2}^2 + C_f\|e^{n,\alpha}\|_{L^2}^2 + \|P^n\|_{L^2}\|e^{n,\alpha}\|_{L^2} \\
&\leq C_1\|e^n\|_{L^2}^2 + C_2\|e^{n-1}\|_{L^2}^2 + C_3\|e^{n-2}\|_{L^2}^2 + \|P^n\|_{L^2}\|e^{n,\alpha}\|_{L^2},
\end{aligned}$$

where

$$C_1 = \Big(1 - \frac{\alpha}{2}\Big)^2\left(\frac{1+4C_f}{2}\right), \quad C_2 = \frac{9C_f^2K_1^4}{4} + \frac{\alpha^2}{8}(1+4C_f), \quad C_3 = \frac{C_f^2K_1^4}{4}.$$

Applying Lemmas 3.3-3.6 and the discrete Grönwall inequality, we have there exists an $\tilde{N}_2 \geq (2\pi_A\Gamma(2-\alpha)(C_1+C_2+C_3))^\alpha$ such that

$$\begin{aligned}
\|e^n\|_{L^2} &\leq 4E_\alpha\big(4\pi_A(C_1+C_2+C_3)t_n^\alpha\big)\max_{1\leq k\leq n}\sum_{j=1}^k P_{k-j}^{(k)}\|P^j\|_{L^2} \\
&\leq 4E_\alpha\big(4\pi_A(C_2+C_3)t_n^\alpha\big)\max_{1\leq k\leq n}\sum_{j=1}^k P_{k-j}^{(k)}\|P^j\|_{L^2} \\
&\leq C_4\Big(C_v\big(T^\alpha N^{-\delta\alpha} + T^\alpha\delta^{3-\alpha}8^{\delta-1}N^{-\min\{\delta\alpha,3-\alpha\}}\big) + T^{2\alpha}N^{-\min\{\delta\alpha,2\}}\Big) \\
&\leq C_5\big(N^{-\min\{\delta\alpha,2\}}\big),
\end{aligned} \tag{3.31}$$

where C_5 is a positive constant independent of n, N and h.

To estimate $\|e^n\|_{H^1}$, Eq. (3.22) is multiplied by $D_{\Delta t}^\alpha e^{n-\alpha/2}$ and integrated over Ω. Considering resulting equation's real part and applying Green's equality, we arrive at

$$\mathrm{Re}\big(\nabla D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}, \nabla e^{n,\alpha}\big) \leq \big|\big(R_1^n, D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}\big)\big| + \big|\big(P^n, D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}\big)\big|. \tag{3.32}$$

Similar to the analysis of (3.30), by using (3.29) we can obtain

$$\begin{aligned}
\|\nabla R_1^n\|_{L^2} &\leq \big\|\big(\nabla f(|\hat{u}^n|^2)\big)u^{n,\alpha} - \big(\nabla f(|\hat{U}^n|^2)\big)U^{n,\alpha}\big\|_{L^2} + \big\|f(|\hat{u}^n|^2)\nabla u^{n,\alpha} - f(|\hat{U}^n|^2)\nabla U^{n,\alpha}\big\|_{L^2} \\
&\leq \big\|\big(\nabla f(|\hat{u}^n|^2)\big)u^{n,\alpha} - \big(\nabla f(|\hat{u}^n|^2)\big)U^{n,\alpha} + \big(\nabla f(|\hat{u}^n|^2)\big)U^{n,\alpha} - \big(\nabla f(|\hat{U}^n|^2)\big)U^{n,\alpha}\big\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+\big\|f(|\hat{u}^n|^2)\nabla u^{n,\alpha}-f(|\hat{U}^n|^2)\nabla u^{n,\alpha}\big\|_{L^2}+\big\|f(|\hat{U}^n|^2)\nabla u^{n,\alpha}-f(|\hat{U}^n|^2)\nabla U^{n,\alpha}\big\|_{L^2}\\
\le{}& C_f\|e^{n,\alpha}\|_{L^2}+\big\|\big(f''(\xi_1)(|\hat{u}^n|^2-|\hat{U}^n|^2)\nabla|\hat{u}^n|^2+f'(|\hat{U}^n|^2)\nabla|\hat{u}^n|^2-\nabla|\hat{U}^n|^2\big)U^{n,\alpha}\big\|_{L^2}\\
&+\big\|f'(\xi_2)\nabla u^{n,\alpha}\big(|\hat{u}^n|^2-|\hat{U}^n|^2\big)+f(|\hat{U}^n|^2)\nabla e^{n,\alpha}\big\|_{L^2}\\
\le{}& C_k^*\big(\|\nabla e^n\|_{L^2}+\|\nabla e^{n-1}\|_{L^2}+\|\nabla e^{n-2}\|_{L^2}\big),
\end{aligned}
\tag{3.33}
$$

where C_k^* is a constant independent of n but dependent on K_1, C_f and C_Ω. Together with (3.22), (3.33) and (3.31), we obtain that

$$
\begin{aligned}
\big|\big(R_1^n, D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}\big)\big| &\le \big|\mathrm{Re}\big(\nabla R_1^n, i\nabla e^{n,\alpha}\big)\big|+\big|\mathrm{Re}\big(R_1^n, iP^n\big)\big|\\
&\le \frac{1}{2}\|\nabla e^{n,\alpha}\|_{L^2}^2+\frac{1}{2}\|\nabla R_1^n\|_{L^2}^2+\frac{1}{2}\|R_1^n\|_{L^2}^2+\frac{1}{2}\|P^n\|_{L^2}^2\\
&\le \tilde{C}_1\|\nabla e^n\|_{L^2}^2+\tilde{C}_2\|\nabla e^{n-1}\|_{L^2}^2+(C_K^*)^2\|\nabla e^{n-2}\|_{L^2}^2+\frac{1}{2}\|P^n\|_{L^2}^2,
\end{aligned}
\tag{3.34}
$$

where

$$
\tilde{C}_1=\left[\left(1-\frac{\alpha}{2}\right)^2+\left(C_K^*\right)^2\right],\quad \tilde{C}_2=\left[\frac{\alpha^2}{2}+\left(C_K^*\right)^2\right].
$$

Moreover,

$$
\begin{aligned}
\big|\big(P^n, D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}\big)\big| &\le |\mathrm{Re}(i\nabla e^{n,\alpha}, P^n)|+|\mathrm{Re}(iR_1^n, P^n)|\\
&\le \frac{1}{4}\|\nabla e^{n,\alpha}\|_{L^2}^2+\|\nabla P^n\|_{L^2}^2+\frac{1}{2}\|R_1^n\|_{L^2}^2+\frac{1}{2}\|P^n\|_{L^2}^2\\
&\le \tilde{C}_3\|\nabla e^n\|_{L^2}^2+\tilde{C}_4\|\nabla e^{n-1}\|_{L^2}^2+(C_K^*)^2\|\nabla e^{n-2}\|_{L^2}^2\\
&\quad+\|\nabla P^n\|_{L^2}^2+\frac{1}{2}\|P^n\|_{L^2}^2,
\end{aligned}
\tag{3.35}
$$

where

$$
\tilde{C}_3=\left[\frac{1}{2}\left(1-\frac{\alpha}{2}\right)^2+(C_K^*)^2\right],\quad \tilde{C}_4=\left[\frac{\alpha^2}{8}+(C_K^*)^2\right].
$$

Substituting (3.34) and (3.35) into the inequality (3.32), we get

$$
\begin{aligned}
&\mathrm{Re}\big(D_{\Delta t}^\alpha\nabla e^{n-\frac{\alpha}{2}}, \nabla e^{n,\alpha}\big)\\
\le{}& \big(\tilde{C}_1+\tilde{C}_3\big)\|\nabla e^n\|_{L^2}^2+\big(\tilde{C}_2+\tilde{C}_3\big)\|\nabla e^{n-1}\|_{L^2}^2\\
&+2\big(C_K^*\big)^2\|\nabla e^{n-2}\|_{L^2}^2+\|\nabla P^n\|_{L^2}^2+\|P^n\|_{L^2}^2.
\end{aligned}
$$

Together with Lemmas 3.2-3.4, there exists an $N \ge \tilde{N}_3$,

$$
\|\nabla e^n\|_{L^2} \le C_6 N^{-\min\{\delta\alpha,2\}},
\tag{3.36}
$$

where C_6 is a positive constant only dependent on $C_K^*, C_f, C_\nu, C_\Omega$.

To estimate $\|e^n\|_{H^2}$, we multiply (3.22) by $D_{\Delta t}^\alpha \Delta e^{n-\alpha/2}$ and integrate it over Ω. Taking the real part of the resulting equation and applying Green's equality, we arrive at

$$
\begin{aligned}
&\mathrm{Re}\big(\Delta e^{n,\alpha}, D_{\Delta t}^\alpha\Delta e^{n-\frac{\alpha}{2}}\big)\\
={}& -\mathrm{Re}\big(R_1^n, D_{\Delta t}^\alpha\Delta e^{n-\frac{\alpha}{2}}\big)+\mathrm{Re}\big(P^n, D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}\big)\\
={}& -\mathrm{Re}\big(\Delta R_1^n, D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}\big)+\mathrm{Re}\big(\Delta P^n, D_{\Delta t}^\alpha e^{n-\frac{\alpha}{2}}\big).
\end{aligned}
\tag{3.37}
$$

Using the similar approach as the proof of (3.36), we arrive at that there exists an $N \ge \tilde{N}_4$,

$$
\|\Delta e^n\|_{L^2} \le C_7 N^{-\min\{\delta\alpha,2\}}.
\tag{3.38}
$$

Now, by (3.27), (3.36) and (3.38), we get, when $N \geq \tilde{N}_5 = \max\{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4\}$,

$$\begin{aligned}\|e^n\|_{H^2} &\leq \sqrt{\|e^n\|_{L^2}^2 + \|\nabla e^n\|_{L^2}^2 + \|\Delta e^n\|_{L^2}^2} \\ &\leq \sqrt{C_5 + C_6 + C_7}\, N^{-\min\{\delta\alpha,2\}} \\ &\leq C_c N^{-\min\{\delta\alpha,2\}},\end{aligned}$$

which further implies that

$$\|U^n\|_{H^2} \leq \|u^n\|_{H^2} + \|e^n\|_{H^2} \leq \|u^n\|_{H^2} + C_c N^{-\min\{\delta\alpha,2\}} \leq K_1,$$

when $N \geq \tilde{N}_6 = (C_c)^{1/\min\{\delta\alpha,2\}}$. Taking $N_1^* = \max\{\tilde{N}_1, \tilde{N}_2, \tilde{N}_3, \tilde{N}_4, \tilde{N}_5, \tilde{N}_6\}$, the result (3.27) holds for $m = n$.

Using the definition of $(D_{\Delta t}^{\alpha} v)^{n-\alpha/2}$, and

$$B_0^{(n)} \leq \tilde{z}_0^{(n)} + \rho_{n-1}\tilde{y}_1^{(n)} \leq \frac{24}{11\tau_n}\int_{t_{n-1}}^{t_n} w_{1-\alpha}(t_n - s)ds,$$

which can be found in [25, 36], we arrive at that

$$\begin{aligned}\left\|\left(D_{\Delta t}^{\alpha} e\right)^{n-\frac{\alpha}{2}}\right\|_{H^2} &\leq B_0^{(n)}\|e^n\|_{H^2} + \sum_{k=1}^{n-1}\left(B_{n-k}^{(n)} - B_{n-k-1}^{(n)}\right)\|e^k\|_{H^2} - B_{n-1}^{(n)}\|e^0\|_{H^2} \\ &\leq \left[B_0^{(n)} + \sum_{k=1}^{n-1}\left(B_{n-k}^{(n)} - B_{n-k-1}^{(n)}\right)\right] C_1^* N^{-\min\{\delta\alpha,2\}} \leq B_0^{(n)} C_1^* N^{-\min\{\delta\alpha,2\}} \\ &\leq \frac{24N^{\delta\alpha}}{11\Gamma(2-\alpha)T^{\alpha}} C_1^* N^{-\min\{\delta\alpha,2\}} \leq \frac{24}{11\Gamma(2-\alpha)T^{\alpha}} C_1^*,\end{aligned}$$

where the assumption $\delta\alpha \leq 2$ is used in the last inequality. Therefore

$$\left\|\left(D_{\Delta t}^{\alpha} U\right)^{n-\frac{\alpha}{2}}\right\|_{H^2} \leq \left\|\left(D_{\Delta t}^{\alpha} u\right)^{n-\frac{\alpha}{2}}\right\|_{H^2} + \left\|\left(D_{\Delta t}^{\alpha} e\right)^{n-\frac{\alpha}{2}}\right\|_{H^2} \leq C_1^{**},$$

the mathematical induction is finished and the proof is accomplished. □

3.3. The boundness of $\|U_h^n\|_{L^\infty}$

In this subsection, the boundedness of $\|U_h^n\|_{L^\infty}$ is derived. Firstly, by Theorem 3.1 and $\|R_h v\|_{L^\infty} \leq C_\Omega \|v\|_{H^2}$ for any $v \in H^2(\Omega)$, we can obtain $\|R_h U^n\|_{L^\infty}$ is bounded. Therefore, we define

$$K_2 = \max_{1\leq m\leq N} \|R_h U^m\|_{L^\infty} + 1. \tag{3.39}$$

Eq. (3.12) can be rewritten as

$$i\left((D_{\Delta t}^{\alpha} U)^{n-\frac{\alpha}{2}}, v\right) - (\nabla U^{n,\alpha}, \nabla v) + \left(f(|\hat{U}^n|^2)U^{n,\alpha}, v\right) = 0, \quad n = 2, \ldots, N. \tag{3.40}$$

Let

$$U^n - U_h^n = U^n - R_h U^n + R_h U^n - U_h^n = U^n - R_h U^n + \vartheta_h^n, \quad n = 0, 1, 2, \ldots, N.$$

Subtracting (2.3) from (3.40) and applying (3.9) gives

$$i\left((D_{\Delta t}^{\alpha}\vartheta_h)^{n-\frac{\alpha}{2}}, v\right) - \left(\nabla\vartheta_h^{n,\alpha}, \nabla v\right) + (R_2^n, v) = -i\left(\left(D_{\Delta t}^{\alpha}(U - R_h U)\right)^{n-\frac{\alpha}{2}}, v\right), \quad \forall v \in V_h, \tag{3.41}$$

where

$$R_2^n = f\big(|\hat{U}^n|^2\big)U^{n,\alpha} - f\big(|\hat{U}_h^n|^2\big)U_h^{n,\alpha}. \tag{3.42}$$

Moreover, from (3.23) and (3.15) we get

$$\begin{aligned}
&i\left(\frac{\vartheta_h^1}{\mu\tau_1^\alpha}, v\right) - \big(\nabla\vartheta_h^{1,\alpha}, \nabla v\big) + \Big(f\big(|U^0|^2\big)U^{1,\alpha} - f\big(|U_h^0|^2\big)U_h^{1,\alpha}, v\Big) \\
=& -i\left(\frac{U^1 - R_hU^1}{\mu\tau_1^\alpha}, v\right) + i\left(\frac{U^0 - U_h^0}{\mu\tau_1^\alpha}, v\right).
\end{aligned} \tag{3.43}$$

Lemma 3.8. *Let U^1 and U_h^1 be the solutions of* (3.40) *and* (3.43)*, respectively. Then there exist $N_2^* > 0$, $h_1^* > 0$ such that, when $N \geq N_2^*$, $h \leq h_1^*$,*

$$\big\|\vartheta_h^1\big\|_{L^2} \leq h^{\frac{11}{6}}, \tag{3.44}$$

$$\big\|U_h^1\big\|_{L^\infty} \leq K_2. \tag{3.45}$$

Proof. We leave the proof in the appendix for the reader's convenience. □

Theorem 3.2. *Let U^m and U_h^m be the solutions of* (3.40) *and* (3.41)*, respectively. Then there exist $N_3^* > 0$, $h_1^* > 0$ such that for $m = 1, 2, 3, \ldots, N$, when $N \geq N_3^*$, $h \leq h_1^*$,*

$$\big\|\vartheta_h^m\big\|_{L^2} \leq h^{\frac{11}{6}}, \tag{3.46}$$

$$\big\|U_h^m\big\|_{L^\infty} \leq K_2. \tag{3.47}$$

Proof. Taking $v = U_h^{n,\alpha}$ in (3.40) and the imaginary part of the equation, we get

$$Re\big((D_{\Delta t}^\alpha U_h)^{n-\frac{\alpha}{2}}, U_h^{n,\alpha}\big) = 0.$$

Together with Lemma 3.2 and B1, B2 yields

$$\frac{1}{2}\sum_{k=1}^{n} B_{n-k}^{(n)}\nabla_\tau\big(\|U_h^k\|^2\big) \leq Re\big\langle U_h^{n,\alpha}, (D_{\Delta t}^\alpha v)^{n-\frac{\alpha}{2}}\big\rangle = 0,$$

which further implies

$$\big\|U_h^n\big\|_{L^2} \leq \big\|U_h^0\big\|_{L^2}.$$

It is obvious that the solution of Eq. (3.40) exists and is unique. Next, we still use mathematic induction to prove (3.46). By Lemma 3.8, we have (3.46) holds for $m = 1$. Now, assume that (3.46) is valid for $m = 2, \ldots, n-1$. Then we prove that it holds for $m = n$. By the assumption and (3.39), we have

$$\begin{aligned}
\big\|U_h^m\big\|_{L^\infty} &\leq \big\|R_hU^m\big\|_{L^\infty} + \big\|R_hU^m - U_h^m\big\|_{L^\infty} \\
&\leq \big\|R_hU^m\big\|_{L^\infty} + C_\Omega h^{-\frac{d}{2}}\big\|R_hU^m - U_h^m\big\|_{L^2} \\
&\leq \big\|R_hU^m\big\|_{L^\infty} + C_\Omega h^{-\frac{d}{2}}h^{\frac{11}{6}} \\
&\leq \big\|R_hU^m\big\|_{L^\infty} + 1 \leq K_2
\end{aligned} \tag{3.48}$$

for $d = 2, 3$, and $h \leq h_1 = C_\Omega^{-6/(11-3d)}$.

With the analogous argument for r^n, considering the boundedness of $\|U^n\|_{H^2}, \|U_h^{n-1}\|_{L^\infty}$ and $f \in C^3(\mathcal{R})$, we can obtain that

$$\begin{aligned}\|R_2^n\|_{L^2} &= \|f(|\hat{U}^n|)U^{n,\alpha} - f(|\hat{U}_h^n|^2)U^{n,\alpha} + f(|\hat{U}_h^n|^2)U^{n,\alpha} - f(|\hat{U}_h^n|^2)U_h^{n,\alpha}\|_{L^2} \\ &\le K_2\|f'(\xi)\|_{L^\infty}(\|\hat{U}^n - R_h\hat{U}^n\|_{L^2} + \|\vartheta_h^n\|_{L^2}) + C_f(\|U^{n,\alpha} - R_hU^{n,\alpha}\|_{L^2} + \|\vartheta_h^{n,\alpha}\|_{L^2}) \\ &\le C_K(\|\hat{\vartheta}_h^n\|_{L^2} + \|\vartheta_h^{n,\alpha}\|_{L^2} + h^2), \end{aligned} \tag{3.49}$$

where the classical FEM theory (3.10) is used in the last inequality, and C_K is a constant related to K_2, C_f and C_Ω.

Taking $v = \vartheta_h^{n,\alpha}$ in Eq. (3.41) and the imaginary part of the resulting equation, we obtain

$$\mathrm{Re}\Big(D_{\Delta t}^\alpha \vartheta_h^{n-\frac{\alpha}{2}}, \vartheta_h^{n,\alpha}\Big) + \mathrm{Re}\big(R_2^n, \vartheta_h^{n,\alpha}\big) = -\mathrm{Re}\Big(\big(D_{\Delta t}^\alpha(U - R_hU)\big)^{n-\frac{\alpha}{2}}, \vartheta_h^{n,\alpha}\Big). \tag{3.50}$$

Combining (3.49), (3.50) with Lemma 3.2, we arrive at

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^n B_{n-k}^{(n)}\nabla_\tau\|\vartheta_h^n\|_{L^2}^2 \\ &\le \|R_2^n\|_{L^2}\|\vartheta_h^{n,\alpha}\|_{L^2} + \|\big(D_{\Delta t}^\alpha(U - R_hU)\big)^{n-\frac{\alpha}{2}}\|_{L^2}\|\vartheta_h^{n,\alpha}\|_{L^2} \\ &\le \Big[C_K\Big(\|\hat{\vartheta}_h^n\|_{L^2} + \|\vartheta_h^{n,\alpha}\|_{L^2} + h^2\Big) + C_\Omega\|\big(D_{\Delta t}^\alpha U\big)^{n-\frac{\alpha}{2}}\|_{H^2}h^2\Big]\|\vartheta_h^{n,\alpha}\|_{L^2} \\ &\le C_K\left(\|\hat{\vartheta}_h^n\|_{L^2}^2 + \frac{1}{4}\|\vartheta_h^{n,\alpha}\|_{L^2}^2\right) + C_K\|\vartheta_h^{n,\alpha}\|_{L^2}^2 + (C_K + C_\Omega K)\|\vartheta_h^{n,\alpha}\|_{L^2}h^2 \\ &\le C_8\|\vartheta_h^n\|_{L^2}^2 + C_9\|\vartheta_h^{n-1}\|_{L^2}^2 + C_{10}\|\vartheta_h^{n-2}\|_{L^2}^2 + \big(C_K + C_\Omega C_1^{**}\big)\|\vartheta_h^{n,\alpha}\|_{L^2}h^2,\end{aligned}$$

where

$$C_8 = 2\left(\frac{1}{4} + C_K\right)\left(1 - \frac{\alpha}{2}\right)^2, \quad C_9 = C_K\left(2 - \frac{\alpha}{2}\right)^2, \quad C_{10} = C_K\left(1 - \frac{\alpha}{2}\right)^2.$$

Together with the inequality (3.28), Lemma 3.3 and $U_h^0 = R_hu^0$, we arrive at

$$\begin{aligned}\|\vartheta_h^n\|_{L^2} &\le 2E_\alpha\big(4\pi_A(C_8 + C_9 + C_{10})t_n^\alpha\big)\Big[\|\vartheta_h^0\|_{L^2} + 2\pi_A\Gamma(2-\alpha)t_n^\alpha(C_k + C_\Omega C_1^{**})h^2\Big] \\ &\le 4E_\alpha\big(4\pi_A(C_8 + C_9 + C_{10})t_n^\alpha\big)\big[\pi_A\Gamma(2-\alpha)T^\alpha(C_k + C_\Omega C_1^{**})\big]h^2 \le h^{\frac{11}{6}},\end{aligned}$$

where

$$h \le h_2 = \left[\frac{1}{4E_\alpha(4\pi_A(C_8 + C_9 + C_{10})t_n^\alpha)(\pi_A\Gamma(2-\alpha)T^\alpha(C_k + C_\Omega C_1^{**}))}\right]^{-\frac{1}{6}}.$$

Furthermore,

$$\|U_h^n\|_{L^\infty} \le \|R_hU^n\|_{L^\infty} + \|\vartheta_h^n\|_{L^\infty} \le \|R_hU^n\|_{L^\infty} + C_\Omega h^{-\frac{d}{2}}h^{\frac{11}{6}} \le K_2.$$

Then (3.46) and (3.47) hold for $m = n$. The proof is complete. □

3.4. Error estimates for the fully discrete system

In Section 3.3, the numerical solution U_h^n is proved to be unconditionally bounded in L^∞ norm. Based on the results, we can start to verify our main conclusions.

The weak form of Eq. (3.17) satisfies for all $v \in V_h$ and $n = 2, 3, \ldots, N$,

$$i\big((D^{\alpha}_{\Delta t}u)^{n-\frac{\alpha}{2}}, v\big) + (\Delta u^{n,\alpha}, v) + \big(f(|\hat{u}^n|^2)u^{n,\alpha}, v\big) = (P^n, v). \tag{3.51}$$

Denote

$$u^n - U_h^n = u^n - R_h u^n + R_h u^n - U_h^n = u^n - R_h u^n + \eta_h^n, \quad n = 0, 1, 2, \ldots, N. \tag{3.52}$$

Subtracting (3.51) from (2.3) gives

$$i\big((D^{\alpha}_{\Delta t}\eta_h)^{n-\frac{\alpha}{2}}, v\big) - \big(\nabla \eta_h^{n,\alpha}, \nabla v\big) + (R_3^n, v) = (P^n, v) - (R_4^n, v), \tag{3.53}$$

where

$$R_3^n = f\big(|\hat{u}^n|^2\big)u^{n,\alpha} - f\big(|\hat{U}_h^n|^2\big)U_h^{n,\alpha}, \quad R_4^n = i\big(D^{\alpha}_{\Delta t}(u - R_h u)\big)^{n-\frac{\alpha}{2}}.$$

Moreover, from (3.19) and (2.4) we get

$$\begin{aligned} &i\left(\frac{\eta_h^1}{\mu\tau_1^{\alpha}}, v\right) - \big(\nabla \eta_h^{1,\alpha}, \nabla v\big) + \Big(f\big(|u^0|^2\big)u^{1,\alpha} - f\big(|U_h^0|^2\big)U_h^{1,\alpha}, v\Big) \\ =& -i\left(\frac{u^1 - R_h u^1}{\mu\tau_1^{\alpha}}, v\right) + i\left(\frac{u^0 - U_h^0}{\mu\tau_1^{\alpha}}, v\right). \end{aligned} \tag{3.54}$$

Now prove one of the previous results.

Proof of Theorem 2.1 Taking $v = \eta_h^1$ in Eq. (3.54) and using the same approach applied in Lemmas 3.7 and 3.8, we obtain that (2.6) holds for $n = 1$,

$$\big\|u^1 - U_h^1\big\|_{L^2} \le C_0\big(N^{-\delta\alpha} + h^{r+1}\big).$$

Next, we consider the case of $2 \le n \le N$.

Recalling (3.47) and (3.10), we arrive at

$$\begin{aligned} \big\|R_3^n\big\|_{L^2} &\le \big\|f\big(|\hat{u}^n|^2\big)u^{n,\alpha} - f\big(|\hat{u}^n|^2\big)U_h^{n,\alpha}\big\|_{L^2} + \big\|f\big(|\hat{u}^n|^2\big)U_h^{n,\alpha} - f\big(|\hat{U}_h^n|^2\big)U_h^{n,\alpha}\big\|_{L^2} \\ &\le C_f\big(\|u^{n,\alpha} - R_h u^{n,\alpha}\|_{L^2} + \|\eta_h^{n,\alpha}\|_{L^2}\big) \\ &\quad + \big\|f'(\xi_1)\big\|_{L^\infty}\big(\|\hat{u}^n - R_h\hat{u}^n\|_{L^2} + \|\hat{\eta}_h^n\|_{L^2}\big)\big\|U_h^{n,\alpha}\big\|_{L^2} \\ &\le C_f\big(C_\Omega K h^{r+1} + \|\eta_h^{n,\alpha}\|_{L^2}\big) + C_f C_\Omega K h^{r+1}K_2 + C_f K_2\|\hat{\eta}_h^n\|_{L^2} \\ &\le C_f C_\Omega K(1 + K_2)h^{r+1} + C_f K_2\big\|\hat{\vartheta}_h^n\big\|_{L^2} + C_f\big\|\vartheta_h^{n,\alpha}\big\|_{L^2}. \end{aligned} \tag{3.55}$$

Substituting $v = \eta_h^{n,\alpha}$ into (3.53) and taking the imaginary part, we derive

$$\begin{aligned} \big|\mathrm{Re}\big((D^{\alpha}_{\Delta t}\eta_h)^{n-\frac{\alpha}{2}}, \eta_h^{n,\alpha}\big)\big| &\le \big|\big(R_3^n, \eta_h^{n,\alpha}\big)\big| + \big|\big(P^n, \eta_h^{n,\alpha}\big)\big| + \big|\big(R_4^n, \eta_h^{n,\alpha}\big)\big| \\ &\le \big\|R_3^n\big\|_{L^2}\big\|\vartheta_h^{n,\alpha}\big\|_{L^2} + \|P^n\|_{L^2}\big\|\vartheta_h^{n,\alpha}\big\|_{L^2} + \big\|R_4^n\big\|_{L^2}\big\|\vartheta_h^{n,\alpha}\big\|_{L^2} \\ &\le C_f K_2\big\|\hat{\vartheta}_h^n\big\|_{L^2}^2 + \big(C_f K_2/4 + C_f\big)\big\|\vartheta_h^{n,\alpha}\big\|_{L^2}^2 + \big\|R_4^n\big\|_{L^2}\big\|\vartheta_h^{n,\alpha}\big\|_{L^2} \\ &\le C_{11}\big\|\vartheta_h^n\big\|_{L^2}^2 + C_{12}\big\|\vartheta_h^{n-1}\big\|_{L^2}^2 + C_{13}\big\|\vartheta_h^{n-2}\big\|_{L^2}^2 \\ &\quad + C_{14}h^{r+1}\big\|\vartheta_h^{n,\alpha}\big\|_{L^2} + \big(\|P^n\|_{L^2} + \|R_4^n\|_{L^2}\big)\big\|\vartheta_h^{n,\alpha}\big\|_{L^2}, \end{aligned} \tag{3.56}$$

where Cauchy-Schwarz inequality is used and $C_{11}, C_{12}, C_{13}, C_{14}$ are positive constants dependent on α, K_2, C_f. Together with (2.5), we get

$$\max_{1\le k\le n}\sum_{j=1}^{k} P_{k-j}^{(k)}\big\|R_4^j\big\|_{L^2} \le \max_{1\le k\le n}\sum_{l=1}^{k}\sum_{j=l}^{k} P_{k-j}^{(k)}B_{j-l}^{(j)}\big\|\nabla_\tau\big(u^l - R_h u^l\big)\big\|_{L^2}$$

$$
\begin{aligned}
&\leq \max_{1\leq k\leq n}\sum_{l=1}^{k}\left\|\nabla_\tau\left(u^l-R_hu^l\right)\right\|_{L^2}\\
&\leq \max_{1\leq k\leq n}\sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}\left\|(u-R_hu)'(t)\right\|_{L^2}dt\\
&\leq C_\Omega h^{r+1}\int_0^{t_n}\left\|u'(t)\right\|_{H^{r+1}}dt\leq C_\Omega\left(t_n+t_n^\delta\right)h^{r+1}.
\end{aligned}
$$

By Lemmas 3.3, 3.4 and 3.6, there exists a constant $N_3^*>0$, when $N\geq N_3^*$, it holds

$$
\begin{aligned}
\left\|\eta_h^n\right\|_{L^2}\leq{}& 4E_\alpha\left(2\pi_A3C_{12}t_n^\alpha\right)\\
&\times\left[\left\|\eta_h^0\right\|_{L^2}+\max_{1\leq k\leq n}\sum_{j=1}^{k}P_{k-j}^{(k)}\left(\|P^j\|_{L^2}+\|R_4^j\|_{L^2}\right)+\pi_A\Gamma(2-\alpha)t_n^\alpha C_{12}C_\Omega h^{r+1}\right]\\
\leq{}& 4E_\alpha\left(6\pi_AC_{12}t_n^\alpha\right)\left(C_{12}C_\Omega h^{r+1}+C_pN^{-\min\{\delta\alpha,2\}}\right)\\
\leq{}& C_{13}\left(h^{r+1}+N^{-\min\{\delta\alpha,2\}}\right),
\end{aligned}
$$

where $C_{13}=4E_\alpha(6\pi_AC_{12}t_n^\alpha)(C_{12}C_\Omega+C_p)$. With (3.10), the above inequality further implies that

$$
\begin{aligned}
\left\|u^n-U_h^n\right\|_{L^2}&\leq\left\|u^n-R_hu^n\right\|_{L^2}+\left\|R_hu^n-U_h^n\right\|_{L^2}\\
&\leq(C_\Omega C+C_{13})\left(N^{-\min\{\delta\alpha,2\}}+h^{r+1}\right)
\end{aligned}
\tag{3.57}
$$

for $1\leq n\leq N$. Therefore, (2.6) holds when $N\geq N_0=\max\{N_1^*,N_2^*,N_3^*\},h\leq h_0=h_1^*$ and $C_0\geq C_\Omega C+C_{13}$. Theorem 2.1 is proved. □

4. Numerical Results

In this section, some numerical experiments are presented to illustrate the convergence results. All numerical examples are calculated by using the software FreeFem++. And the errors are computed by L^2-norm with setting $\delta\alpha=2$.

Example 4.1. Consider the following two-dimensional time-fractional Schrödinger equation:

$$
\begin{aligned}
&i{}^CD_t^\alpha u+\Delta u-|u|^2u+|u|^4u=g(x,y,t), && (x,y,t)\in(0,1)\times(0,1)\times(0,1],\\
&u(0,y,t)=u(1,y,t)=u(x,0,t)=u(x,1,t)=0, && t\in[0,1],\\
&u(x,y,0)=x^2(1-x)^3y^2(1-y)^3, && (x,y)\in(0,1)\times[0,1],
\end{aligned}
\tag{4.1}
$$

where g is obtained correspondingly to the exact solution

$$
u(x,y,t)=(1+t^\alpha)x^2(1-x)^3y^2(1-y)^3.
$$

We solve the time-fractional Schrödinger equation (4.1) by using linear finite element method (L-FEM) and quadratic finite element method (Q-FEM) with different stepsizes. In temporal direction, the numerical errors and convergence rates are verified with taking $M=80$ with $N=5,10,20,40$. Here and below, a triangular partition with $M+1$ points in every direction is applied. Similarly, we take the spatial stepsize as $M=5,10,20,40$ and $N=1000$ with $\alpha=0.8$ for $r=1,r=2$, respectively. The numerical results are displayed in Tables 4.1 and 4.2, respectively. All these experimental consequences agree with theoretical findings.

Table 4.1: Errors and convergence rates in temporal direction (Problem (4.1)).

	α=0.4		α=0.6		α=0.8	
N	Errors	Orders	Errors	Orders	Errors	Orders
5	$7.0112e-06$	$*$	$5.2198e-06$	$*$	$2.9147e-06$	$*$
10	$1.9080e-06$	1.8776	$1.1692e-06$	2.1584	$7.7054e-07$	1.9194
20	$4.9848e-07$	1.9364	$2.9833e-07$	1.9706	$1.9776e-07$	1.9621
40	$1.2711e-07$	1.9714	$7.5210e-08$	1.9879	$4.2101e-08$	2.2318

Table 4.2: Errors and convergence rates with $\alpha = 0.8$ (Problem (4.1)).

	L-FEM		Q-FEM	
M	Errors	Orders	Errors	Orders
5	$2.8760e-04$	$*$	$1.9307e-05$	$*$
10	$8.7236e-05$	1.7211	$2.3501e-06$	3.0384
20	$2.3003e-05$	1.9216	$2.8795e-08$	3.0289
40	$5.8384e-06$	1.9782	$3.5770e-09$	3.0090

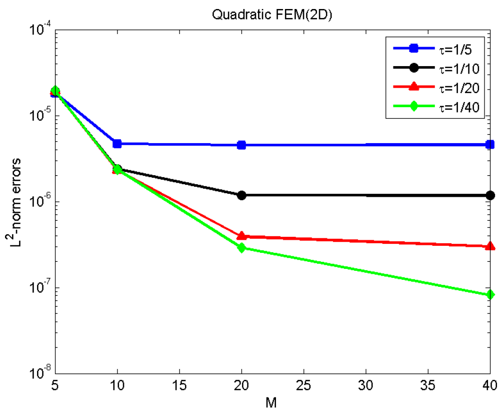

Fig. 4.1. 2D problem: L^2-errors of Q-FEM with fixed τ and different spatial step sizes ($\alpha = 0.6$).

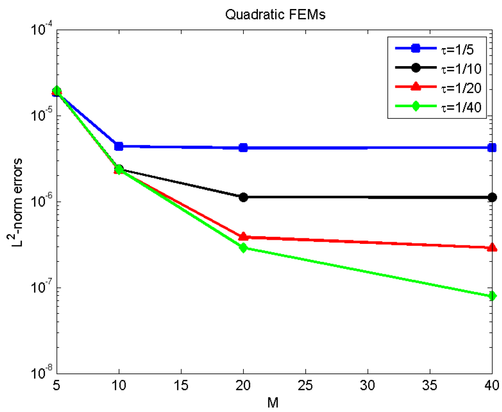

Fig. 4.2. 2D problem: L^2-errors of Q-FEM with fixed τ and different spatial step sizes ($\alpha = 0.4$).

In order to confirm the unconditional convergence of the proposed method, we solve problem (4.1) by using Q-FEM in spatial direction with different stepsizes. The numerical results can be seen in Figs. 4.1 and 4.2, respectively. These results imply that for each fixed τ, the error tends to be a constant. It implies that the errors hold without any time-step restrictions dependent on the spatial mesh size.

Example 4.2. Consider the following three-dimensional time-fractional Schrödinger equation:

$$\begin{aligned} & i^C D_t^\alpha u + \Delta u + |u|^2 u = g(x,y,z,t), \quad && (x,y,z,t) \in (0,\pi)\times(0,\pi)\times(0,\pi)\times(0,1], \\ & u(0,y,z,t) = u(\pi,y,z,t) = u(x,0,z,t) && \\ & \qquad = u(x,\pi,z,t) = u(x,y,0,t) && \\ & \qquad = u(x,y,\pi,t) = 0, && t \in [0,1], \\ & u(x,y,z,0) = 0, && (x,y,z) \in (0,1)\times(0,1)\times(0,1), \end{aligned} \tag{4.2}$$

Table 4.3: Errors and convergent orders with linear element. (Problem (4.2)).

	α=0.4		α=0.6		α=0.8	
$M=N$	Errors	Orders	Errors	Orders	Errors	Orders
$M=5$	$1.1084e-03$	$*$	$1.1284e-03$	$*$	$1.1439e-03$	$*$
$M=10$	$3.0391e-04$	1.8667	$3.1175e-04$	1.8559	$3.2256e-04$	1.8263
$M=20$	$7.7705e-05$	1.9676	$7.9941e-05$	1.9634	$8.1830e-05$	1.9788
$M=40$	$1.9523e-05$	1.9928	$2.0111e-05$	1.9910	$2.0598e-05$	1.9901

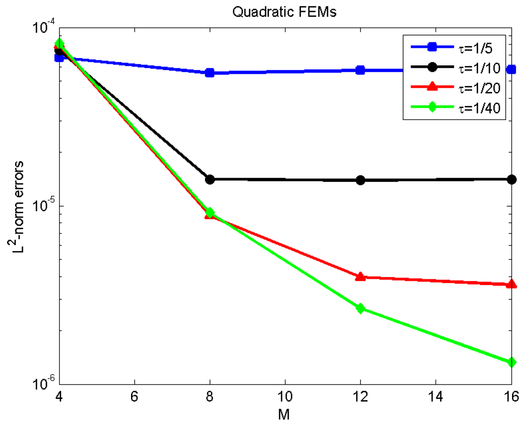

Fig. 4.3. 3D problem: L^2-errors of Q-FEM with fixed τ and different spatial step sizes ($\alpha = 0.6$).

where g is chosen correspondingly to the exact solution

$$u(x, y, z, t) = t^{\alpha} \sin x \sin y \sin z.$$

L-FEM with $M = N$ is applied to solve problem (4.2). The numerical results in Table 4.3 confirm the convergence results of the proposed methods. Moreover, we solve problem (4.2) by using Q-FEM in spatial direction with different stepsizes. The numerical results can be seen in Fig. 4.3. These results imply that for each fixed τ, the error tends to be a constant. It implies that the errors hold without any time-step restrictions dependent on the spatial mesh size, again.

5. Conclusions

In this paper, the nonlinear TFSEs are effectively solved by a linearized Alikhanov FEM. We have applied a discrete Grönwall inequality to obtain the optimal error estimates. Such convergence results hold unconditionally. Numerical results are given to confirm the theoretical results.

Appendix A

Proof of Lemma 3.7. Multiplying (3.23) by e^1, integrating it over Ω, and considering the imaginary term of the corresponding equation, we obtain

$$\|e^1\|_{L^2} \leq \mu C_K \tau_1^{2\alpha},$$

where the result (3.21) is used. Moreover, we multiply (3.23) by Δe^1 and integrate it over Ω to get

$$-i\frac{\|\nabla e^1\|_{L^2}^2}{\mu\tau_1^\alpha} + \Big(1-\frac{\alpha}{2}\Big)\|\Delta e^1\|_{L^2}^2 + \Big(1-\frac{\alpha}{2}\Big)\big(f(|u^0|^2)e^1, \Delta e^1\big) = \big(P^1, e^1\big). \tag{A.1}$$

We consider the imaginary and real part to obtain

$$\|\nabla e^1\|_{L^2} + \tau_1^\alpha \|\Delta e^1\|_{L^2} \leq C\tau_1^{2\alpha},$$

which further implies

$$\|e^1\|_{H^2} \leq C_1^* N^{-\delta\alpha}.$$

Applying above inequality, we arrive at

$$\begin{aligned}
&\|U^1\|_{H^2} \leq \|u^1\|_{H^2} + \|e^1\|_{H^2} \leq \|u^1\|_{H^2} + C_1^* N^{-\delta\alpha} \leq K_1,\\
&\big\|(D_{\Delta t}^\alpha U)^{1-\frac{\alpha}{2}}\big\|_{H^2} \leq \big\|(D_{\Delta t}^\alpha u)^{1-\frac{\alpha}{2}}\big\|_{H^2} + \big\|(D_{\Delta t}^\alpha e)^{1-\frac{\alpha}{2}}\big\|_{H^2} \leq C_1^{**},
\end{aligned}$$

when $N \geq \tilde{N}_1 = (C_\Omega C_1^*)^{1/(\delta\alpha)}$. Thus, the proof is complete. $\square$

Proof of Lemma 3.8. To estimate $\|\vartheta_h^1\|_{L^2}$, we take $v = \vartheta_h^1$ in Eq. (3.43) and the imaginary term to arrive at

$$\begin{aligned}
\big\|\vartheta_h^1\big\|_{L^2}^2 = &-\mu\tau_1^\alpha \mathrm{Im}\big(f(|U^0|^2)U^{1,\alpha} - f(|U_h^0|^2)U_h^{1,\alpha}, \vartheta_h^1\big)\\
&- \mathrm{Re}\big(U^1 - R_h U^1, \vartheta_h^1\big) + \mathrm{Re}\big(U^0 - U_h^0, \vartheta_h^1\big).
\end{aligned}$$

As for the boundedness of $\|U_h^0\|_{L^\infty}$, we consider the fact $U_h^0 = \Pi_h u^0$ and inequality (3.11) to obtain

$$\begin{aligned}
\big\|U_h^0\big\|_{L^\infty} &\leq \big\|R_h U^0\big\|_{L^\infty} + \big\|R_h U^0 - U_h^0\big\|_{L^\infty} \leq \big\|R_h U^0\big\|_{L^\infty} + Ch^{-\frac{d}{2}}\big\|R_h U^0 - U_h^0\big\|_{L^2}\\
&\leq \big\|R_h U^0\big\|_{L^\infty} + 2C_\Omega^2 K_1 h^{-\frac{d}{2}} h^2 \leq K_2.
\end{aligned}$$

Taking the boundedness of $\|U^1\|_{L^\infty}$ and $\|U_h^0\|_{L^\infty}$ into account, we get

$$\begin{aligned}
&\big|\mathrm{Re}\big(U^1 - R_h U^1, \vartheta_h^1\big)\big| \leq 4C_\Omega^2 K_1^2 h^4 + \frac{1}{4}\big\|\vartheta_h^1\big\|_{L^2}^2,\\
&\big|\mathrm{Re}\big(U^0 - U_h^0, \vartheta_h^1\big)\big| \leq 4C_\Omega^2 K_1^2 h^4 + \frac{1}{4}\big\|\vartheta_h^1\big\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
&\mu\tau_1^\alpha \big|\mathrm{Im}\big(f(|U^0|^2)U^{1,\alpha} - f(|U_h^0|^2)U_h^{1,\alpha}, \vartheta_h^1\big)\big|\\
=\;& \mu\tau_1^\alpha \big|\mathrm{Im}\big(f'(\xi_3)(|U^0|^2 - |U_h^0|^2)U^{1,\alpha} + f(|U_h^0|^2)(U^{1,\alpha} - U_h^{1,\alpha}), \vartheta_h^1\big)\big|\\
\leq\;& C_{K,1}\tau_1^\alpha \big\|\vartheta_h^1\big\|_{L^2}^2 + C_{K,2}h^4 \leq \frac{1}{4}\big\|\vartheta_h^1\big\|_{L^2}^2 + C_{K,2}h^4,
\end{aligned}$$

when $N \geq N_6 = (C_{K,1}T)^{1/(\delta\alpha)}$. Then, we can easily get

$$\big\|\vartheta_h^1\big\|_{L^2} \leq 2\sqrt{8C_\Omega^2 K_1^2 + C_{K,2}}\, h^2 \leq h^{\frac{11}{6}},$$

when

$$h \leq h_3 = \left(\frac{1}{2\sqrt{8C_\Omega^2 K_1^2 + C_{K,2})}} \right)^{-\frac{1}{6}}.$$

Furthermore,

$$\left\| U_h^1 \right\|_{L^\infty} \leq \left\| R_h U^1 \right\|_{L^\infty} + \left\| \vartheta_h^1 \right\|_{L^\infty} \leq \left\| R_h U^1 \right\|_{L^\infty} + C_\Omega h^{-\frac{d}{2}} h^{\frac{11}{6}} \leq K_2.$$

Then the proof is complete. □

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