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The Exact Limits and Improved Decay Estimates for All Order Derivatives of Global Weak Solutions of Three Incompressible Fluid Dynamics Equations

Linghai Zhang[1,†]

Abstract First of all, the author accomplishes the exact limits for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations, the n-dimensional incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation. Secondly, by making use of the exact limits, he establishes the improved decay estimates with sharp rates for all order derivatives of the global weak solutions, for all sufficiently large t. The author proves these results by making use of existing ideas, existing results and several new, novel ideas.

Keywords Incompressible fluid dynamics equations, global weak solutions, all order derivatives, exact limits, improved decay estimates with sharp rates

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1. Introduction

Consider the n-dimensional incompressible magnetohydrodynamics equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{u} - \frac{1}{\mathrm{RE}}\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{A}\cdot\nabla)\mathbf{A} + \nabla P = \mathbf{f}(\mathbf{x},t),\\
&\frac{\partial}{\partial t}\mathbf{A} - \frac{1}{\mathrm{RM}}\triangle\mathbf{A} + (\mathbf{u}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{u} = \mathbf{g}(\mathbf{x},t),\\
&\nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{f} = 0, \qquad \nabla\cdot\mathbf{A} = 0, \qquad \nabla\cdot\mathbf{g} = 0,
\end{aligned}$$

the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x},t), \qquad \nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{f} = 0,$$

and the two-dimensional incompressible dissipative quasi-geostrophic equation

$$\frac{\partial}{\partial t}u + \alpha(-\triangle)^{\rho}u + J(u,(-\triangle)^{-1/2}u) = f(\mathbf{x},t).$$

These incompressible fluid dynamics equations play very important roles in applied mathematics. See Caffarelli, Kohn, Nirenberg [5], Leray [25], Temam [40] and [41] for the physical backgrounds. It is well known that there exists a unique

†The Corresponding Author
Email Address: liz5@lehigh.edu
[1]Department of Mathematics, Lehigh University. 17 Memorial Drive East, Bethlehem, PA 18015, USA

global smooth solution or a global weak solution to each equation, under certain conditions. Even if the dimension is high, the nonlinear couplings are strong and the initial functions and the external forces are large, the global weak solutions exist. Moreover, there hold some elementary uniform energy estimates for the global weak solutions. The global weak solutions become small enough and sufficiently smooth after a long time T.

The elementary decay estimates with sharp rates have been established very well for the global weak solutions of these equations. For very similar n-dimensional incompressible fluid dynamics equations, the existence of a global smooth solution or the existence of a global weak solution, the elementary uniform energy estimates and the elementary decay estimates with sharp rates have also been established. For more results on the incompressible magnetohydrodynamics equations, the incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation, please see all the references [1]-[44] for these known related results.

Let n be a positive integer, such that $2 \leq n \leq 5$. Let $\alpha > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, $0 < \delta < 4$, $0 < \varepsilon < 1$, $0 < \rho < 1$ and $T > 0$ be positive constants. Let $m \geq 0$ be a constant.

We will accomplish the exact limits for all order derivatives of the global weak solutions of these incompressible fluid dynamics equations. We may use the global smooth solution of the corresponding linear problem to approximate the global weak solutions of the nonlinear problem. We also establish the improved decay estimates with sharp rates for all order derivatives of the global weak solutions.

The exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions are very challenging problems, because if the spatial dimension is high and the nonlinear functions are strong, the existence and uniqueness of the global smooth solution may be unknown and there exist no available uniform energy estimates for any order derivatives of the global weak solutions.

The main purposes of the next three sections are to accomplish the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions, which are also local smooth solutions on some unbounded interval (T, ∞), for all of the incompressible fluid dynamics equations mentioned above. In each section, we will make mathematical assumptions, provide the precise statements of the main results. All the results on the global weak solutions of these incompressible fluid dynamics equations are completely new.

The main ideas and the main steps in the proofs of the main results for the n-dimensional incompressible magnetohydrodynamics equations, the n-dimensional incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation are almost the same, although some details may be slightly different. To keep the paper from being too long, we will only provide a sketch of the proofs of the main results for the n-dimensional incompressible magnetohydrodynamics equations. We will skip the proofs of the main results for the n-dimensional incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation.

Definition 1.1. Let $\phi \in L^1(\mathbb{R}^n)$. Define the Fourier transformation

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} \exp(-\mathrm{i}\mathbf{x} \cdot \xi)\phi(\mathbf{x})\mathrm{d}\mathbf{x}, \qquad \xi \in \mathbb{R}^n.$$

Definition 1.2. Let $m \in \mathbb{R}$ be a real constant. Define the fractional order derivative of ϕ by using the Fourier transformation

$$\widehat{(-\triangle)^m}\phi(\xi) = |\xi|^{2m}\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R}^n.$$

We will use the following Banach spaces and Hilbert spaces

$$\begin{aligned}
&L^1(\mathbb{R}^n) \stackrel{\text{def}}{=} \left\{\phi : \int_{\mathbb{R}^n} |\phi(\mathbf{x})| \mathrm{d}\mathbf{x} < \infty\right\},\\
&L^2(\mathbb{R}^n) \stackrel{\text{def}}{=} \left\{\phi : \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^2 \mathrm{d}\mathbf{x} < \infty\right\},\\
&L^\infty(\mathbb{R}^n) \stackrel{\text{def}}{=} \left\{\phi : \sup_{\mathbf{x}\in\mathbb{R}^n} |\phi(\mathbf{x})| < \infty\right\},\\
&L^1(\mathbb{R}^n \times \mathbb{R}^+) \stackrel{\text{def}}{=} \left\{\psi : \int_0^\infty \int_{\mathbb{R}^n} |\psi(\mathbf{x},t)| \mathrm{d}\mathbf{x}\mathrm{d}t < \infty\right\},\\
&L^2(\mathbb{R}^n \times \mathbb{R}^+) \stackrel{\text{def}}{=} \left\{\psi : \int_0^\infty \int_{\mathbb{R}^n} |\psi(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t < \infty\right\},\\
&L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)) \stackrel{\text{def}}{=} \left\{\psi : \int_0^\infty \left[\int_{\mathbb{R}^n} |\psi(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right]^{1/2} \mathrm{d}t < \infty\right\},\\
&L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)) \stackrel{\text{def}}{=} \left\{\psi : \sup_{t>0} \int_{\mathbb{R}^n} |\psi(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} < \infty\right\}.
\end{aligned}$$

2. The incompressible magnetohydrodynamics equations

2.1. The mathematical model equations and known related results

Consider the Cauchy problems for the n-dimensional incompressible magnetohydrodynamics equations

$$\frac{\partial}{\partial t}\mathbf{u} - \frac{1}{\mathrm{RE}}\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{A}\cdot\nabla)\mathbf{A} + \nabla P = \mathbf{f}(\mathbf{x},t), \tag{2.1}$$

$$\frac{\partial}{\partial t}\mathbf{A} - \frac{1}{\mathrm{RM}}\triangle\mathbf{A} + (\mathbf{u}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{u} = \mathbf{g}(\mathbf{x},t), \tag{2.2}$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{f} = 0, \qquad \nabla\cdot\mathbf{A} = 0, \qquad \nabla\cdot\mathbf{g} = 0, \tag{2.3}$$

$$\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}), \qquad \mathbf{A}(\mathbf{x},0) = \mathbf{A}_0(\mathbf{x}), \qquad \nabla\cdot\mathbf{u}_0 = 0, \qquad \nabla\cdot\mathbf{A}_0 = 0. \tag{2.4}$$

The real vector valued function $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ represents the velocity of the fluid at position $\mathbf{x} \in \mathbb{R}^n$ and time $t \in \mathbb{R}^+$, the real vector valued function $\mathbf{A} = \mathbf{A}(\mathbf{x},t)$ represents the magnetic field at position $\mathbf{x} \in \mathbb{R}^n$ and time $t \in \mathbb{R}^+$. The real scalar function

$$P(\mathbf{x},t) \stackrel{\text{def}}{=} p(\mathbf{x},t) + \frac{M^2}{2\mathrm{RE}\cdot\mathrm{RM}}|\mathbf{A}(\mathbf{x},t)|^2$$

represents the total pressure, where the real scalar function $p = p(\mathbf{x},t)$ represents the pressure of the fluid and $\frac{1}{2}|\mathbf{A}(\mathbf{x},t)|^2$ represents the magnetic pressure. Additionally,

$M > 0$ represents the Hartman constant, $\mathrm{RE} > 0$ represents the Reynolds constant and $\mathrm{RM} > 0$ represents the magnetic Reynolds constant.

It is well known that there exists a global smooth solution

$$\begin{gathered}\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\ \ \nabla\mathbf{u} \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\\ \mathbf{A} \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\ \ \nabla\mathbf{A} \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\\ \mathbf{u} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+),\ \ \mathbf{A} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+),\end{gathered}$$

if the dimension $n = 2$ and if the initial functions and the external forces are divergence free and satisfy

$$\begin{gathered}\mathbf{u}_0 \in C^1(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2),\ \ \mathbf{f} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\\ \mathbf{A}_0 \in C^1(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2),\ \ \mathbf{g} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\end{gathered}$$

for all real constants $m \geq 0$.

There exists a global weak solution

$$\begin{gathered}\mathbf{u} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)),\ \ \nabla\mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)),\\ \mathbf{A} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)),\ \ \nabla\mathbf{A} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)),\end{gathered}$$

if the dimension $n = 3, 4, 5$ and if the initial functions and the external forces are divergence free and satisfy the following conditions

$$\begin{gathered}\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),\ \ \mathbf{f} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)),\\ \mathbf{A}_0 \in C^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),\ \ \mathbf{g} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)).\end{gathered}$$

Moreover, there holds the following elementary uniform energy estimate for the global weak solutions

$$\begin{aligned}&\bigg\{\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\\ &+ \frac{2}{\mathrm{RE}}\int_0^t \int_{\mathbb{R}^n} |\nabla\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau + \frac{2}{\mathrm{RM}}\int_0^t \int_{\mathbb{R}^n} |\nabla\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}\mathrm{d}\tau\bigg\}^{1/2}\\ &\leq \bigg\{\int_{\mathbb{R}^n} |\mathbf{u}_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |\mathbf{A}_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\bigg\}^{1/2}\\ &+ \int_0^\infty \bigg\{\int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} |\mathbf{g}(\mathbf{x},t)|^2\bigg\}^{1/2} \mathrm{d}t.\end{aligned}$$

Additionally, the global weak solutions become small enough and sufficiently smooth after a long time. That is, there exists a sufficiently large positive constant T, such that

$$\begin{gathered}\sup_{t>T}\bigg\{\int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\bigg\} < \infty,\\ \sup_{t>T}\bigg\{\int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\bigg\} < \infty,\end{gathered}$$

for all positive constants $m > 0$. Furthermore, there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0}\bigg\{t^{1+n/2}\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\bigg\} < \infty,$$

$$\sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty.$$

Also consider the Cauchy problems for the corresponding linear equations

$$\frac{\partial}{\partial t}\mathbf{v}-\frac{1}{\mathrm{RE}}\triangle\mathbf{v}=\mathbf{f}(\mathbf{x},t),\qquad \nabla\cdot\mathbf{v}=0,\qquad \nabla\cdot\mathbf{f}=0,\tag{2.5}$$

$$\frac{\partial}{\partial t}\mathbf{B}-\frac{1}{\mathrm{RM}}\triangle\mathbf{B}=\mathbf{g}(\mathbf{x},t),\qquad \nabla\cdot\mathbf{B}=0,\qquad \nabla\cdot\mathbf{g}=0,\tag{2.6}$$

$$\mathbf{v}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\qquad \nabla\cdot\mathbf{u}_0=0,\tag{2.7}$$

$$\mathbf{B}(\mathbf{x},0)=\mathbf{A}_0(\mathbf{x}),\qquad \nabla\cdot\mathbf{A}_0=0.\tag{2.8}$$

2.2. The main purposes - the main difficulties - the main strategies

For the n-dimensional incompressible magnetohydrodynamics equations, we will accomplish the following limits

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},$$

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\},$$

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},$$

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\},$$

in terms of certain known information, and establish the following improved decay estimates with sharp rates for all order derivatives of the global weak solutions

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}_1+\mathcal{B}_1t^{-n},$$

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\le\mathcal{C}_1+\mathcal{D}_1t^{-n},$$

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}_2+\mathcal{B}_2t^{-n},$$

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\le\mathcal{C}_2+\mathcal{D}_2t^{-n},$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, and for all sufficiently large $t>T$, T is a sufficiently large positive constant, where

$$\mathcal{A}_1=\mathcal{A}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{A}_2=\mathcal{A}_2(\alpha_2,\delta,\varepsilon,m,n),$$
$$\mathcal{B}_1=\mathcal{B}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{B}_2=\mathcal{B}_2(\alpha_2,\delta,\varepsilon,m,n),$$
$$\mathcal{C}_1=\mathcal{C}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{C}_2=\mathcal{C}_2(\alpha_2,\delta,\varepsilon,m,n),$$
$$\mathcal{D}_1=\mathcal{D}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{D}_2=\mathcal{D}_2(\alpha_2,\delta,\varepsilon,m,n),$$

are positive constants, independent of $(\mathbf{u}(\mathbf{x},t),\mathbf{A}(\mathbf{x},t))$ and its derivatives, independent of $(\mathbf{v}(\mathbf{x},t),\mathbf{B}(\mathbf{x},t))$ and its derivatives, they are also independent of $(\mathbf{x},t)$. In the above, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ represent the global smooth solutions of the corresponding heat equations, which are obtained by dropping the nonlinear terms in the corresponding nonlinear equations.

To study the influence of α_1, α_2, m, n, $\mathbf{u}_0$, $\mathbf{f}$ on the solutions, we will make the constants $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ as explicit as possible. These exact limits and the positive constants $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, for $k = 1, 2$, will be represented explicitly in terms of certain known information, such as the order m of the derivative, the dimension n, the diffusion coefficients $\alpha_1 = \frac{1}{\mathrm{RE}}$, $\alpha_2 = \frac{1}{\mathrm{RM}}$, and

(1) the integrals of functions related to the initial functions,

(2) the integrals of functions related to the external forces,

(3) and the integrals of the nonlinear functions $u_i u_j$, $A_i A_j$, $A_i u_j$, $u_i A_j$, for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

The most difficult technical problems in the mathematical analysis are the control of the following integrals

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{2ij}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i u_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i A_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

for all real constants $m \geq 0$, where $0 < \varepsilon < 1$ is a positive constant, ψ_{1ij} and ψ_{2ij} are auxiliary functions which will be made clear shortly.

We are able to use novel ideas to establish the optimal estimates for these integrals. In particular, we are able to use the following singular integrals

$$\int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha_1 \varepsilon |\eta|^2)]^2}{\alpha_1^2 |\eta|^{n+\delta}} \mathrm{d}\eta, \qquad \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha_2 \varepsilon |\eta|^2)]^2}{\alpha_2^2 |\eta|^{n+\delta}} \mathrm{d}\eta,$$

and the primary decay estimates with sharp rates to obtain the best possible estimates for these integrals.

First of all, we will make complete use of the elementary decay estimates, the comprehensive analysis and iteration technique to establish the primary decay estimates with sharp sates for all order derivatives of the global weak solutions. Secondly, we will couple together various ideas, methods and techniques to accomplish the exact limits. Thirdly, we will couple together the exact limits and novel ideas to establish the improved decay estimates with sharp rates for all order derivatives of the global weak solutions.

2.3. The mathematical assumptions

Let us make the following assumptions for the n-dimensional incompressible magnetohydrodynamics equations.

(A1) Suppose that the initial functions are divergence free and satisfy the conditions

$$\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \qquad \mathbf{A}_0 \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Suppose that the external forces are divergence free and satisfy the conditions

$$\begin{aligned} &\mathbf{f} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)), \\ &\mathbf{g} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)). \end{aligned}$$

(A2) Suppose that there exist real scalar smooth functions

$$\begin{aligned} &\phi_{1ij} \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),\ \ \psi_{1ij} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+), \\ &\phi_{2ij} \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),\ \ \psi_{2ij} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+), \end{aligned}$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$, such that

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \phi_{1ij}(\mathbf{x}) = 0,\ \ \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \psi_{1ij}(\mathbf{x}, t) = 0, \\ &\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \phi_{2ij}(\mathbf{x}) = 0,\ \ \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \psi_{2ij}(\mathbf{x}, t) = 0, \end{aligned}$$

and that the initial functions and the external forces satisfy

$$\begin{aligned} \mathbf{u}_0(\mathbf{x}) &= \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{11j}(\mathbf{x}), \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{12j}(\mathbf{x}), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{1nj}(\mathbf{x}) \right), \\ \mathbf{f}(\mathbf{x}, t) &= \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{11j}(\mathbf{x}, t), \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{12j}(\mathbf{x}, t), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{1nj}(\mathbf{x}, t) \right), \\ \mathbf{A}_0(\mathbf{x}) &= \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{21j}(\mathbf{x}), \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{22j}(\mathbf{x}), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{2nj}(\mathbf{x}) \right), \\ \mathbf{g}(\mathbf{x}, t) &= \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{21j}(\mathbf{x}, t), \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{22j}(\mathbf{x}, t), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{2nj}(\mathbf{x}, t) \right), \end{aligned}$$

for all $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

These assumptions are motivated by the special structures of the nonlinear functions

$$(\mathbf{u}\cdot\nabla)\mathbf{u}, \qquad (\mathbf{A}\cdot\nabla)\mathbf{A}, \qquad (\mathbf{u}\cdot\nabla)\mathbf{A}, \qquad (\mathbf{A}\cdot\nabla)\mathbf{u}.$$

These assumptions are also motivated by the incompressible conditions

$$\nabla\cdot\mathbf{u}_0 = 0, \qquad \nabla\cdot\mathbf{f} = 0, \qquad \nabla\cdot\mathbf{A}_0 = 0, \qquad \nabla\cdot\mathbf{g} = 0,$$

which imply that

$$\begin{gathered}\int_{\mathbb{R}^n}\mathbf{u}_0(\mathbf{x})\mathrm{d}\mathbf{x} = \mathbf{0},\ \int_{\mathbb{R}^n}\mathbf{f}(\mathbf{x},t)\mathrm{d}\mathbf{x} = \mathbf{0},\\ \int_{\mathbb{R}^n}\mathbf{A}_0(\mathbf{x})\mathrm{d}\mathbf{x} = \mathbf{0},\ \int_{\mathbb{R}^n}\mathbf{g}(\mathbf{x},t)\mathrm{d}\mathbf{x} = \mathbf{0},\end{gathered}$$

for all $t > 0$.

(A3) Suppose that there exist the following limits

$$\begin{aligned}\lim_{t\to\infty}\left\{\sum_{i=1}^n\sum_{j=1}^n\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\} &\overset{\text{def}}{=} \mathrm{E}_4(m),\\ \lim_{t\to\infty}\left\{\sum_{i=1}^n\sum_{j=1}^n\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\} &\overset{\text{def}}{=} \mathrm{F}_4(m),\end{aligned}$$

for all real constants $m \geq 0$.

Here are some slightly weaker conditions than (A3):

$$\begin{aligned}\sup_{t>0}\left\{\sum_{i=1}^n\sum_{j=1}^n\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\} &< \infty,\\ \sup_{t>0}\left\{\sum_{i=1}^n\sum_{j=1}^n\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\} &< \infty,\end{aligned}$$

for all real constants $m \geq 0$.

(A4) Suppose that there exists a unique global smooth solution

$$\begin{gathered}\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\ \nabla\mathbf{u} \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\\ \mathbf{A} \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\ \nabla\mathbf{A} \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\end{gathered}$$

if $n = 2$ and if the initial functions and the external forces are divergence free and satisfy the conditions

$$\begin{gathered}\mathbf{u}_0 \in C^1(\mathbb{R}^n)\cap H^{2m}(\mathbb{R}^2), \qquad \mathbf{A}_0 \in C^1(\mathbb{R}^n)\cap H^{2m}(\mathbb{R}^2),\\ \mathbf{f} \in C^\infty(\mathbb{R}^2\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2))\cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),\\ \mathbf{g} \in C^\infty(\mathbb{R}^2\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2))\cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)).\end{gathered}$$

Suppose that there exists a global weak solution

$$\mathbf{u} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)),\ \nabla\mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)),$$

$$\mathbf{A} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)),\ \ \nabla\mathbf{A} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)),$$

if $n = 3, 4, 5$ and if the initial functions and the external forces are divergence free and satisfy the conditions

$$\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),\ \ \mathbf{f} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)),$$
$$\mathbf{A}_0 \in C^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),\ \ \mathbf{g} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)).$$

(A5) Suppose that the global weak solutions become small enough and sufficiently smooth after a long time. Namely, there exists a sufficiently large positive constant T, such that

$$\mathbf{u} \in L^\infty((T,\infty), H^{2m}(\mathbb{R}^n)),\ \ \nabla\mathbf{u} \in L^2((T,\infty), H^{2m}(\mathbb{R}^n)),$$
$$\mathbf{A} \in L^\infty((T,\infty), H^{2m}(\mathbb{R}^n)),\ \ \nabla\mathbf{A} \in L^2((T,\infty), H^{2m}(\mathbb{R}^n)),$$

for all positive constants $m > 0$. That is, there hold the following estimates

$$\sup_{t>T}\left\{\int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\} < \infty,$$
$$\sup_{t>T}\left\{\int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\} < \infty,$$

and

$$\int_T^\infty \int_{\mathbb{R}^n} |(-\triangle)^{m+1/2}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t < \infty,$$
$$\int_T^\infty \int_{\mathbb{R}^n} |(-\triangle)^{m+1/2}\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t < \infty,$$

for all positive constants $m > 0$.

(A6) Suppose that there hold the following representations for the Fourier transformations of the global weak solutions of the Cauchy problems for the n-dimensional incompressible magnetohydrodynamics equations

$$\begin{aligned}
&\widehat{u}_i(\xi,t) = \mathrm{i}\exp(-\alpha_1|\xi|^2 t)\sum_{j=1}^n \xi_j \widehat{\phi}_{1ij}(\xi)\\
&+ \mathrm{i}\int_0^t \exp[-\alpha_1|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j \widehat{\psi}_{1ij}(\xi,\tau)\mathrm{d}\tau\\
&- \mathrm{i}\int_0^t \exp[-\alpha_1|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j \widehat{u_i u_j}(\xi,\tau)\mathrm{d}\tau\\
&+ \mathrm{i}\int_0^t \exp[-\alpha_1|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j \widehat{A_i A_j}(\xi,\tau)\mathrm{d}\tau\\
&+ \mathrm{i}\int_0^t \exp[-\alpha_1|\xi|^2(t-\tau)]\frac{\xi_i}{|\xi|^2}\sum_{k=1}^n\sum_{l=1}^n \xi_k\xi_l \widehat{u_k u_l}(\xi,\tau)\mathrm{d}\tau\\
&- \mathrm{i}\int_0^t \exp[-\alpha_1|\xi|^2(t-\tau)]\frac{\xi_i}{|\xi|^2}\sum_{k=1}^n\sum_{l=1}^n \xi_k\xi_l \widehat{A_k A_l}(\xi,\tau)\mathrm{d}\tau,
\end{aligned}$$

$$\widehat{A}_i(\xi,t) = \mathrm{i}\exp(-\alpha_2|\xi|^2 t)\sum_{j=1}^n \xi_j\widehat{\phi}_{2ij}(\xi)$$
$$+\mathrm{i}\int_0^t \exp[-\alpha_2|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j\widehat{\psi}_{2ij}(\xi,\tau)\mathrm{d}\tau$$
$$-\mathrm{i}\int_0^t \exp[-\alpha_2|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j\widehat{A_i u_j}(\xi,\tau)\mathrm{d}\tau$$
$$+\mathrm{i}\int_0^t \exp[-\alpha_2|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j\widehat{u_i A_j}(\xi,\tau)\mathrm{d}\tau,$$
$$\widehat{P}(\xi,t) = -\frac{1}{|\xi|^2}\sum_{k=1}^n\sum_{l=1}^n \xi_k\xi_l\widehat{u_k u_l}(\xi,t) + \frac{1}{|\xi|^2}\sum_{k=1}^n\sum_{l=1}^n \xi_k\xi_l\widehat{A_k A_l}(\xi,t),$$

for all $(\xi,t)\in\mathbb{R}^n\times\mathbb{R}^+$ and for all $i=1,2,3,\cdots,n$. However, $\xi\neq\mathbf{0}$ in the representations of $\widehat{u}_i(\xi,t)$, $\widehat{A}_i(\xi,t)$ and $\widehat{P}(\xi,t)$.

(A7) Suppose that there hold the elementary decay estimates with the sharp rate $r=1+n/2$

$$\sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,\qquad \sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty.$$

2.4. The main results

Suppose that the mathematical assumptions (A1) - (A7) on the initial functions $(\mathbf{u}_0,\mathbf{A}_0)=(\mathbf{u}_0(\mathbf{x}),\mathbf{A}_0(\mathbf{x}))$, the external forces $(\mathbf{f},\mathbf{g})=(\mathbf{f}(\mathbf{x},t),\mathbf{g}(\mathbf{x},t))$, and the global weak solutions $(\mathbf{u},\mathbf{A})=(\mathbf{u}(\mathbf{x},t),\mathbf{A}(\mathbf{x},t))$ of the incompressible magnetohydrodynamics equations are true.

There are two parts in the main results.

Part 1: The exact limits for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations.

Theorem 2.1. *There hold the following exact limits*

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} = \frac{1}{n(n+2)}\mathcal{I}_1(m)\mathcal{J}_1,$$
$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\} = \frac{1}{n(n+2)}\mathcal{I}_1(m)\mathcal{K}_1,$$

for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations (2.1)-(2.4)*. In Theorem 2.1,*

$$\mathcal{I}_1(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta,$$
$$\mathcal{J}_1 = n\sum_{i=1}^n\sum_{j=1}^n\lambda_{1ij}^2 + 2\sum_{i=1}^n\sum_{j=1}^n\alpha_{1ij}^2 + \left[\sum_{i=1}^n\alpha_{1ii}\right]^2 - \left[\sum_{i=1}^n\lambda_{1ii}\right]^2,$$
$$\mathcal{K}_1 = n\sum_{i=1}^n\sum_{j=1}^n\rho_{1ij}^2 - \left[\sum_{i=1}^n\rho_{1ii}\right]^2,$$

where

$$\alpha_{1ij} = \int_{\mathbb{R}^n} \phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$\lambda_{1ij} = \int_{\mathbb{R}^n} \phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$\rho_{1ij} = \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i = 1,2,3,\cdots,n$ *and* $j = 1,2,3,\cdots,n$.

Obviously

$$\alpha_{1ij} = \lambda_{1ij} + \rho_{1ij}, \ \ \rho_{1ji} = \rho_{1ij}.$$

Theorem 2.2. *There hold the following exact limits*

$$\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} = \frac{1}{n}\mathcal{I}_2(m)\mathcal{J}_2,$$

$$\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{A}(\mathbf{x},t) - \mathbf{B}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right\} = \frac{1}{n}\mathcal{I}_2(m)\mathcal{K}_2,$$

for all order derivatives of the global weak solutions of the incompressible magneto-hydrodynamics equations (2.1)-(2.4).

In Theorem 2.2,

$$\mathcal{I}_2(m) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta,$$

$$\mathcal{J}_2 = \sum_{i=1}^n \sum_{j=1}^n \lambda_{2ij}^2,$$

$$\mathcal{K}_2 = \sum_{i=1}^n \sum_{j=1}^n \rho_{2ij}^2,$$

where

$$\alpha_{2ij} = \int_{\mathbb{R}^n} \phi_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{2ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$\lambda_{2ij} = \int_{\mathbb{R}^n} \phi_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^n} \psi_{2ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

$$\rho_{2ij} = \int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i = 1,2,3,\cdots,n$ *and* $j = 1,2,3,\cdots,n$.

Obviously

$$\alpha_{2ij} = \lambda_{2ij} + \rho_{2ij}, \ \ \rho_{2ji} = -\rho_{2ij}.$$

Theorem 2.3. *The ratio of the exact limits for* (2.1)-(2.4) *is the same as the ratio of the exact limits for the linear equations, for each constant* $m \geq 0$*. That is, there hold the following results*

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{4m+n+2}{4\alpha_1},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{(4m+n+2)(4m+n+4)}{(4\alpha_1)^2},
\end{aligned}$$

for all constants $m \geq 0$*; and*

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{4m+n+2}{4\alpha_1},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{(4m+n+2)(4m+n+4)}{(4\alpha_1)^2},
\end{aligned}$$

for all constants $m \geq 0$.

Moreover

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}
\end{aligned}$$

$$= \left\{ \lim_{t\to\infty} \left[t^{2m+2+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+1/2}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} = \frac{4m+n+2}{4\alpha_2},$$
$$\left\{ \lim_{t\to\infty} \left[t^{2m+3+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+1}\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$= \left\{ \lim_{t\to\infty} \left[t^{2m+3+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+1}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} = \frac{(4m+n+2)(4m+n+4)}{(4\alpha_2)^2},$$

for all constants $m \geq 0$; and

$$\left\{ \lim_{t\to\infty} \left[t^{2m+2+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+1/2}[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$= \left\{ \lim_{t\to\infty} \left[t^{2m+2+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+1/2}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} = \frac{4m+n+2}{4\alpha_2},$$
$$\left\{ \lim_{t\to\infty} \left[t^{2m+3+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+1}[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$= \left\{ \lim_{t\to\infty} \left[t^{2m+3+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+1}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{B}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} = \frac{(4m+n+2)(4m+n+4)}{(4\alpha_2)^2},$$

for all constants $m \geq 0$.

Part 2: The primary decay estimates and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations.

Theorem 2.4. *There hold the following primary decay estimates with sharp rates*

$$\sup_{t>T} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty,$$
$$\sup_{t>T} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty,$$

for all positive constants $m > 0$. Here T is a sufficiently large positive constant. It is the same constant as in (A5).

Theorem 2.5. *There hold the following improved decay estimates with sharp rates*

$$\begin{aligned} & t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \\ \leq\ & \mathcal{A}_1(\alpha_1,\delta,\varepsilon,m,n) + \mathcal{B}_1(\alpha_1,\delta,\varepsilon,m,n) t^{-n}, \\ & t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{u}(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \\ \leq\ & \mathcal{C}_1(\alpha_1,\delta,\varepsilon,m,n) + \mathcal{D}_1(\alpha_1,\delta,\varepsilon,m,n) t^{-n}, \end{aligned}$$

for all order derivatives of the global weak solutions of (2.1)-(2.4)*, for all positive constants* $0<\delta<4$ *and* $0<\varepsilon<1$*, and for all sufficiently large* t*. In Theorem 2.5, the positive constants are given by*

$$\begin{aligned} \mathcal{A}_1 &= \mathcal{A}_1(\alpha_1,\delta,\varepsilon,m,n) \\ &= 11\mathcal{I}_1(m)\left\{\mathrm{E}_1(\mathbf{u}_0) + \frac{\mathrm{E}_2(\mathbf{f}) + 2\mathrm{E}_3(\mathbf{u}) + 2\mathrm{F}_3(\mathbf{A})}{\varepsilon^{2m+1+n/2}}\right\}, \\ \mathcal{B}_1 &= \mathcal{B}_1(\alpha_1,\delta,\varepsilon,m,n) \\ &= 22C_0(m,n)\mathcal{S}_1(\alpha_1,\delta,\varepsilon,n)\left\{\mathrm{E}_4(m+(n-2+\delta)/4)\right. \\ &\quad \left. + \frac{4\mathcal{I}_1(0)\mathcal{I}_1(m+(n-2+\delta)/4)}{[n(n+2)]^2}\mathcal{J}_1^2 + \frac{4\mathcal{I}_2(0)\mathcal{I}_2(m+(n-2+\delta)/4)}{n^2}\mathcal{J}_2^2\right\}, \\ \mathcal{C}_1 &= \mathcal{C}_1(\alpha_1,\delta,\varepsilon,m,n) = 16\mathcal{I}_1(m)\frac{\mathrm{E}_3(\mathbf{u}) + \mathrm{F}_3(\mathbf{A})}{\varepsilon^{2m+1+n/2}}, \\ \mathcal{D}_1 &= \mathcal{D}_1(\alpha_1,\delta,\varepsilon,m,n) = 64C_0(m,n)\mathcal{S}_1(\alpha_1,\delta,\varepsilon,n) \\ &\quad \cdot\left\{\frac{\mathcal{I}_1(0)\mathcal{I}_1(m+(n-2+\delta)/4)}{[n(n+2)]^2}\mathcal{J}_1^2 + \frac{\mathcal{I}_2(0)\mathcal{I}_2(m+(n-2+\delta)/4)}{n^2}\mathcal{J}_2^2\right\}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{E}_1(\mathbf{u}_0) &= \sum_{i=1}^n\sum_{j=1}^n\left\{\int_{\mathbb{R}^n}|\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2, \\ \mathrm{E}_2(\mathbf{f}) &= \sum_{i=1}^n\sum_{j=1}^n\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2, \\ \mathrm{E}_3(\mathbf{u}) &= \left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2, \\ \mathrm{E}_4(m) &= \lim_{t\to\infty}\left\{\sum_{i=1}^n\sum_{j=1}^n\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\}, \\ \mathrm{F}_3(\mathbf{A}) &= \left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2, \\ \mathcal{I}_1(m) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta, \\ \mathcal{I}_2(m) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta, \\ \mathcal{J}_1 &= n\sum_{i=1}^n\sum_{j=1}^n\lambda_{1ij}^2 + 2\sum_{i=1}^n\sum_{j=1}^n\alpha_{1ij}^2 + \left[\sum_{i=1}^n\alpha_{1ii}\right]^2 - \left[\sum_{i=1}^n\lambda_{1ii}\right]^2, \end{aligned}$$

$$\mathcal{J}_2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{2ij}^2,$$

$$\mathcal{S}_1(\alpha_1,\delta,\varepsilon,n) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta,$$

$$\mathcal{S}_2(\alpha_2,\delta,\varepsilon,n) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta.$$

Theorem 2.6. *There hold the following improved decay estimates with sharp rates*

$$\begin{aligned}
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
&\le \mathcal{A}_2(\alpha_2,\delta,\varepsilon,m,n)+\mathcal{B}_2(\alpha_2,\delta,\varepsilon,m,n)t^{-n},\\
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
&\le \mathcal{C}_2(\alpha_2,\delta,\varepsilon,m,n)+\mathcal{D}_2(\alpha_2,\delta,\varepsilon,m,n)t^{-n},
\end{aligned}$$

for all order derivatives of the global weak solutions of (2.1)-(2.4), *for all positive constants* $0<\delta<4$ *and* $0<\varepsilon<1$, *and for all sufficiently large* t.

In Theorem 2.6, the positive constants are given by

$$\begin{aligned}
\mathcal{A}_2 &= \mathcal{A}_2(\alpha_2,\delta,\varepsilon,m,n)\\
&= 7\mathcal{I}_2(m)\left\{\mathrm{F}_1(\mathbf{A}_0)+\frac{\mathrm{F}_2(\mathbf{g})+\mathrm{F}_3(\mathbf{A})+\mathrm{E}_3(\mathbf{u})}{\varepsilon^{2m+1+n/2}}\right\},\\
\mathcal{B}_2 &= \mathcal{B}_2(\alpha_2,\delta,\varepsilon,m,n)\\
&= 14C_0(m,n)\mathcal{S}_2(\alpha_2,\delta,\varepsilon,n)\left\{\mathrm{F}_4(m+(n-2+\delta)/4)\right.\\
&\quad\left.+\frac{2}{n^2(n+2)}[\mathcal{I}_1(0)\mathcal{I}_2(m+(n-2+\delta)/4)+\mathcal{I}_2(0)\mathcal{I}_1(m+(n-2+\delta)/4)]\mathcal{J}_1\mathcal{J}_2\right\},\\
\mathcal{C}_2 &= \mathcal{C}_2(\alpha_2,\delta,\varepsilon,m,n) = 4\mathcal{I}_2(m)\frac{\mathrm{E}_3(\mathbf{u})+\mathrm{F}_3(\mathbf{A})}{\varepsilon^{2m+1+n/2}},\\
\mathcal{D}_2 &= \mathcal{D}_2(\alpha_2,\delta,\varepsilon,m,n) = \frac{16C_0(m,n)}{n^2(n+2)}\mathcal{S}_2(\alpha_2,\delta,\varepsilon,n)\\
&\quad\{\mathcal{I}_1(0)\mathcal{I}_2(m+(n-2+\delta)/4)+\mathcal{I}_2(0)\mathcal{I}_1(m+(n-2+\delta)/4)\}\mathcal{J}_1\mathcal{J}_2,
\end{aligned}$$

where

$$\mathrm{F}_1(\mathbf{A}_0) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|\phi_{2ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2,$$

$$\mathrm{F}_2(\mathbf{g}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{F}_3(\mathbf{A}) = \left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{F}_4(m) = \lim_{t\to\infty}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\},$$

$$\mathrm{E}_3(\mathbf{u}) = \left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathcal{I}_1(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta,$$
$$\mathcal{I}_2(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta,$$
$$\mathcal{J}_1 = n\sum_{i=1}^n\sum_{j=1}^n \lambda_{1ij}^2 + 2\sum_{i=1}^n\sum_{j=1}^n \alpha_{1ij}^2 + \left[\sum_{i=1}^n \alpha_{1ii}\right]^2 - \left[\sum_{i=1}^n \lambda_{1ii}\right]^2,$$
$$\mathcal{J}_2 = \sum_{i=1}^n\sum_{j=1}^n \lambda_{2ij}^2,$$
$$\mathcal{S}_1(\alpha_1,\delta,\varepsilon,n) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta,$$
$$\mathcal{S}_2(\alpha_2,\delta,\varepsilon,n) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta.$$

3. The mathematical analysis and the proofs of the main results

The main purposes of this section are to accomplish the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations (2.1)-(2.4). The main difficulties are that the existence and uniqueness of the global smooth solution are unknown, if the dimension $n \geq 3$ and if the initial function and the external force are large. There exists no available uniform energy estimate for any order derivatives. To overcome the main difficulties, we will make complete use of the special structures and the semi-explicit representations of the Fourier transformations of the global weak solutions, to establish the primary decay estimates with sharp rates for all order derivatives on the interval (T,∞), where T is a sufficiently large positive constant. Then we will couple together existing ideas, existing results and a few novel ideas to accomplish the main results.

If there exists a unique global smooth solution to the Cauchy problems for the n-dimensional incompressible magnetohydrodynamics equations (2.1)-(2.4), then the energies

$$\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}, \qquad \int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$$

are finite, for all $m > 0$ and for all $t > 0$. If there exists a global weak solution, then the energies

$$\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}, \qquad \int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$$

are finite, for all $m > 0$ and for all sufficiently large $t > T$. It is unknown if any of these energies is equal to infinity at some finite time $0 < t_0 < T$.

Let

$$\alpha_1 = \frac{1}{\mathrm{RE}}, \qquad \alpha_2 = \frac{1}{\mathrm{RM}}.$$

Let the positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, and let the real constant $m \geq 0$. This is only a *sketch* of the proofs of the main results for the n-dimensional incompressible magnetohydrodynamics equations. We provide the main ideas and the main steps here. Because of space limit, all easy and simple details of the proofs will be skipped. There are several lemmas in the proofs of the main results. In this part, we will use $C_0 = C_0(m, n)$ to represent any positive constant, which is independent of (α_1, α_2), (δ, ε), (ϕ_{1ij}, ϕ_{2ij}), (ψ_{1ij}, ψ_{2ij}), $(\mathbf{u}_0, \mathbf{A}_0)$, $(\mathbf{f}, \mathbf{g})$, $(\mathbf{u}, \mathbf{A})$ and $(\mathbf{x}, t)$.

3.1. The elementary estimates

First of all, we will establish a series of elementary estimates.

Lemma 3.1. *There hold the following eighteen elementary estimates:*
(1)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \exp(-\alpha_1 |\eta|^2) \sum_{j=1}^{n} \eta_j \widehat{\phi}_{1ij}(t^{-1/2}\eta) \right|^2 \mathrm{d}\eta \\ &\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\int_{\mathbb{R}^n} |\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x} \right]^2 \right\}.\end{aligned}$$

(2)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \exp(-\alpha_2 |\eta|^2) \sum_{j=1}^{n} \eta_j \widehat{\phi}_{2ij}(t^{-1/2}\eta) \right|^2 \mathrm{d}\eta \\ &\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\int_{\mathbb{R}^n} |\phi_{2ij}(\mathbf{x})|\mathrm{d}\mathbf{x} \right]^2 \right\}.\end{aligned}$$

(3)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1 |\eta|^2 (1-\frac{\tau}{t})\right] \sum_{j=1}^{n} \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\ &\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\int_0^{\infty}\int_{\mathbb{R}^n} |\psi_{1ij}(\mathbf{x}, t)|\mathrm{d}\mathbf{x}\mathrm{d}t \right]^2 \right\}.\end{aligned}$$

(4)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2 |\eta|^2 (1-\frac{\tau}{t})\right] \sum_{j=1}^{n} \eta_j \widehat{\psi}_{2ij}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\ &\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\int_0^{\infty}\int_{\mathbb{R}^n} |\psi_{2ij}(\mathbf{x}, t)|\mathrm{d}\mathbf{x}\mathrm{d}t \right]^2 \right\}.\end{aligned}$$

(5)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.$$

(6)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.$$

(7)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.$$

(8)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2.$$

(9)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i u_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta$$
$$\cdot \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\} \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}.$$

(10)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i A_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta$$

$$\cdot \left\{\int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t\right\}\left\{\int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t\right\}.$$

(11)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left|\int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta\\
&\le C_0 t^{-n}\left\{\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sum_{i=1}^n\sum_{j=1}^n \sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4}\psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}.\end{aligned}$$

(12)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left|\int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta\\
&\le C_0 t^{-n}\left\{\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sum_{i=1}^n\sum_{j=1}^n \sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4}\psi_{2ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}.\end{aligned}$$

(13)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left|\int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta\\
&\le C_0 t^{-n}\left\{\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}.\end{aligned}$$

(14)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left|\int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta\\
&\le C_0 t^{-n}\left\{\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\end{aligned}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

(15)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t}) \right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\le C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha_1 \varepsilon |\eta|^2)]^2}{\alpha_1^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

(16)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t}) \right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^{n} \sum_{l=1}^{n} \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\le C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha_1 \varepsilon |\eta|^2)]^2}{\alpha_1^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

(17)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_2 |\eta|^2 (1 - \frac{\tau}{t}) \right] \sum_{j=1}^{n} \eta_j \widehat{A_i u_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\le C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha_2 \varepsilon |\eta|^2)]^2}{\alpha_2^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$+ C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha_2 \varepsilon |\eta|^2)]^2}{\alpha_2^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

(18)

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^{n} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^{n} \eta_j \widehat{u_i A_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\leq C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$+ C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

These estimates are true for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, for all real constants $m \geq 0$ and for all sufficiently large t. In (11) - (18), the positive constant $C_0 = C_0(m,n)$ is independent of (α_1,α_2), (δ,ε), (ϕ_{1ij},ϕ_{2ij}), (ψ_{1ij},ψ_{2ij}), $(\mathbf{u}_0,\mathbf{A}_0)$, $(\mathbf{f},\mathbf{g})$, $(\mathbf{u},\mathbf{A})$ and $(\mathbf{x},t)$.

Proof. By using the standard Cauchy-Schwartz's inequality and a few very simple properties of the Fourier transformation, then by using the change of variables $\eta = t^{1/2}\xi$, so that $|\eta|^2 = |\xi|^2 t$, we may establish the following basic estimates.

(1')

$$\sum_{i=1}^{n} \left| \exp(-\alpha_1|\eta|^2) \sum_{j=1}^{n} \eta_j \widehat{\phi}_{1ij}(t^{-1/2}\eta) \right|^2$$
$$\leq |\eta|^2 \exp(-2\alpha_1|\eta|^2) \sum_{i=1}^{n}\sum_{j=1}^{n} \left\{ \int_{\mathbb{R}^n} |\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x} \right\}^2.$$

(2')

$$\sum_{i=1}^{n} \left| \exp(-\alpha_2|\eta|^2) \sum_{j=1}^{n} \eta_j \widehat{\phi}_{2ij}(t^{-1/2}\eta) \right|^2$$
$$\leq |\eta|^2 \exp(-2\alpha_2|\eta|^2) \sum_{i=1}^{n}\sum_{j=1}^{n} \left\{ \int_{\mathbb{R}^n} |\phi_{2ij}(\mathbf{x})|\mathrm{d}\mathbf{x} \right\}^2.$$

(3')

$$\sum_{i=1}^{n} \left| \int_{0}^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^{n} \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2$$

$$\leq |\eta|^2 \exp(-2\alpha_1\varepsilon|\eta|^2) \sum_{i=1}^{n}\sum_{j=1}^{n} \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\psi_{ij}(\mathbf{x}, t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .$$

(4')

$$\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t}) \right] \sum_{j=1}^{n} \eta_j \widehat{\psi}_{2ij}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2$$
$$\leq |\eta|^2 \exp(-2\alpha_2\varepsilon|\eta|^2) \sum_{i=1}^{n}\sum_{j=1}^{n} \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\psi_{2ij}(\mathbf{x}, t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .$$

(5')

$$\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t}) \right] \sum_{j=1}^{n} \eta_j \widehat{u_i u_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2$$
$$\leq |\eta|^2 \exp(-2\alpha_1\varepsilon|\eta|^2) \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .$$

(6')

$$\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t}) \right] \sum_{j=1}^{n} \eta_j \widehat{A_i A_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2$$
$$\leq |\eta|^2 \exp(-2\alpha_1\varepsilon|\eta|^2) \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .$$

(7')

$$\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t}) \right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^{n}\sum_{l=1}^{n} \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2$$
$$\leq |\eta|^2 \exp(-2\alpha_1\varepsilon|\eta|^2) \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .$$

(8')

$$\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t}) \right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^{n}\sum_{l=1}^{n} \eta_k\eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2$$
$$\leq |\eta|^2 \exp(-2\alpha_1\varepsilon|\eta|^2) \left\{ \int_0^{\infty} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 .$$

(9')

$$\sum_{i=1}^{n} \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t}) \right] \sum_{j=1}^{n} \eta_j \widehat{A_i u_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2$$

$$\leq |\eta|^2 \exp(-2\alpha_2\varepsilon|\eta|^2) \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\} \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}.$$

(10')

$$\sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t}) \right] \sum_{j=1}^n \eta_j \widehat{u_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2$$

$$\leq |\eta|^2 \exp(-2\alpha_2\varepsilon|\eta|^2) \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\} \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}.$$

(11')

$$\sum_{i=1}^n \sup_{(1-\varepsilon)t\leq\tau\leq t} \left| |\eta|^{2m-2+(n+\delta)/2} t \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau) \right|^2$$

$$\leq C_0 t^{-n} \left\{ \sum_{i=1}^n \sum_{j=1}^n \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{m+(3n+2+\delta)/4} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x} \right]^2 \right\}.$$

(12')

$$\sum_{i=1}^n \sup_{(1-\varepsilon)t\leq\tau\leq t} \left| |\eta|^{2m-2+(n+\delta)/2} t \sum_{j=1}^n \eta_j \widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau) \right|^2$$

$$\leq C_0 t^{-n} \left\{ \sum_{i=1}^n \sum_{j=1}^n \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{m+(3n+2+\delta)/4} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \psi_{2ij}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x} \right]^2 \right\}.$$

(13')

$$\sum_{i=1}^n \sup_{(1-\varepsilon)t\leq\tau\leq t} \left| |\eta|^{2m-2+(n+\delta)/2} t \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau) \right|^2$$

$$\leq C_0 t^{-n} \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

(14')

$$\sum_{i=1}^n \sup_{(1-\varepsilon)t\leq\tau\leq t} \left| |\eta|^{2m-2+(n+\delta)/2} t \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau) \right|^2$$

$$\leq C_0 t^{-n} \sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

(15')

$$
\begin{aligned}
&\sum_{i=1}^{n} \sup_{(1-\varepsilon)t\le\tau\le t} \left| |\eta|^{2m-2+(n+\delta)/2} t \frac{\eta_i}{|\eta|^2} \sum_{k=1}^{n}\sum_{l=1}^{n} \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau) \right|^2 \\
&\le C_0 t^{-n} \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.
\end{aligned}
$$

(16')

$$
\begin{aligned}
&\sum_{i=1}^{n} \sup_{(1-\varepsilon)t\le\tau\le t} \left| |\eta|^{2m-2+(n+\delta)/2} t \frac{\eta_i}{|\eta|^2} \sum_{k=1}^{n}\sum_{l=1}^{n} \eta_k\eta_l \widehat{A_k A_l}(t^{-1/2}\eta,\tau) \right|^2 \\
&\le C_0 t^{-n} \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.
\end{aligned}
$$

(17')

$$
\begin{aligned}
&\sum_{i=1}^{n} \sup_{(1-\varepsilon)t\le\tau\le t} \left| |\eta|^{2m-2+(n+\delta)/2} t \sum_{j=1}^{n} \eta_j \widehat{A_i u_j}(t^{-1/2}\eta,\tau) \right|^2 \\
&\le C_0 t^{-n} \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&+ C_0 t^{-n} \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.
\end{aligned}
$$

(18')

$$
\begin{aligned}
&\sum_{i=1}^{n} \sup_{(1-\varepsilon)t\le\tau\le t} \left| |\eta|^{2m-2+(n+\delta)/2} t \sum_{j=1}^{n} \eta_j \widehat{u_i A_j}(t^{-1/2}\eta,\tau) \right|^2 \\
&\le C_0 t^{-n} \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}
\end{aligned}
$$

$$+ C_0 t^{-n} \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.$$

The details of the proofs are skipped because they are very easy. These estimates are true for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, for all $m \ge 0$ and for all $(\eta, t) \in \mathbb{R}^n \times \mathbb{R}^+$. In estimates (11'), (12'), (13'), up to (17'), (18'), $C_0 = C_0(m,n) > 0$ is a positive constant, independent of (α_1, α_2), (δ, ε), (ϕ_{1ij}, ϕ_{2ij}), (ψ_{1ij}, ψ_{2ij}), $(\mathbf{u}_0, \mathbf{A}_0)$ $(\mathbf{f}, \mathbf{g})$, $(\mathbf{u}, \mathbf{A})$ and $(\mathbf{x}, t)$.

Now by using (11') - (18'), we get the following estimates

(11")

$$\alpha_1^2 |\eta|^{4m+n+\delta} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2$$

$$\le C_0 [1 - \exp(-\alpha_1 \varepsilon |\eta|^2)]^2 t^{-n}$$

$$\cdot \left\{ \sum_{i=1}^n \sum_{j=1}^n \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{m+(3n+2+\delta)/4} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \psi_{1ij}(\mathbf{x},\tau)| \mathrm{d}\mathbf{x} \right]^2 \right\},$$

(12")

$$\alpha_2^2 |\eta|^{4m+n+\delta} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{2ij}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2$$

$$\le C_0 [1 - \exp(-\alpha_2 \varepsilon |\eta|^2)]^2 t^{-n}$$

$$\cdot \left\{ \sum_{i=1}^n \sum_{j=1}^n \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{m+(3n+2+\delta)/4} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \psi_{2ij}(\mathbf{x},\tau)| \mathrm{d}\mathbf{x} \right]^2 \right\},$$

(13")

$$\alpha_1^2 |\eta|^{4m+n+\delta} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2$$

$$\le C_0 [1 - \exp(-\alpha_1 \varepsilon |\eta|^2)]^2 t^{-n}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},$$

(14")

$$\alpha_1^2 |\eta|^{4m+n+\delta} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2$$

$$\leq C_0[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2 t^{-n}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},$$

(15")

$$\alpha_1^2|\eta|^{4m+n+\delta}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\,\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2$$
$$\leq C_0[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2 t^{-n}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},$$

(16")

$$\alpha_1^2|\eta|^{4m+n+\delta}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\,\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_k A_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2$$
$$\leq C_0[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2 t^{-n}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},$$

(17")

$$\alpha_2^2|\eta|^{4m+n+\delta}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2$$
$$\leq C_0[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2 t^{-n}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$+ C_0[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2 t^{-n}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},$$

(18")

$$
\begin{aligned}
&\alpha_2^2|\eta|^{4m+n+\delta}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n \eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\\
&\le C_0[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2t^{-n}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+C_0[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2t^{-n}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}
$$

for all real constants $m\ge 0$ and for all sufficiently large t.

The desired elementary estimates follow from the basic estimates (1') - (10') and (11") - (18") immediately. □

Remark 3.1. (1) The upper bounds for the integrals

$$
\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,
$$

and

$$
\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta
$$

are the same. The upper bound is

$$
\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,
$$

for all real constants $m\ge 0$ and for all $t>0$.

(2) The upper bounds for the integrals

$$
\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,
$$

and

$$
\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta
$$

are the same. The upper bound is

$$\int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

for all real constants $m \geq 0$ and for all $t > 0$.

(3) The upper bounds for the integrals

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta,$$

and

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta$$

are the same. The upper bound is

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta \\ &\cdot \left\{\int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t\right\} \left\{\int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t\right\},\end{aligned}$$

for all real constants $m \geq 0$ and for all $t > 0$.

(4) The upper bounds for the integrals

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta,$$

and

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta$$

are the same. The upper bound is

$$\begin{aligned}&C_0 t^{-n}\left\{\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\} \\ &\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\} \\ &\cdot \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},\end{aligned}$$

for all real constants $m \geq 0$ and for all sufficiently large t.

(5) The upper bounds for the integrals

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta,$$

and

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$

are the same. The upper bound is

$$\begin{aligned}
&C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
\end{aligned}$$

for all real constants $m \ge 0$ and for all sufficiently large t.

(6) The upper bounds for the integrals

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i u_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

and

$$\int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i A_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$

are the same. The upper bound is

$$\begin{aligned}
&C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&+ C_0 t^{-n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
\end{aligned}$$

for all real constants $m \ge 0$ and for all sufficiently large t.

Now let us do some further preparations.

Lemma 3.2. *Define the following complex auxiliary functions*

$$\begin{aligned}
&\Lambda_{1i}(\eta,t)=\exp(-\alpha_1|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta)\\
&+\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Lambda_{2i}(\eta,t)=-\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Lambda_{3i}(\eta,t)=\exp(-\alpha_2|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta)\\
&+\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Lambda_{4i}(\eta,t)=-\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^{n}\eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta, t) \in \mathbb{R}^n \times \mathbb{R}^+$ *and for all* $i = 1, 2, 3, \cdots, n,$ *and*

$$\begin{aligned}
&\Gamma_{1i}(\eta,t) = \exp(-\alpha_1|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta) \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau, \\
&\Gamma_{2i}(\eta,t) = -\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau, \\
&\Gamma_{3i}(\eta,t) = \exp(-\alpha_2|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta) \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau, \\
&\Gamma_{4i}(\eta,t) = -\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta, t) \in \mathbb{R}^n \times \mathbb{R}^+$ *and for all* $i = 1, 2, 3, \cdots, n.$

Then there hold the following estimates.

(1)

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{1i}(\eta,t)|^2\mathrm{d}\eta-\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{1i}(\eta,t)|^2\mathrm{d}\eta\right|\\
\le&\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{1i}(\eta,t)-\Gamma_{1i}(\eta,t)|^2\mathrm{d}\eta\\
&+2\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{1i}(\eta,t)-\Gamma_{1i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\\
&\cdot\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{1i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}.\end{aligned}$$

(2)

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{2i}(\eta,t)|^2\mathrm{d}\eta-\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{2i}(\eta,t)|^2\mathrm{d}\eta\right|\\
\le&\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{2i}(\eta,t)-\Gamma_{2i}(\eta,t)|^2\mathrm{d}\eta\\
&+2\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{2i}(\eta,t)-\Gamma_{2i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\\
&\cdot\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{2i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}.\end{aligned}$$

(3)

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{3i}(\eta,t)|^2\mathrm{d}\eta-\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{3i}(\eta,t)|^2\mathrm{d}\eta\right|\\
\le&\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{3i}(\eta,t)-\Gamma_{3i}(\eta,t)|^2\mathrm{d}\eta\\
&+2\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{3i}(\eta,t)-\Gamma_{3i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\\
&\cdot\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{3i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}.\end{aligned}$$

(4)

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{4i}(\eta,t)|^2\mathrm{d}\eta-\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{4i}(\eta,t)|^2\mathrm{d}\eta\right|\\
\le&\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{4i}(\eta,t)-\Gamma_{4i}(\eta,t)|^2\mathrm{d}\eta\end{aligned}$$

$$+\ 2\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{4i}(\eta,t)-\Gamma_{4i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}$$
$$\cdot\ \left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{4i}(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}.$$

These estimates are true for all real constants $m \geq 0$ and for all sufficiently large t.

Proof. It follows from the standard Cauchy-Schwartz's inequality. The details are very easy and they are skipped. □

3.2. The comprehensive analysis

The main purposes of this subsection are to make use of the elementary estimates and the representations of the Fourier transformations of the global weak solutions of the Cauchy problems for the n-dimensional incompressible magnetohydrodynamics equations (2.1)-(2.4) to establish optimal estimates for

$$\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x},\ \int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x},$$
$$\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x},\ \int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x},$$

for all constants $m \geq 0$.

For each of the four integrals, there will be two parts in the upper bound. The terms in the first part come from the integrals

$$\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha|\eta|^2(1-\frac{\tau}{t})\right]\cdots\cdots\mathrm{d}\tau\right|^2\mathrm{d}\eta.$$

The terms in the second part come from the integrals

$$\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha|\eta|^2(1-\frac{\tau}{t})\right]\cdots\cdots\mathrm{d}\tau\right|^2\mathrm{d}\eta.$$

The comprehensive analysis will play very important roles when we accomplish the primary decay estimates and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the magnetohydrodynamics equations (2.1)-(2.4).

Lemma 3.3. *Let the positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$. Let the constant $m \geq 0$. Then there hold the following estimates*

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$$
$$\leq \frac{11}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2$$
$$+\ \frac{11}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_0^{\infty}\int_{\mathbb{R}^n}|\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2$$

$$
\begin{aligned}
&+\frac{22}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{22}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{11C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{22C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{22C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
&\le\frac{16}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{16}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{16C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{16C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}
$$

for all positive constants $0 < \delta < 4$ *and* $0 < \varepsilon < 1$*, for all* $m \geq 0$ *and for all sufficiently large* t*, the positive constant* $C_0 = C_0(m, n)$ *is independent of* (α_1, α_2)*,* (δ, ε)*,* (ϕ_{1ij}, ϕ_{2ij})*,* (ψ_{1ij}, ψ_{2ij})*,* $(\mathbf{u}_0, \mathbf{A}_0)$*,* $(\mathbf{f}, \mathbf{g})$*,* $(\mathbf{u}, \mathbf{A})$ *and* $(\mathbf{x}, t)$*.*

Proof. First of all, there hold the following representations for the Fourier transformations of the global weak solutions

$$\begin{aligned}
& t^{1/2}\widehat{u}_i(t^{-1/2}\eta, t) = \mathrm{i}\exp(-\alpha_1|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta) \\
&+ \mathrm{i}\int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \\
&- \mathrm{i}\int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{u_iu_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \\
&+ \mathrm{i}\int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{A_iA_j}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \\
&+ \mathrm{i}\int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \\
&- \mathrm{i}\int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^{n}\sum_{l=1}^{n}\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta, \tau)\mathrm{d}\tau, \\
& t^{1/2}\widehat{v}_i(t^{-1/2}\eta, t) = \mathrm{i}\exp(-\alpha_1|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta) \\
&+ \mathrm{i}\int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^{n}\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta, \tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta, t) \in \mathbb{R}^n \times \mathbb{R}^+$ and for all $i = 1, 2, 3, \cdots, n$. The representation of the Fourier transformation of the global weak solutions follows from Assumption (A6) and the change of variables $\eta = t^{1/2}\xi$, where $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$.

Let the positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$. Let the constant $m \geq 0$. Let time $t > 0$. Then by coupling together the Parseval's identity, a simple property of the Fourier transformation, the change of variables $\eta = t^{1/2}\xi$, the representation of the Fourier transformation $t^{1/2}\widehat{u}_i(t^{-1/2}\eta, t)$, the decomposition $[0, t] = [0, (1-\varepsilon)t] \cup [(1-\varepsilon)t, t]$ and the eighteen elementary estimates, we have the following computations and estimates

$$\begin{aligned}
& t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x}, t)|^2\mathrm{d}\mathbf{x} \\
&= \frac{t^{2m+1+n/2}}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi|^{4m}|\widehat{\mathbf{u}}(\xi, t)|^2\mathrm{d}\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}|t^{1/2}\widehat{\mathbf{u}}(t^{-1/2}\eta, t)|^2\mathrm{d}\eta \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|t^{1/2}\widehat{u}_i(t^{-1/2}\eta, t)|^2\mathrm{d}\eta
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \Bigg| \exp(-\alpha_1|\eta|^2) \sum_{j=1}^n \eta_j \widehat{\phi}_{1ij}(t^{-1/2}\eta) \\
&+ \int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \Bigg|^2 \mathrm{d}\eta \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \Bigg| \exp(-\alpha_1|\eta|^2) \sum_{j=1}^n \eta_j \widehat{\phi}_{1ij}(t^{-1/2}\eta) \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+ \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha_1|\eta_1|^2(1-\frac{\tau}{t})\Big] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&- \int_{(1-\varepsilon)t}^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \Bigg|^2 \mathrm{d}\eta
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \exp(-\alpha_1|\eta|^2) \sum_{j=1}^n \eta_j \widehat{\phi}_{1ij}(t^{-1/2}\eta) \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \right. \\
&\cdot \left. \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \right. \\
&\cdot \left. \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \right. \\
&\cdot \left. \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&+ \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \right. \\
&\cdot \left. \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&\le \frac{11}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta \sum_{i=1}^n \sum_{j=1}^n \left\{ \int_{\mathbb{R}^n} |\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x} \right\}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{11}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\sum_{i=1}^n\sum_{j=1}^n\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{22}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{22}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{11C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{22C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{22C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}
$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, for all real constants $m\ge 0$ and for all sufficiently large t.

Recall that there holds the following representation for the Fourier transformation

$$
\begin{aligned}
&t^{1/2}[\widehat{u}_i(t^{-1/2}\eta,t)-\widehat{v}_i(t^{-1/2}\eta,t)]\\
&=-\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta, t) \in \mathbb{R}^n \times \mathbb{R}^+$ and for all $i = 1, 2, 3, \cdots, n$. Now

$$
\begin{aligned}
& t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \\
= & \frac{t^{2m+1+n/2}}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{4m} |\widehat{\mathbf{u}}(\xi, t) - \widehat{\mathbf{v}}(\xi, t)|^2 \mathrm{d}\xi \\
= & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} |t^{1/2} [\widehat{\mathbf{u}}(t^{-1/2}\eta, t) - \widehat{\mathbf{v}}(t^{-1/2}\eta, t)]|^2 \mathrm{d}\eta \\
= & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n |t^{1/2} [\widehat{u}_i(t^{-1/2}\eta, t) - \widehat{v}_i(-1/2\eta, t)]|^2 \mathrm{d}\eta \\
= & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| - \int_0^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right. \\
+ & \int_0^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
+ & \int_0^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
- & \left. \int_0^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
= & \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m} \sum_{i=1}^n \left| - \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right. \\
+ & \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
+ & \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
- & \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
- & \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{u_i u_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
+ & \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
+ & \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{u_k u_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \\
- & \left. \int_{(1-\varepsilon)t}^t \exp\left[-\alpha_1 |\eta|^2 (1 - \frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \eta_k \eta_l \widehat{A_k A_l}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\right.\\
&\cdot\left.\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\right.\\
&\cdot\left.\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&+\frac{8}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^{t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\right.\\
&\cdot\left.\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\\
&\le\frac{16}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{16}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{16C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{16C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}
\end{aligned}
$$

$$
\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}
$$

$$
\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
$$

for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, for all real constants $m \ge 0$ and for all sufficiently large t. The proof of the lemma is finished now. □

Lemma 3.4. *There hold the following estimates*

$$
\begin{aligned}
& t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \\
\le\ & \frac{7}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2|\eta|^2) \mathrm{d}\eta \sum_{i=1}^n \sum_{j=1}^n \left\{ \int_{\mathbb{R}^n} |\phi_{2ij}(\mathbf{x})| \mathrm{d}\mathbf{x} \right\}^2 \\
+\ & \frac{7}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2\varepsilon|\eta|^2) \mathrm{d}\eta \sum_{i=1}^n \sum_{j=1}^n \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\psi_{2ij}(\mathbf{x},t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
+\ & \frac{7}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
+\ & \frac{7}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
+\ & \frac{7C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
\cdot\ & \left\{ \sum_{i=1}^n \sum_{j=1}^n \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{m+(3n+2+\delta)/4} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \psi_{2ij}(\mathbf{x},\tau)| \mathrm{d}\mathbf{x} \right]^2 \right\} \\
+\ & \frac{7C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
\cdot\ & \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
\cdot\ & \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
+\ & \frac{7C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
\cdot\ & \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
\cdot\ & \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
\end{aligned}
$$

and

$$
t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{A}(\mathbf{x},t) - \mathbf{B}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x}
$$

$$
\begin{aligned}
&\leq \frac{4}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{4}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{4C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{4C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}
$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, for all $m\geq 0$ and for all sufficiently large t, where the positive constant $C_0=C_0(m,n)$ is independent of (α_1,α_2), (δ,ε), (ϕ_{1ij},ϕ_{2ij}), (ψ_{1ij},ψ_{2ij}), $(\mathbf{u}_0,\mathbf{A}_0)$, $(\mathbf{f},\mathbf{g})$, $(\mathbf{u},\mathbf{A})$ and $(\mathbf{x},t)$.

Proof. First of all, there holds the following representation for the Fourier transformation

$$
\begin{aligned}
t^{1/2}\widehat{A}_i(t^{-1/2}\eta,t)&=\mathrm{i}\exp(-\alpha_2|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta)\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\mathrm{i}\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_ju_i}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
t^{1/2}\widehat{B}_i(t^{-1/2}\eta,t)&=\mathrm{i}\exp(-\alpha_2|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta)\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta,t)\in\mathbb{R}^n\times\mathbb{R}^+$ and for all $i=1,2,3,\cdots,n$. The representation follows from Assumption (A6). The proof of the lemma is very similar to that of Lemma 3.3. The details are skipped. □

3.3. The primary decay estimates

The main purpose of this subsection is to make use of the comprehensive analysis and the elementary decay estimates to prove the primary decay estimates with sharp rates for all order derivatives of the global weak solutions of the Cauchy problems for the n-dimensional incompressible magnetohydrodynamics equations (2.1)-(2.4).
The Proof of Theorem 2.4: Recall that there hold the elementary decay estimates with a sharp rate:

$$\sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty, \qquad \sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty.$$

Recall also that there exists a sufficiently large positive constant T, such that there hold the following estimates

$$\sup_{t>T}\left\{\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$
$$\sup_{t>T}\left\{\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all positive constants $m>0$.

Lemma 3.5. *Suppose that Assumptions (A1) - (A7) hold. Then there hold the following primary decay estimates with sharp rates*

$$\sup_{t>T}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$
$$\sup_{t>T}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all positive constants $m>0$.

Proof. We make use of the estimates in the comprehensive analysis. Let $\kappa\geq 0$ be a real constant. Multiplying the first estimate in Lemma 3.3 and the first estimate in Lemma 3.4 by $t^{-\kappa}$, we have the following estimates

$$\begin{aligned}
&t^{2m+1-\kappa+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
\leq{}&\frac{11t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\sum_{i=1}^n\sum_{j=1}^n\left\{\int_{\mathbb{R}^n}|\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{11t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\sum_{i=1}^n\sum_{j=1}^n\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{22t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{22t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{11C_0t^{-\kappa}}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{22C_0t^{-\kappa}}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{22C_0t^{-\kappa}}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}
$$

and the following estimates

$$
\begin{aligned}
&t^{2m+1-\kappa+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
&\le\frac{7t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_{\mathbb{R}^n}|\phi_{2ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{7t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{7t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{7t^{-\kappa}}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{7C_0t^{-\kappa}}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{2ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{7C_0t^{-\kappa}}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{7C_0t^{-\kappa}}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}
\end{aligned}
$$

$$
\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}
$$
$$
\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, for all real constants $m\ge 0$ and for all sufficiently large t. The constants $C_0=C_0(m,n)$ in all the above estimates have the same value. In particular, C_0 is independent of (α_1,α_2), (δ,ε), (ϕ_{1ij},ϕ_{2ij}), (ψ_{1ij},ψ_{2ij}), $(\mathbf{u}_0,\mathbf{A}_0)$, $(\mathbf{f},\mathbf{g})$, $(\mathbf{u},\mathbf{A})$ and $(\mathbf{x},t)$. Moreover, it is independent of κ.

Let us iterate these estimates for finitely many times to establish the primary decay estimates with sharp rates for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations. Even though the order of the derivatives in

$$
\int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}, \qquad \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x},
$$

which appear on the left hand sides of the above estimates is not the same as the order of derivative in

$$
\int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x}, \qquad \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x},
$$

which appear on the right hand sides of the above estimates, we may use the decay rate $r_i=i(1+\frac{n-\delta}{2})$ obtained in each step, because the rate obtained in each step is independent of the order m and the order $m+(n-2+\delta)/4$. Each step of iteration will increase the decay rate by $1+\frac{1}{2}(n-\delta)>0$. By the iterations, we may establish the primary decay estimates with sharp rates for each positive constant $m>0$. Precisely, let r_i be the rate of decay we will obtain, when we let $\kappa=\kappa_i$, where $i=1,2,3,\cdots,M$, and M is a finite positive integer. We have the following iteration process

$$
\begin{array}{llllll}
\kappa_1 = 2m+\dfrac{\delta}{2}, & r_1 = \left(1+\dfrac{n-\delta}{2}\right), \\
\kappa_2 = 2m+\dfrac{\delta}{2}-\left(1+\dfrac{n-\delta}{2}\right), & r_2 = 2\left(1+\dfrac{n-\delta}{2}\right), \\
\kappa_3 = 2m+\dfrac{\delta}{2}-2\left(1+\dfrac{n-\delta}{2}\right), & r_3 = 3\left(1+\dfrac{n-\delta}{2}\right), \\
\kappa_4 = 2m+\dfrac{\delta}{2}-3\left(1+\dfrac{n-\delta}{2}\right), & r_4 = 4\left(1+\dfrac{n-\delta}{2}\right), \\
\cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \\
\kappa_M = 2m+\dfrac{\delta}{2}-(M-1)\left(1+\dfrac{n-\delta}{2}\right), & r_M = M\left(1+\dfrac{n-\delta}{2}\right),
\end{array}
$$

where M is the largest positive integer, such that

$$
M\left(1+\frac{n-\delta}{2}\right) \le 2m+1+\frac{n}{2}.
$$

If

$$
M\left(1+\frac{n-\delta}{2}\right) = 2m+1+\frac{n}{2},
$$

then we have finished the iteration process and we will stop right there.

If

$$M\left(1+\frac{n-\delta}{2}\right) < 2m+1+\frac{n}{2} < (M+1)\left(1+\frac{n-\delta}{2}\right),$$

then

$$\begin{aligned}
\kappa_M &= 2m+1+\frac{n}{2} - M\left(1+\frac{n-\delta}{2}\right) > 0,\\
r_M &= M\left(1+\frac{n-\delta}{2}\right) < 2m+1+\frac{n}{2},\\
2m+\frac{\delta}{2} &< M\left(1+\frac{n-\delta}{2}\right).
\end{aligned}$$

We will iterate once more by letting $\kappa_{M+1} = 0$, to get the optimal rate of decay $r_{M+1} = 2m+1+n/2$. The proof of the lemma is finished now. □

The primary decay estimates with sharp rates for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations are established now. The proof of Theorem 2.4 is finished now.

3.4. The fundamental limits

Now we are ready to make use of the elementary estimates and the primary decay estimates to establish several fundamental limits for the n-dimensional incompressible magnetohydrodynamics equations. These limits are true for all constants $m \geq 0$.

Lemma 3.6. *Let $\eta \in \mathbb{R}^n$. Then there hold the following limits.*
(1)

$$\begin{aligned}
&\lim_{t\to\infty}\Bigg\{\exp(-\alpha_1|\eta|^2)\sum_{j=1}^n \eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta)\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n \eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n \eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n \eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg\}\\
&=\exp(-\alpha_1|\eta|^2)\sum_{j=1}^n\eta_j\left\{\int_{\mathbb{R}^n}\phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right.
\end{aligned}$$

$$- \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\Big\}$$
$$+ \exp(-\alpha_1|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l$$
$$\cdot \ \Big\{\int_0^\infty \int_{\mathbb{R}^n} u_k(\mathbf{x},t)u_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} A_k(\mathbf{x},t)A_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\Big\}$$
$$= \exp(-\alpha_1|\eta|^2)\sum_{j=1}^n \lambda_{1ij}\eta_j + \exp(-\alpha_1|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \rho_{1kl}\eta_k\eta_l.$$

(2)

$$\lim_{t\to\infty}\Big\{- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big\}$$
$$= \exp(-\alpha_1|\eta|^2)\sum_{j=1}^n \eta_j\Big\{- \int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\Big\}$$
$$+ \exp(-\alpha_1|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l$$
$$\cdot \ \Big\{\int_0^\infty \int_{\mathbb{R}^n} u_k(\mathbf{x},t)u_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^n} A_k(\mathbf{x},t)A_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\Big\}$$
$$= - \exp(-\alpha_1|\eta|^2)\sum_{j=1}^n \rho_{1ij}\eta_j + \exp(-\alpha_1|\eta|^2)\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \rho_{1kl}\eta_k\eta_l.$$

(3)

$$\lim_{t\to\infty}\Big\{\exp(-\alpha_2|\eta|^2)\sum_{j=1}^n \eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta)$$
$$+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j\widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$- \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$+ \int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big\}$$

$$
\begin{aligned}
&= \exp(-\alpha_2|\eta|^2)\sum_{j=1}^n \eta_j \left\{\int_{\mathbb{R}^n}\phi_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^n}\psi_{2ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right.\\
&\left. - \int_0^\infty\int_{\mathbb{R}^n} A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^\infty\int_{\mathbb{R}^n} u_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}\\
&= \exp(-\alpha_2|\eta|^2)\sum_{j=1}^n \lambda_{2ij}\eta_j.
\end{aligned}
$$

(4)

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right.\\
&\left.+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right\}\\
&= \exp(-\alpha_2|\eta|^2)\sum_{j=1}^n\eta_j\\
&\cdot\left\{-\int_0^\infty\int_{\mathbb{R}^n} A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^\infty\int_{\mathbb{R}^n} u_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}\\
&= -\exp(-\alpha_2|\eta|^2)\sum_{j=1}^n\rho_{2ij}\eta_j.
\end{aligned}
$$

Proof. By using the elementary estimates in Lemma 3.1 and Lebesgue's dominated convergence theorem, it is easy to prove these limits. □

Lemma 3.7. *Define the following complex auxiliary functions*

$$
\begin{aligned}
&\Gamma_{1i}(\eta,t) = \exp(-\alpha_1|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta)\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Gamma_{2i}(\eta,t) = -\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau
\end{aligned}
$$

$$+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j \widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l \widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$-\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l \widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

$$\Gamma_{3i}(\eta,t)=\exp(-\alpha_2|\eta|^2)\sum_{j=1}^n \eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta)$$

$$+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j \widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$-\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j \widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j \widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

$$\Gamma_{4i}(\eta,t)=-\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j \widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$+\int_0^{(1-\varepsilon)t} \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j \widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

for all $(\eta,t)\in\mathbb{R}^n\times\mathbb{R}^+$ *and for all* $i=1,2,3,\cdots,n$, *with* $\eta\neq\mathbf{0}$.

Then there hold the following limits

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n|\Gamma_{1i}(\eta,t)|^2\mathrm{d}\eta\right\}$$

$$=\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha_1|\eta|^2)\sum_{i=1}^n\left|\sum_{j=1}^n\lambda_{1ij}\eta_j+\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\rho_{1kl}\eta_k\eta_l\right|^2\mathrm{d}\eta,$$

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n|\Gamma_{2i}(\eta,t)|^2\mathrm{d}\eta\right\}$$

$$=\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha_1|\eta|^2)\sum_{i=1}^n\left|-\sum_{j=1}^n\rho_{1ij}\eta_j+\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\rho_{1kl}\eta_k\eta_l\right|^2\mathrm{d}\eta,$$

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n|\Gamma_{3i}(\eta,t)|^2\mathrm{d}\eta\right\}$$

$$=\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha_2|\eta|^2)\sum_{i=1}^n\left|\sum_{j=1}^n\lambda_{2ij}\eta_j\right|^2\mathrm{d}\eta,$$

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{4i}(\eta,t)|^2\mathrm{d}\eta\right\}$$
$$=\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha_2|\eta|^2)\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\rho_{2ij}\eta_j\right|^2\mathrm{d}\eta,$$

where we recall that the parameters

$$\begin{aligned}\alpha_{1ij} &= \int_{\mathbb{R}^n}\phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\ \lambda_{1ij} &= \int_{\mathbb{R}^n}\phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\\ &-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\ \rho_{1ij} &= \int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned}$$

and

$$\begin{aligned}\alpha_{2ij} &= \int_{\mathbb{R}^n}\phi_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{2ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\ \lambda_{2ij} &= \int_{\mathbb{R}^n}\phi_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{2ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\\ &-\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x})A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\ \rho_{2ij} &= \int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned}$$

for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$.

Proof. It is easy to apply the elementary estimates (1) - (10), the limits in Lemma 3.6 and the Lebesgue's dominated convergence theorem to prove these limits. The details of the proof in this lemma are very simple and they are omitted. □

Lemma 3.8. *There hold the following limits*

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{1i}(\eta,t)|^2\mathrm{d}\eta\right\}=\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{1i}(\eta,t)|^2\mathrm{d}\eta\right\},$$
$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{2i}(\eta,t)|^2\mathrm{d}\eta\right\}=\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{2i}(\eta,t)|^2\mathrm{d}\eta\right\},$$
$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{3i}(\eta,t)|^2\mathrm{d}\eta\right\}=\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{3i}(\eta,t)|^2\mathrm{d}\eta\right\},$$
$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Lambda_{4i}(\eta,t)|^2\mathrm{d}\eta\right\}=\lim_{t\to\infty}\left\{\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^{n}|\Gamma_{4i}(\eta,t)|^2\mathrm{d}\eta\right\},$$

for all real constants $m\geq 0$.

Proof. The proof of these limits follows from the application of very simple ideas, such as the squeeze theorem, the estimates in Lemma 3.2, the elementary estimates (11)-(18) in Lemma 3.1 and the limits in Lemma 3.7. The details are skipped here. □

3.5. The computations of special integrals

Lemma 3.9. *There hold the following identities.*
(1)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \exp(-2\alpha_1|\eta|^2) \sum_{i=1}^n \left| \sum_{j=1}^n \lambda_{1ij}\eta_j + \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \rho_{1kl}\eta_k\eta_l \right|^2 \mathrm{d}\eta \\ =& \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta \\ &\cdot \left\{ n\sum_{i=1}^n \sum_{j=1}^n \lambda_{1ij}^2 + 2\sum_{i=1}^n \sum_{j=1}^n \alpha_{1ij}^2 + \left[\sum_{i=1}^n \alpha_{1ii}\right]^2 - \left[\sum_{i=1}^n \lambda_{1ii}\right]^2 \right\}.\end{aligned}$$

(2)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \exp(-2\alpha_1|\eta|^2) \sum_{i=1}^n \left| -\sum_{j=1}^n \rho_{1ij}\eta_j + \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n \sum_{l=1}^n \rho_{1kl}\eta_k\eta_l \right|^2 \mathrm{d}\eta \\ =& \frac{1}{n(n+2)} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta \left\{ n\sum_{i=1}^n \sum_{j=1}^n \rho_{1ij}^2 - \left[\sum_{i=1}^n \rho_{1ii}\right]^2 \right\}.\end{aligned}$$

(3)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \exp(-2\alpha_2|\eta|^2) \sum_{i=1}^n \left| \sum_{j=1}^n \lambda_{2ij}\eta_j \right|^2 \mathrm{d}\eta \\ =& \frac{1}{n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta \left\{ \sum_{i=1}^n \sum_{j=1}^n \lambda_{2ij}^2 \right\}.\end{aligned}$$

(4)

$$\begin{aligned}&\int_{\mathbb{R}^n} |\eta|^{4m} \exp(-2\alpha_2|\eta|^2) \sum_{i=1}^n \left| \sum_{j=1}^n \rho_{2ij}\eta_j \right|^2 \mathrm{d}\eta \\ =& \frac{1}{n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta \left\{ \sum_{i=1}^n \sum_{j=1}^n \rho_{2ij}^2 \right\}.\end{aligned}$$

Proof. They follow from very technical, tedious computations. To save space, the details are omitted. □

3.6. The exact limits

The main purposes of this subsection are to make complete use of the fundamental limits and the computations of the special integrals to accomplish the exact limits for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations:

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n(n+2)}\mathcal{I}_1(m)\mathcal{J}_1,$$
$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n(n+2)}\mathcal{I}_1(m)\mathcal{K}_1,$$

and

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(m)\mathcal{J}_2,$$
$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(m)\mathcal{K}_2,$$

for all constants $m\geq 0$.

Recall that $(\mathbf{u},\mathbf{A})=(\mathbf{u}(\mathbf{x},t),\mathbf{A}(\mathbf{x},t))$ represents the global weak solutions of the magnetohydrodynamics equations (2.1)-(2.4), and $(\mathbf{v},\mathbf{B})=(\mathbf{v}(\mathbf{x},t),\mathbf{B}(\mathbf{x},t))$ represents the global smooth solution of the linear heat equations (2.5)-(2.8), with the same initial functions $(\mathbf{u}_0,\mathbf{A}_0)=(\mathbf{u}_0(\mathbf{x}),\mathbf{A}_0(\mathbf{x}))$ and the same external forces $(\mathbf{f},\mathbf{g})=(\mathbf{f}(\mathbf{x},t),\mathbf{g}(\mathbf{x},t))$.

The Proof of Theorem 2.1: Recall that there hold the following semi-explicit representations for the Fourier transformations $t^{1/2}\widehat{u}_i(t^{-1/2}\eta,t)$ and $t^{1/2}\widehat{A}_i(t^{-1/2}\eta,t)$:

$$\begin{aligned}
&t^{1/2}\widehat{u}_i(t^{-1/2}\eta,t)=\mathrm{i}\exp(-\alpha_1|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta)\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\mathrm{i}\int_0^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&t^{1/2}\widehat{A}_i(t^{-1/2}\eta,t)=\mathrm{i}\exp(-\alpha_2|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta)\\
&+\mathrm{i}\int_0^t\exp\left[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau
\end{aligned}$$

$$- \mathrm{i}\int_0^t \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j\widehat{A_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$+ \mathrm{i}\int_0^t \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n \eta_j\widehat{u_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

for all $(\eta,t)\in\mathbb{R}^n\times\mathbb{R}^+$ and for all $i=1,2,3,\cdots,n$.

By coupling together the Parseval's identity, a simple property of the Fourier transformation, the change of variables $\eta=t^{1/2}\xi$, so that $|\eta|^2=|\xi|^2t$ and $\mathrm{d}\eta=t^{n/2}\mathrm{d}\xi$, where $\xi\in\mathbb{R}^n$ and $\eta\in\mathbb{R}^n$, the representation of the Fourier transformation $t^{1/2}\widehat{u}_i(t^{-1/2}\eta,t)$, and the results of Subsection 3.1, Subsection 3.3 and Subsection 3.4, we have the following computations

$$\begin{aligned}
&\lim_{t\to\infty}\Big\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\Big\}\\
=&\lim_{t\to\infty}\Big\{\frac{t^{2m+1+n/2}}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi|^{4m}|\widehat{\mathbf{u}}(\xi,t)|^2\mathrm{d}\xi\Big\}\\
=&\lim_{t\to\infty}\Big\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}|t^{1/2}\widehat{\mathbf{u}}(t^{-1/2}\eta,t)|^2\mathrm{d}\eta\Big\}\\
=&\lim_{t\to\infty}\Big\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n|t^{1/2}\widehat{u}_i(t^{-1/2}\eta,t)|^2\mathrm{d}\eta\Big\}\\
=&\lim_{t\to\infty}\Bigg\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\Bigg|\exp(-\alpha_1|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta)\\
+&\int_0^t\exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
-&\int_0^t\exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
+&\int_0^t\exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
+&\int_0^t\exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
-&\int_0^t\exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\Bigg\}\\
=&\lim_{t\to\infty}\Bigg\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\Bigg|\exp(-\alpha_1|\eta|^2)\sum_{j=1}^n\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta)\\
+&\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
-&\int_0^{(1-\varepsilon)t}\exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau
\end{aligned}$$

$$+ \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \sum_{j=1}^n \eta_j \widehat{A_i A_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$+ \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l \widehat{u_k u_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$- \int_0^{(1-\varepsilon)t} \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right] \frac{\eta_i}{|\eta|^2} \sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l \widehat{A_k A_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2 \mathrm{d}\eta\Bigg\}$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m}\exp(-2\alpha_1|\eta|^2)$$

$$\cdot \sum_{i=1}^n \Bigg|\sum_{j=1}^n \eta_j \left[\int_{\mathbb{R}^n}\phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^n}\psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right.$$

$$\left. - \int_0^\infty\int_{\mathbb{R}^n} u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^\infty\int_{\mathbb{R}^n} A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]$$

$$+ \frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l \left[\int_0^\infty\int_{\mathbb{R}^n} u_k(\mathbf{x},t)u_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}^n} A_k(\mathbf{x},t)A_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]\Bigg|^2 \mathrm{d}\eta$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m}\exp(-2\alpha_1|\eta|^2) \sum_{i=1}^n \left|\sum_{j=1}^n \lambda_{1ij}\eta_j + \frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \rho_{1kl}\eta_k\eta_l\right|^2 \mathrm{d}\eta$$

$$= \frac{1}{n(n+2)}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta$$

$$\cdot \left\{ n\sum_{i=1}^n\sum_{j=1}^n \lambda_{1ij}^2 + 2\sum_{i=1}^n\sum_{j=1}^n \alpha_{1ij}^2 + \left[\sum_{i=1}^n \alpha_{1ii}\right]^2 - \left[\sum_{i=1}^n \lambda_{1ii}\right]^2 \right\}$$

$$= \frac{1}{n(n+2)}\mathcal{I}_1(m)\mathcal{J}_1.$$

The main ideas and the main steps of the proof of the second exact limit are very similar to the proof of the first exact limit in Theorem 2.1. We provide all the details for completeness of the paper. We have the following computations

$$\lim_{t\to\infty}\left\{ t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}$$

$$= \lim_{t\to\infty}\left\{ \frac{t^{2m+1+n/2}}{(2\pi)^n}\int_{\mathbb{R}^n} |\xi|^{4m}|\widehat{\mathbf{u}}(\xi,t)-\widehat{\mathbf{v}}(\xi,t)|^2\mathrm{d}\xi\right\}$$

$$= \lim_{t\to\infty}\left\{ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m}|t^{1/2}[\widehat{\mathbf{u}}(t^{-1/2}\eta,t)-\widehat{\mathbf{v}}(t^{-1/2}\eta,t)]|^2\mathrm{d}\eta\right\}$$

$$= \lim_{t\to\infty}\left\{ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m}\sum_{i=1}^n |t^{1/2}[\widehat{u}_i(t^{-1/2}\eta,t)-\widehat{v}_i(t^{-1/2}\eta,t)]|^2\mathrm{d}\eta\right\}$$

$$= \lim_{t\to\infty}\left\{ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m}\sum_{i=1}^n \Bigg| - \int_0^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n \eta_j\widehat{u_i u_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right.$$

$$+\int_0^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n \eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$+\int_0^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$\left.\left.-\int_0^t \exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n \eta_k\eta_k\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}$$

$$=\lim_{t\to\infty}\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|-\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right.\right.$$

$$+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{A_iA_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$+\int_0^{(1-\varepsilon)t}\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$\left.\left.-\int_{(1-\varepsilon)t}^t\exp\left[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{A_kA_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}$$

$$=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m}\exp(-2\alpha_1|\eta|^2)$$

$$\cdot\sum_{i=1}^n\left|\sum_{j=1}^n\eta_j\left[-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]\right.$$

$$\left.+\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\left[\int_0^\infty\int_{\mathbb{R}^n}u_k(\mathbf{x},t)u_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}A_k(\mathbf{x},t)A_l(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right]\right|^2\mathrm{d}\eta$$

$$=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\eta|^{4m}\exp(-2\alpha_1|\eta|^2)\sum_{i=1}^n\left|-\sum_{j=1}^n\rho_{1ij}\eta_j+\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\rho_{1kl}\eta_k\eta_l\right|^2\mathrm{d}\eta$$

$$=\frac{1}{n(n+2)}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\left\{n\sum_{i=1}^n\sum_{j=1}^n\rho_{1ij}^2-\left[\sum_{i=1}^n\rho_{1ii}\right]^2\right\}$$

$$=\frac{1}{n(n+2)}\mathcal{I}_1(m)\mathcal{K}_1.$$

The Proof of Theorem 2.2: The main ideas and the main steps in the proofs of the exact limits

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(m)\mathcal{J}_2,$$
$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(m)\mathcal{K}_2,$$

are the same as those of Theorem 2.1. The details are skipped. The proof of Theorem 2.2 is finished. □

3.7. The ratios of the exact limits

The main purposes of this subsection are to compute the ratios of the exact limits for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations (2.1)-(2.4).

Lemma 3.10. *Let the positive integer $n \geq 1$, let the positive constant $\alpha > 0$ and let the real constant $m \geq 0$. Then there hold the following identities*

$$\int_{\mathbb{R}^n} |\eta|^{4m+4} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \tfrac{4m+n+2}{4\alpha} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta,$$
$$\int_{\mathbb{R}^n} |\eta|^{4m+6} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \tfrac{(4m+n+2)(4m+n+4)}{(4\alpha)^2} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta.$$

The proof of the lemma is skipped.

The Proof of Theorem 2.3: The ratios of the exact limits for all order derivatives of the global weak solutions follow from the results in Theorem 2.1, Theorem 2.2 and these identities. In fact, we have

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n} |(-\triangle)^{m+1/2}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&=\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+4}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\right\}\\
&/\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\right\} = \frac{4m+n+2}{4\alpha_1},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n} |(-\triangle)^{m+1}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^{m}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&=\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+6}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\right\}\\
&/\left\{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\right\} = \frac{(4m+n+2)(4m+n+4)}{(4\alpha_1)^2},
\end{aligned}$$

for all constants $m \geq 0$. Other ratios may be proved very similarly. The details are skipped. The proof of Theorem 2.3 is completely finished now. □

3.8. The improved decay estimates with sharp rates

The main purposes of this subsection are to make complete use of the comprehensive analysis and the exact limits for all order derivatives of the global weak solutions to establish the following improved decay estimates with sharp rates

$$\begin{aligned}
&t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \leq \mathcal{A}_1 + \mathcal{B}_1 t^{-n},\\
&t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x} \leq \mathcal{C}_1 + \mathcal{D}_1 t^{-n},
\end{aligned}$$

and

$$t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \leq \mathcal{A}_2 + \mathcal{B}_2 t^{-n},$$

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\le\mathcal{C}_2+\mathcal{D}_2t^{-n},$$

for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations (2.1)-(2.4), for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, for all real constants $m\ge 0$ and for all sufficiently large t, where

$$\begin{aligned}
&\mathcal{A}_1=\mathcal{A}_1(\alpha_1,\delta,\varepsilon,m,n), &&\mathcal{A}_2=\mathcal{A}_2(\alpha_2,\delta,\varepsilon,m,n),\\
&\mathcal{B}_1=\mathcal{B}_1(\alpha_1,\delta,\varepsilon,m,n), &&\mathcal{B}_2=\mathcal{B}_2(\alpha_2,\delta,\varepsilon,m,n),\\
&\mathcal{C}_1=\mathcal{C}_1(\alpha_1,\delta,\varepsilon,m,n), &&\mathcal{C}_2=\mathcal{C}_2(\alpha_2,\delta,\varepsilon,m,n),\\
&\mathcal{D}_1=\mathcal{D}_1(\alpha_1,\delta,\varepsilon,m,n), &&\mathcal{D}_2=\mathcal{D}_2(\alpha_2,\delta,\varepsilon,m,n),
\end{aligned}$$

are positive constants, which have been specified in the main results.

Recall that there hold the following estimates

$$\begin{aligned}
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
\le&\frac{11}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta\sum_{i=1}^n\sum_{j=1}^n\left\{\int_{\mathbb{R}^n}|\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
&+\frac{11}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\sum_{i=1}^n\sum_{j=1}^n\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{22}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{22}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
&+\frac{11C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
&+\frac{22C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&+\frac{22C_0}{(2\pi t)}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}$$

$$
\begin{aligned}
& t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\\
\le\ & \frac{16}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
+\ & \frac{16}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
+\ & \frac{16C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
+\ & \frac{16C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\},
\end{aligned}
$$

$$
\begin{aligned}
& t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\\
\le\ & \frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta\sum_{i=1}^n\sum_{j=1}^n\left\{\int_{\mathbb{R}^n}|\phi_{2ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2\\
+\ & \frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\sum_{i=1}^n\sum_{j=1}^n\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
+\ & \frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
+\ & \frac{7}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2\\
+\ & \frac{7C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
\cdot\ & \left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{2ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}\\
+\ & \frac{7C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&+ \frac{7C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2 \varepsilon |\eta|^2)]^2}{\alpha_2^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{A}(\mathbf{x},t) - \mathbf{B}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \\
\le {} & \frac{4}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2 \varepsilon |\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
& + \frac{4}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha_2 \varepsilon |\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
& + \frac{4C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2 \varepsilon |\eta|^2)]^2}{\alpha_2^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
& \cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
& \cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
& + \frac{4C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2 \varepsilon |\eta|^2)]^2}{\alpha_2^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
& \cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
& \cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
\end{aligned}
$$

for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, for all real constants $m \ge 0$ and for all sufficiently large t.

Recall that there exist the following exact limits

$$
\begin{aligned}
& \lim_{t\to\infty} \left\{ \sum_{i=1}^n \sum_{j=1}^n \left[t^{m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \psi_{1ij}(\mathbf{x},t)| \mathrm{d}\mathbf{x} \right]^2 \right\} = \mathrm{E}_4(m), \\
& \lim_{t\to\infty} \left\{ \sum_{i=1}^n \sum_{j=1}^n \left[t^{m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \psi_{2ij}(\mathbf{x},t)| \mathrm{d}\mathbf{x} \right]^2 \right\} = \mathrm{F}_4(m), \\
& \lim_{t\to\infty} \left\{ t^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} = \frac{1}{n(n+2)} \mathcal{I}_1(0) \mathcal{J}_1,
\end{aligned}
$$

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n(n+2)}\mathcal{I}_1(m)\mathcal{J}_1,$$

and

$$\lim_{t\to\infty}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(0)\mathcal{J}_2,$$
$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(m)\mathcal{J}_2,$$

for all real constants $m\geq 0$.

By using the squeeze theorem, we have the following limits

$$\lim_{t\to\infty}\left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}$$
$$\overset{\text{def}}{=}\mathrm{E}_4(m+(n-2+\delta)/4),$$
$$\lim_{t\to\infty}\left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{2ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]\right\}$$
$$\overset{\text{def}}{=}\mathrm{F}_4(m+(n-2+\delta)/4),$$
$$\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{1}{n(n+2)}\mathcal{I}_1(0)\mathcal{J}_1,$$
$$\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$=\frac{1}{n(n+2)}\mathcal{I}_1(m+(n-2+\delta)/4)\mathcal{J},$$

and

$$\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{1}{n}\mathcal{I}_2(0)\mathcal{J}_2,$$
$$\lim_{t\to\infty}\left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$=\frac{1}{n}\mathcal{I}_2(m+(n-2+\delta)/4)\mathcal{J}_2,$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, and for all real constants $m\geq 0$.

Therefore, there exists a sufficiently large positive constant T, such that

$$\left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{1ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}$$
$$\leq 2\mathrm{E}_4(m+(n-2+\delta)/4),$$
$$\left\{\sum_{i=1}^n\sum_{j=1}^n\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[\tau^{m+(3n+2+\delta)/4}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\psi_{2ij}(\mathbf{x},\tau)|\mathrm{d}\mathbf{x}\right]^2\right\}$$

$$\le 2\mathrm{F}_4(m+(n-2+\delta)/4),$$

$$\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\le\frac{2}{n(n+2)}\mathcal{I}_1(0)\mathcal{J}_1,$$

$$\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$

$$\le\frac{2}{n(n+2)}\mathcal{I}_1(m+(n-2+\delta)/4)\mathcal{J}_1,$$

$$\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\le\frac{2}{n}\mathcal{I}_2(0)\mathcal{J}_2,$$

$$\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{A}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}$$

$$\le\frac{2}{n}\mathcal{I}_2(m+(n-2+\delta)/4)\mathcal{J}_2,$$

for all $t>T$.

Define

$$\mathrm{E}_1(\mathbf{u}_0)=\sum_{i=1}^n\sum_{j=1}^n\left\{\int_{\mathbb{R}^n}|\phi_{1ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2,$$

$$\mathrm{E}_2(\mathbf{f})=\sum_{i=1}^n\sum_{j=1}^n\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{E}_3(\mathbf{u})=\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{E}_4(m)=\lim_{t\to\infty}\left\{\sum_{i=1}^n\sum_{j=1}^n\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{1ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\},$$

$$\mathrm{F}_1(\mathbf{A}_0)=\sum_{i=1}^n\sum_{j=1}^n\left\{\int_{\mathbb{R}^n}|\phi_{2ij}(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2,$$

$$\mathrm{F}_2(\mathbf{g})=\sum_{i=1}^n\sum_{j=1}^n\left\{\int_0^\infty\int_{\mathbb{R}^n}|\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{F}_3(\mathbf{A})=\left\{\int_0^\infty\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{F}_4(m)=\lim_{t\to\infty}\left\{\sum_{i=1}^n\sum_{j=1}^n\left[t^{m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_{2ij}(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\},$$

$$\mathcal{I}(m)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,$$

$$\mathcal{I}_1(m)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta,$$

$$\mathcal{I}_2(m)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta,$$

$$\begin{aligned}
\mathcal{J}_1 &= n\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{1ij}^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{1ij}^2 + \left[\sum_{i=1}^{n}\alpha_{1ii}\right]^2 - \left[\sum_{i=1}^{n}\lambda_{1ii}\right]^2,\\
\mathcal{J}_2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{2ij}^2,\\
\mathcal{S}_1 &= \mathcal{S}_1(\alpha_1,\delta,\varepsilon,n) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta,\\
\mathcal{S}_2 &= \mathcal{S}_2(\alpha_2,\delta,\varepsilon,n) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta.
\end{aligned}$$

Then define

$$\begin{aligned}
\mathcal{A}_1 &= \mathcal{A}_1(\alpha_1,\delta,\varepsilon,m,n)\\
&= 11\mathcal{I}_1(m)\left\{\mathrm{E}_1(\mathbf{u}_0) + \frac{\mathrm{E}_2(\mathbf{f}) + 2\mathrm{E}_3(\mathbf{u}) + 2\mathrm{F}_3(\mathbf{A})}{\varepsilon^{2m+1+n/2}}\right\},\\
\mathcal{B}_1 &= \mathcal{B}_1(\alpha_1,\delta,\varepsilon,m,n)\\
&= 22C_0(m,n)\mathcal{S}_1(\alpha_1,\delta,\varepsilon,n)\{\mathrm{E}_4(m+(n-2+\delta)/4)\\
&\quad + \frac{4\mathcal{I}_1(0)\mathcal{I}_1(m+(n-2+\delta)/4)}{[n(n+2)]^2}\mathcal{J}_1^2 + \frac{4\mathcal{I}_2(0)\mathcal{I}_2(m+(n-2+\delta)/4)}{n^2}\mathcal{J}_2^2\right\},\\
\mathcal{C}_1 &= \mathcal{C}_1(\alpha_1,\delta,\varepsilon,m,n) = 16\mathcal{I}_1(m)\frac{\mathrm{E}_3(\mathbf{u}) + \mathrm{F}_3(\mathbf{A})}{\varepsilon^{2m+1+n/2}},\\
\mathcal{D}_1 &= \mathcal{D}_1(\alpha_1,\delta,\varepsilon,m,n) = 64C_0(m,n)\mathcal{S}_2(\alpha_2,\delta,\varepsilon,n)\\
&\quad\cdot\left\{\frac{\mathcal{I}_1(0)\mathcal{I}_1(m+(n-2+\delta)/4)}{[n(n+2)]^2}\mathcal{J}_1^2 + \frac{\mathcal{I}_2(0)\mathcal{I}_2(m+(n-2+\delta)/4)}{n^2}\mathcal{J}_2^2\right\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_2 &= \mathcal{A}_2(\alpha_2,\delta,\varepsilon,m,n)\\
&= 7\mathcal{I}_2(m)\left\{\mathrm{F}_1(\mathbf{A}_0) + \frac{\mathrm{F}_2(\mathbf{g}) + \mathrm{F}_3(\mathbf{A}) + \mathrm{E}_3(\mathbf{u})}{\varepsilon^{2m+1+n/2}}\right\},\\
\mathcal{B}_2 &= \mathcal{B}_2(\alpha_2,\delta,\varepsilon,m,n)\\
&= 14C_0(m,n)\mathcal{S}_2(\alpha_2,\delta,\varepsilon,n)\{\mathrm{F}_4(m+(n-2+\delta)/4)\\
&\quad + \frac{2}{n^2(n+2)}[\mathcal{I}_1(0)\mathcal{I}_2(m+(n-2+\delta)/4) + \mathcal{I}_2(0)\mathcal{I}_1(m+(n-2+\delta)/4)]\mathcal{J}_1\mathcal{J}_2\right\},\\
\mathcal{C}_2 &= \mathcal{C}_2(\alpha_2,\delta,\varepsilon,m,n) = 4\mathcal{I}_2(m)\frac{\mathrm{E}_3(\mathbf{u}) + \mathrm{F}_3(\mathbf{A})}{\varepsilon^{2m+1+n/2}},\\
\mathcal{D}_2 &= \mathcal{D}_2(\alpha_2,\delta,\varepsilon,m,n) = \frac{16C_0(m,n)}{n^2(n+2)}\mathcal{S}_2(\alpha_2,\delta,\varepsilon,n)\\
&\quad\cdot\{\mathcal{I}_1(0)\mathcal{I}_2(m+(n-2+\delta)/4) + \mathcal{I}_2(0)\mathcal{I}_1(m+(n-2+\delta)/4)\}\mathcal{J}_1\mathcal{J}_2.
\end{aligned}$$

Remark 3.2. There hold the following relationships

$$\begin{aligned}
\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1\varepsilon|\eta|^2)\mathrm{d}\eta &= \frac{1}{\varepsilon^{2m+1+n/2}}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta,\\
\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2\varepsilon|\eta|^2)\mathrm{d}\eta &= \frac{1}{\varepsilon^{2m+1+n/2}}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta,
\end{aligned}$$

$$\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_1\varepsilon|\eta|^2)]^2}{\alpha_1^2|\eta|^{n+\delta}}\mathrm{d}\eta = \int_{\mathbb{R}^n} \frac{[1-\exp(-|\eta|^2)]^2}{|\eta|^{n+\delta}}\mathrm{d}\eta,$$
$$\int_{\mathbb{R}^n} \frac{[1-\exp(-\alpha_2\varepsilon|\eta|^2)]^2}{\alpha_2^2|\eta|^{n+\delta}}\mathrm{d}\eta = \int_{\mathbb{R}^n} \frac{[1-\exp(-|\eta|^2)]^2}{|\eta|^{n+\delta}}\mathrm{d}\eta,$$

for all positive constants $\alpha_1 > 0$, $\alpha_2 > 0$, $0 < \delta < 4$, $0 < \varepsilon < 1$ and for all $m \geq 0$.

The Proof of Theorem 2.5: By coupling together the estimates in Lemma 3.3 and all of the above estimates, the proof of the improved decay estimates with sharp rates

$$t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \leq \mathcal{A}_1 + \mathcal{B}_1 t^{-n},$$
$$t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x} \leq \mathcal{C}_1 + \mathcal{D}_1 t^{-n},$$

for all order derivatives of the global weak solutions, for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, and for all sufficiently large t, are finished now. $\square$

The Proof of Theorem 2.6: By coupling together the estimates in Lemma 3.4 and all of the above estimates, the proof of the improved decay estimates with sharp rates

$$t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \leq \mathcal{A}_2 + \mathcal{B}_2 t^{-n},$$
$$t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x} \leq \mathcal{C}_2 + \mathcal{D}_2 t^{-n},$$

for all order derivatives of the global weak solutions, for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, and for all sufficiently large t, are finished now. $\square$

3.9. The linear results

The main purposes of this subsection are to establish the exact limits for the global smooth solutions for the Cauchy problems for the heat equations, and to completely finish the proofs Theorem 2.3.

Lemma 3.11. *There hold the following exact limits for the linear equations*

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} = \frac{1}{n}\mathcal{I}_1(m)\mathcal{J}_{01},$$
$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n} |(-\triangle)^m\mathbf{B}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} = \frac{1}{n}\mathcal{I}_2(m)\mathcal{J}_{02},$$

for all real constants $m \geq 0$, where

$$\mathcal{I}_1(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta,\ \ \mathcal{J}_{01} = \sum_{i=1}^n\sum_{j=1}^n \alpha_{1ij}^2,$$
$$\mathcal{I}_2(m) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta,\ \ \mathcal{J}_{02} = \sum_{i=1}^n\sum_{j=1}^n \alpha_{2ij}^2.$$

Proof. First of all, there hold the following representations for the Fourier transformations of the global smooth solutions

$$
\begin{aligned}
t^{1/2}\widehat{v}_i(t^{-1/2}\eta, t) &= \mathrm{i}\exp(-\alpha_1|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{1ij}(t^{-1/2}\eta, t)\\
+\,\mathrm{i}\int_0^t \exp\Big[-\alpha_1|\eta|^2(1-\frac{\tau}{t})\Big]&\sum_{j=1}^{n}\eta_j\widehat{\psi}_{1ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
t^{1/2}\widehat{B}_i(t^{-1/2}\eta, t) &= \mathrm{i}\exp(-\alpha_2|\eta|^2)\sum_{j=1}^{n}\eta_j\widehat{\phi}_{2ij}(t^{-1/2}\eta, t)\\
+\,\mathrm{i}\int_0^t \exp\Big[-\alpha_2|\eta|^2(1-\frac{\tau}{t})\Big]&\sum_{j=1}^{n}\eta_j\widehat{\psi}_{2ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta, t)\in\mathbb{R}^n\times\mathbb{R}^+$, where $i=1,2,3,\cdots,n$. The rest of the proof follows from the estimates in Lemma 3.1 and Lemma 3.2 and the exact limits in Lemma 3.7 and Lemma 3.8. The details are skipped because they are very easy. □

Lemma 3.12. *There hold the following ratios*

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{4m+n+2}{4\alpha_1},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{(4m+n+2)(4m+n+4)}{(4\alpha_1)^2},
\end{aligned}
$$

for all real constants $m\geq 0$. Moreover

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{B}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{B}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{4m+n+2}{4\alpha_2},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{B}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{B}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{(4m+n+2)(4m+n+4)}{(4\alpha_2)^2},
\end{aligned}
$$

for all real constants $m\geq 0$.

Proof. The proof follows from the results of Lemma 3.10 and Lemma 3.11. The details are skipped because they are very easy. □

The proofs of Theorem 2.1, Theorem 2.2, Theorem 2.3, Theorem 2.4, Theorem 2.5 and Theorem 2.6 are finished. The proofs of the main results for the magnetohydrodynamics equations are finished now.

3.10. Summary

Consider the n-dimensional incompressible magnetohydrodynamics equations. Suppose that the initial functions are divergence free and satisfy the following conditions

$$\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$
$$\mathbf{A}_0 \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

Suppose that the external forces are divergence free and satisfy the following conditions

$$\mathbf{f} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)),$$
$$\mathbf{g} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)).$$

Suppose that there exist real scalar smooth functions

$$\phi_{1ij} \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),\ \ \psi_{1ij} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+),$$
$$\phi_{2ij} \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),\ \ \psi_{2ij} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+),$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$, such that

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \phi_{1ij}(\mathbf{x}) = 0,\ \ \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \psi_{1ij}(\mathbf{x}, t) = 0,$$
$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \phi_{2ij}(\mathbf{x}) = 0,\ \ \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \psi_{2ij}(\mathbf{x}, t) = 0,$$

and that the initial functions and the external forces satisfy

$$\mathbf{u}_0(\mathbf{x}) = \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{11j}(\mathbf{x}), \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{12j}(\mathbf{x}), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{1nj}(\mathbf{x}) \right),$$
$$\mathbf{f}(\mathbf{x}, t) = \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{11j}(\mathbf{x}, t), \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{12j}(\mathbf{x}, t), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{1nj}(\mathbf{x}, t) \right),$$
$$\mathbf{A}_0(\mathbf{x}) = \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{21j}(\mathbf{x}), \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{22j}(\mathbf{x}), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{2nj}(\mathbf{x}) \right),$$
$$\mathbf{g}(\mathbf{x}, t) = \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{21j}(\mathbf{x}, t), \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{22j}(\mathbf{x}, t), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{2nj}(\mathbf{x}, t) \right),$$

for all $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

Then there hold the exact limits

$$\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\} = \frac{1}{n(n+2)} \mathcal{I}_1(m) \mathcal{J}_1,$$
$$\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \right\} = \frac{1}{n(n+2)} \mathcal{I}_1(m) \mathcal{K}_1,$$

for all real constants $m \geq 0$.

Moreover, there hold the following exact limits

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(m)\mathcal{J}_2,\\
&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n}\mathcal{I}_2(m)\mathcal{K}_2,
\end{aligned}
$$

for all real constants $m\geq 0$.

In these limits,

$$
\begin{aligned}
\mathcal{I}_1(m)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_1|\eta|^2)\mathrm{d}\eta,\\
\mathcal{J}_1&=n\sum_{i=1}^n\sum_{j=1}^n\lambda_{1ij}^2+2\sum_{i=1}^n\sum_{j=1}^n\alpha_{1ij}^2+\left[\sum_{i=1}^n\alpha_{1ii}\right]^2-\left[\sum_{i=1}^n\lambda_{1ii}\right]^2,\\
\mathcal{K}_1&=n\sum_{i=1}^n\sum_{j=1}^n\rho_{1ij}^2-\left[\sum_{i=1}^n\rho_{1ii}\right]^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{1ij}&=\int_{\mathbb{R}^n}\phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\lambda_{1ij}&=\int_{\mathbb{R}^n}\phi_{1ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{1ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\quad-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\rho_{1ij}&=\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned}
$$

for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$, and

$$
\begin{aligned}
\mathcal{I}_2(m)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha_2|\eta|^2)\mathrm{d}\eta,\\
\mathcal{J}_2&=\sum_{i=1}^n\sum_{j=1}^n\lambda_{2ij}^2,\\
\mathcal{K}_2&=\sum_{i=1}^n\sum_{j=1}^n\rho_{2ij}^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{2ij}&=\int_{\mathbb{R}^n}\phi_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{2ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\lambda_{2ij}&=\int_{\mathbb{R}^n}\phi_{2ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{2ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\quad-\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\rho_{2ij}&=\int_0^\infty\int_{\mathbb{R}^n}A_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)A_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned}
$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Additionally, there hold the following primary decay estimates with sharp rates

$$\sup_{t>T}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$
$$\sup_{t>T}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all real constants $m \geq 0$, where T is a sufficiently large positive constant.

Furthermore, there hold the following improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq\mathcal{A}_1+\mathcal{B}_1t^{-n},$$
$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\leq\mathcal{C}_1+\mathcal{D}_1t^{-n},$$
$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\leq\mathcal{A}_2+\mathcal{B}_2t^{-n},$$
$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{A}(\mathbf{x},t)-\mathbf{B}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\leq\mathcal{C}_2+\mathcal{D}_2t^{-n},$$

for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, and for all sufficiently large t, where

$$\mathcal{A}_1=\mathcal{A}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{A}_2=\mathcal{A}_2(\alpha_2,\delta,\varepsilon,m,n),$$
$$\mathcal{B}_1=\mathcal{B}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{B}_2=\mathcal{B}_2(\alpha_2,\delta,\varepsilon,m,n),$$
$$\mathcal{C}_1=\mathcal{C}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{C}_2=\mathcal{C}_2(\alpha_2,\delta,\varepsilon,m,n),$$
$$\mathcal{D}_1=\mathcal{D}_1(\alpha_1,\delta,\varepsilon,m,n),\qquad \mathcal{D}_2=\mathcal{D}_2(\alpha_2,\delta,\varepsilon,m,n).$$

3.11. Remarks and open problems

Remark 3.3. The exact limits depend on the integrals of ϕ_{1ij} and ϕ_{2ij}, the integrals of ψ_{1ij} and ψ_{2ij}, and the integrals of the nonlinear functions u_iu_j, A_iA_j, A_iu_j and u_iA_j. However, they are independent of

(1) the integrals of any order derivatives of the functions ϕ_{1ij} and ϕ_{2ij},

(2) the integrals of any order derivatives of the functions ψ_{1ij} and ψ_{2ij},

(3) the integrals of any order derivatives of the functions u_iu_j, A_iA_j, A_iu_j and u_iA_j,

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Remark 3.4. The exact limits are increasing functions of the order m of derivatives. The exact limits are decreasing functions of the diffusion coefficients α_1 and α_2.

Remark 3.5. The exact limits of the global weak solutions of the n-dimensional incompressible magnetohydrodynamics equations reduce to the exact limits of the global smooth solutions of the corresponding linear equation, when the nonlinear functions are dropped.

Remark 3.6. It is known that the diffusion coefficients satisfy

$$\alpha_1 = \frac{1}{\mathrm{RE}} \gg \alpha_2 = \frac{1}{\mathrm{RM}} > 0.$$

Therefore, the exact limits

$$\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\}$$

increases much slower than the exact limits

$$\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\},$$

as the order of derivatives m increases.

Remark 3.7. The primary decay estimates with sharp rates for all order derivatives of the global weak solutions of the incompressible magnetohydrodynamics equations are true for all time, if there exists a global smooth solution; the decay estimates are true for all sufficiently large time, if there exists a global weak solution.

Remark 3.8. Recall that there hold the elementary decay estimates with a sharp rate for the n-dimensional incompressible magnetohydrodynamics equations

$$\sup_{t>0} \left\{ (1+t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty,$$
$$\sup_{t>0} \left\{ (1+t)^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{A}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty.$$

Therefore, there exist the following integrals

$$\int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t) u_j(\mathbf{x},t) \mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t) A_j(\mathbf{x},t) \mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\int_0^\infty \int_{\mathbb{R}^n} A_i(\mathbf{x},t) u_j(\mathbf{x},t) \mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\int_0^\infty \int_{\mathbb{R}^n} u_i(\mathbf{x},t) A_j(\mathbf{x},t) \mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Remark 3.9. To make the positive constants

$$\mathcal{A}_k = \mathcal{A}_k(\alpha_k, \delta, \varepsilon, m, n), \qquad \mathcal{C}_k = \mathcal{C}_k(\alpha_k, \delta, \varepsilon, m, n)$$

small, we would like to choose ε close to 1, such as $\varepsilon = 0.99$, for $k = 1, 2$. To make the positive constants

$$\mathcal{B}_k = \mathcal{B}_k(\alpha_k, \delta, \varepsilon, m, n), \qquad \mathcal{D}_k = \mathcal{D}_k(\alpha_k, \delta, \varepsilon, m, n)$$

small, we would like to choose δ close to 1, such as $\delta = 0.99$, for $k = 1, 2$.

Remark 3.10. The decay results are called improved decay estimates with sharp rates, because after some time, t^{-n} becomes arbitrarily small, so that we may almost ignore

$$\mathcal{B}_k(m)t^{-n}, \qquad \mathcal{D}_k(m)t^{-n}.$$

The positive constants

$$\mathcal{A}_k = \mathcal{A}_k(\alpha_k, \delta, \varepsilon, m, n), \qquad \mathcal{C}_k = \mathcal{C}_k(\alpha_k, \delta, \varepsilon, m, n)$$

have been represented explicitly in terms of simple integrals, for $k = 1, 2$. Therefore, the new estimates would have positive impacts to numerical simulations.

Open Problem: Suppose that the initial functions are divergence free and satisfy the conditions

$$\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap H^{2m}(\mathbb{R}^n), \qquad \mathbf{A}_0 \in C^1(\mathbb{R}^n) \cap H^{2m}(\mathbb{R}^n).$$

Suppose that the external forces are divergence free and satisfy the conditions

$$\begin{aligned} \mathbf{f} &\in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)), \\ \mathbf{g} &\in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)), \end{aligned}$$

where m is a sufficiently large positive constant. The existence and uniqueness of the global smooth solution

$$\begin{aligned} \mathbf{u} &\in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)),\ \ \nabla\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)), \\ \mathbf{A} &\in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)),\ \ \nabla\mathbf{A} \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^n)), \end{aligned}$$

have been open for a long time.

4. The incompressible Navier-Stokes equations

The Mathematical Model Equations and Known Related Results

Consider the Cauchy problem for the n-dimensional incompressible Navier-Stokes equations

$$\frac{\partial}{\partial t}\mathbf{u} - \alpha\triangle\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(\mathbf{x}, t), \qquad \nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{f} = 0, \quad (4.1)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \qquad \nabla\cdot\mathbf{u}_0 = 0. \quad (4.2)$$

In the Navier-Stokes equations, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ represents the velocity of the fluid at position $\mathbf{x} \in \mathbb{R}^n$ and time $t \in \mathbb{R}^+$, $p = p(\mathbf{x}, t)$ represents the pressure of the fluid at $\mathbf{x} \in \mathbb{R}^n$ and time $t \in \mathbb{R}^+$. The positive constant $\alpha > 0$ represents the diffusion coefficient.

Suppose that the initial function and the external force are divergence free and satisfy the following conditions

$$\mathbf{u}_0 \in \mathcal{S}(\mathbb{R}^2), \qquad \mathbf{f} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).$$

Then there exists a unique global smooth solution

$$\mathbf{u} \in C^{\infty}(\mathbb{R}^2 \times \mathbb{R}^+).$$

It is well known that there exists a global weak solution

$$\mathbf{u} \in L^{\infty}(\mathbb{R}^+, L^2(\mathbb{R}^n)), \qquad \nabla \mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)),$$

if the dimension $n = 3, 4, 5$ and if the initial function and the external force are divergence free and satisfy the following conditions

$$\begin{aligned} &\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \\ &\mathbf{f} \in C^{\infty}(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)). \end{aligned}$$

Moreover, there holds the following elementary uniform energy estimate for the global weak solutions

$$\begin{aligned} &\left\{ \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t \int_{\mathbb{R}^n} |\nabla \mathbf{u}(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \mathrm{d}\tau \right\}^{1/2} \\ &\le \left\{ \int_{\mathbb{R}^n} |\mathbf{u}_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} + \int_0^{\infty} \left\{ \int_{\mathbb{R}^n} |\mathbf{f}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} \mathrm{d}t. \end{aligned}$$

Additionally, the global weak solutions become small enough and sufficiently smooth after a long time. That is, there exists a sufficiently large positive constant T, such that

$$\sup_{t>T} \left\{ \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty,$$

for all positive constants $m > 0$. Furthermore, there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0} \left\{ t^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty.$$

Also consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}\mathbf{v} - \alpha \triangle \mathbf{v} = \mathbf{f}(\mathbf{x}, t), \qquad \nabla \cdot \mathbf{v} = 0, \qquad \nabla \cdot \mathbf{f} = 0, \tag{4.3}$$

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \qquad \nabla \cdot \mathbf{u}_0 = 0. \tag{4.4}$$

The Main Purposes

For the n-dimensional incompressible Navier-Stokes equations, we will accomplish the following exact limits

$$\begin{aligned} &\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\}, \\ &\lim_{t\to\infty} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \right\}, \end{aligned}$$

and then apply them to establish the improved decay estimates with sharp rates

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le\mathcal{A}(\alpha,\delta,\varepsilon,m,n)+\mathcal{B}(\alpha,\delta,\varepsilon,m,n)t^{-n},$$
$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\le\mathcal{C}(\alpha,\delta,\varepsilon,m,n)+\mathcal{D}(\alpha,\delta,\varepsilon,m,n)t^{-n},$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, for all order derivatives of the global weak solutions and for all $t>T$, where T is a sufficiently large positive constant.

For these purposes, we will prove the primary decay estimates with sharp rates for all order derivatives of the global weak solutions

$$\sup_{t>T}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

for all positive constant $m>0$.

The Main Difficulties

The most difficult technical problems in the mathematical analysis are the control of the following integrals

$$\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^t\exp\left[-\alpha|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{\psi}_{ij}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,$$
$$\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^t\exp\left[-\alpha|\eta|^2(1-\frac{\tau}{t})\right]\sum_{j=1}^n\eta_j\widehat{u_iu_j}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,$$
$$\int_{\mathbb{R}^n}|\eta|^{4m}\sum_{i=1}^n\left|\int_{(1-\varepsilon)t}^t\exp\left[-\alpha|\eta|^2(1-\frac{\tau}{t})\right]\frac{\eta_i}{|\eta|^2}\sum_{k=1}^n\sum_{l=1}^n\eta_k\eta_l\widehat{u_ku_l}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,$$

for all real constants $m\ge 0$, where $0<\varepsilon<1$ is a positive constant.

We are able to use new ideas to establish the optimal estimates for these integrals. In particular, we are able to use the following singular integral

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta$$

and the primary decay estimates with sharp rates to obtain the best possible estimates for these integrals.

First of all, we will apply iteration technique to get the primary decay estimates with sharp sates for all order derivatives of the global weak solutions. Secondly, we will couple together various ideas, methods and techniques to accomplish the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions.

The Mathematical Assumptions

First of all, let us make the following assumptions for the Navier-Stokes equations. Let $0 < \delta < 4$ and $0 < \varepsilon < 1$ be positive constants. Let $m \geq 0$ be a real constant.

(A1) Suppose that the initial function and the external force are divergence free and satisfy the following conditions

$$\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n),$$
$$\mathbf{f} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)).$$

(A2) Suppose that there exist real scalar smooth functions

$$\phi_{ij} \in C^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), \qquad \psi_{ij} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^n \times \mathbb{R}^+),$$

where $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$, such that

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \phi_{ij}(\mathbf{x}) = 0, \qquad \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} \psi_{ij}(\mathbf{x}, t) = 0,$$

and that

$$\mathbf{u}_0(\mathbf{x}) = \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{1j}(\mathbf{x}), \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{2j}(\mathbf{x}), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \phi_{nj}(\mathbf{x}) \right),$$
$$\mathbf{f}(\mathbf{x}, t) = \left(\sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{1j}(\mathbf{x}, t), \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{2j}(\mathbf{x}, t), \cdots, \sum_{j=1}^n \frac{\partial}{\partial x_j} \psi_{nj}(\mathbf{x}, t) \right),$$

for all $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

This assumption is motivated by the incompressible conditions $\nabla \cdot \mathbf{u}_0 = 0$ and $\nabla \cdot \mathbf{f} = 0$, which imply that

$$\int_{\mathbb{R}^n} \mathbf{u}_0(\mathbf{x}) = \mathbf{0}, \qquad \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x}, t) \mathrm{d}\mathbf{x} = \mathbf{0},$$

for all $t > 0$, if $\mathbf{u}_0 \in L^1(\mathbb{R}^n)$ and $\mathbf{f} \in L^1(\mathbb{R}^n)$, for all $t > 0$. This assumption is also motivated by the special structure of the nonlinear function $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

(A3) Suppose that there exist the following limits

$$\lim_{t \to \infty} \left\{ \sum_{i=1}^n \sum_{j=1}^n \left[t^{m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \psi_{ij}(\mathbf{x}, t)| \mathrm{d}\mathbf{x} \right]^2 \right\} \stackrel{\text{def}}{=} \mathrm{E}_4(m),$$

for all real constants $m \geq 0$.

Here is a slightly weaker condition

$$\sup_{t > 0} \left\{ \sum_{i=1}^n \sum_{j=1}^n \left[t^{m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \psi_{ij}(\mathbf{x}, t)| \mathrm{d}\mathbf{x} \right]^2 \right\} < \infty,$$

for all real constants $m \geq 0$.

(A4) Suppose that there exists a global smooth solution

$$\mathbf{u} \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)), \qquad \nabla \mathbf{u} \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)),$$

if the dimension $n = 2$ and if the initial function and the external force are divergence free and satisfy the conditions

$$\mathbf{u}_0 \in C^1(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2), \qquad \mathbf{f} \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)).$$

Suppose that there exists a global weak solution

$$\mathbf{u} \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^n)), \qquad \nabla\mathbf{u} \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^n)),$$

if $n \geq 3$ and if the initial function and the external force are divergence free and satisfy the conditions

$$\mathbf{u}_0 \in C^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n), \qquad \mathbf{f} \in C^1(\mathbb{R}^n \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^n)).$$

(A5) Suppose that the global weak solutions become small enough and sufficiently smooth after a long time. Namely, there exists a sufficiently large positive constant T, such that

$$\mathbf{u} \in L^\infty((T,\infty), H^{2m}(\mathbb{R}^n)), \qquad \nabla\mathbf{u} \in L^2((T,\infty), H^{2m}(\mathbb{R}^n)),$$

for all positive constants $m > 0$. That is

$$\sup_{T<t<\infty}\left\{\int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\right\} < \infty,$$
$$\int_T^\infty \int_{\mathbb{R}^n} |(-\triangle)^{m+1/2}\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t < \infty.$$

(A6) Suppose that there hold the following representations for the Fourier transformations of the global weak solutions

$$\begin{aligned}
&\widehat{u}_i(\xi,t) = \mathrm{i}\exp(-\alpha|\xi|^2 t)\sum_{j=1}^n \xi_j \widehat{\phi}_{ij}(\xi)\\
&+\mathrm{i}\int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j \widehat{\psi}_{ij}(\xi,\tau)\mathrm{d}\tau\\
&-\mathrm{i}\int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\sum_{j=1}^n \xi_j \widehat{u_i u_j}(\xi,\tau)\mathrm{d}\tau\\
&+\mathrm{i}\int_0^t \exp[-\alpha|\xi|^2(t-\tau)]\frac{\xi_i}{|\xi|^2}\sum_{k=1}^n\sum_{l=1}^n \xi_k\xi_l \widehat{u_k u_l}(\xi,\tau)\mathrm{d}\tau,\\
&\widehat{p}(\xi,t) = -\frac{1}{|\xi|^2}\sum_{k=1}^n\sum_{l=1}^n \xi_k\xi_l\widehat{u_k u_l}(\xi,t),
\end{aligned}$$

for all $(\xi,t) \in \mathbb{R}^n \times \mathbb{R}^+$ and for all $i = 1,2,3,\cdots,n$. However, $\xi \neq \mathbf{0}$ in the representations of $\widehat{u}_i(\xi,t)$ and $\widehat{p}(\xi,t)$.

(A7) Suppose that there holds the elementary decay estimate with the sharp rate $r = 1 + n/2$

$$\sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty.$$

The existence of the global smooth solution, the existence of the global weak solutions, the existence of the local smooth solution on some unbounded interval (T, ∞) and the elementary decay estimate with the sharp rate $r = 1 + n/2$ and other related results have been proved very well by many mathematicians. See Beirao da Veiga [2], Bradshaw, Kukavia and Tsai [3], Fefferman [12], Feichtinger, Grochenig, Li and Wang [13], Fujita and Kato [15], Heywood [16], Ji, Wu and Yang, [17], Kato [20], Ladyzhenskaya [22], Lei and Lin [23], Lei, Lin and Zhou [24], Leray [25], Li, Ozawa and Wang [26], Li, Tan and Xu [27], Lin [28], Lin, Suo and Wu [29], Liu and Zhang [30], Miyakawa and Sohr [31], Oliver and Titi [32], Peng and Zhou [33], Robinson [34], Maria E. Schonbek [35], Maria E. Schonbek and Tomas P. Schonbek [36], Maria E. Schonbek, Tomas P. Schonbek and Suli [37], Maria E. Schonbek and Michael Wiegner [38], Sverak [39], Temam [40] and [41], Zhang [43]- [44].

The Main Results

Suppose that the assumptions (A1) - (A7) on the initial function, the external force and the global weak solutions are true. There are two parts in the main results.

Part 1: The exact limits for all order derivatives of the global weak solutions of the n-dimensional incompressible Navier-Stokes equations.

Theorem 4.1. *There hold the following exact limits*

$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n(n+2)}\mathcal{I}(m)\mathcal{J},$$
$$\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n(n+2)}\mathcal{I}(m)\mathcal{K},$$

for all order derivatives of the global weak solutions of the incompressible Navier-Stokes equations.

In Theorem 4.1, the parameters are given by

$$\mathcal{I}(m)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,$$
$$\mathcal{J}=n\sum_{i=1}^n\sum_{j=1}^n\lambda_{ij}^2+2\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}^2+\left[\sum_{i=1}^n\alpha_{ii}\right]^2-\left[\sum_{i=1}^n\lambda_{ii}\right]^2,$$
$$\mathcal{K}=n\sum_{i=1}^n\sum_{j=1}^n\rho_{ij}^2-\left[\sum_{i=1}^n\rho_{ii}\right]^2,$$

where

$$\alpha_{ij}=\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\lambda_{ij}=\int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$
$$\rho_{ij}=\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,$$

for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$.

Clearly

$$\alpha_{ij} = \lambda_{ij} + \rho_{ij}, \qquad \rho_{ji} = \rho_{ij},$$

for all $i = 1, 2, 3, \cdots, n$ *and* $j = 1, 2, 3, \cdots, n$.

Theorem 4.2. *The ratio of the exact limits for the global weak solutions of the Navier-Stokes equations is the same as the ratio of the exact limits for the global smooth solution of the corresponding linear equation, for each constant* $m \geq 0$. *That is*

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&=\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{4m+n+2}{4\alpha},
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&=\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{(4m+n+2)(4m+n+4)}{(4\alpha^2)^2},
\end{aligned}$$

for all real constants $m \geq 0$. *Moreover*

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&=\left\{\lim_{t\to\infty}\left[t^{2m+2+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1/2}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}=\frac{4m+n+2}{4\alpha},
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&/\left\{\lim_{t\to\infty}\left[t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m}[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&=\left\{\lim_{t\to\infty}\left[t^{2m+3+n/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+1}\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}
\end{aligned}$$

$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{v}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right] \right\} = \frac{(4m+n+2)(4m+n+4)}{(4\alpha)^2},$$

for all real constants $m \geq 0$.

Part 2: The primary decay estimates and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the n**-dimensional incompressible Navier-Stokes equations.**

Theorem 4.3. *There hold the following primary decay estimates with sharp rates*

$$\sup_{t>T} \left\{ t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty,$$

for all order derivatives of the global weak solutions of (4.1-4.2).

Theorem 4.4. *There hold the following improved decay estimates with sharp rates*

$$t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \leq \mathcal{A}(\alpha,\delta,\varepsilon,m,n) + \mathcal{B}(\alpha,\delta,\varepsilon,m,n) t^{-n},$$
$$t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{u}(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \leq \mathcal{C}(\alpha,\delta,\varepsilon,m,n) + \mathcal{D}(\alpha,\delta,\varepsilon,m,n) t^{-n},$$

for all positive constants $0 < \delta < 4$ *and* $0 < \varepsilon < 1$, *for all order derivatives of the global weak solutions of (4.1-4.2), and for all sufficiently large* t.

In Theorem 4.4, the positive constants are given explicitly by

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}(\alpha,\delta,\varepsilon,m,n) = 7\mathcal{I}(m) \left\{ \mathrm{E}_1(\mathbf{u}_0) + \frac{\mathrm{E}_2(\mathbf{f}) + 2\mathrm{E}_3(\mathbf{u})}{\varepsilon^{2m+1+n/2}} \right\}, \\
\mathcal{B} &= \mathcal{B}(\alpha,\delta,\varepsilon,m,n) = 14C_0(m,n)\mathcal{S}(\alpha,\delta,\varepsilon,n) \\
&\quad \cdot \left\{ \mathrm{E}_4(m + (n-2+\delta)/4) + \frac{4\mathcal{J}^2}{[n(n+2)]^2} \mathcal{I}(0)\mathcal{I}(m + (n-2+\delta)/4) \right\}, \\
\mathcal{C} &= \mathcal{C}(\alpha,\delta,\varepsilon,m,n) = 8\mathcal{I}(m) \frac{\mathrm{E}_3(\mathbf{u})}{\varepsilon^{2m+1+n/2}}, \\
\mathcal{D} &= \mathcal{D}(\alpha,\delta,\varepsilon,m,n) \\
&= 64C_0(m,n)\mathcal{S}(\alpha,\delta,\varepsilon,n) \left\{ \frac{\mathcal{J}^2}{[n(n+2)]^2} \mathcal{I}(0)\mathcal{I}(m + (n-2+\delta)/4) \right\},
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{E}_1(\mathbf{u}_0) &= \sum_{i=1}^n \sum_{j=1}^n \left\{ \int_{\mathbb{R}^n} |\phi_{ij}(\mathbf{x})| \mathrm{d}\mathbf{x} \right\}^2, \\
\mathrm{E}_2(\mathbf{f}) &= \sum_{i=1}^n \sum_{j=1}^n \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\psi_{ij}(\mathbf{x},t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2, \\
\mathrm{E}_3(\mathbf{u}) &= \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2, \\
\mathrm{E}_4(m) &= \lim_{t\to\infty} \left\{ \sum_{i=1}^n \sum_{j=1}^n \left[t^{m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \psi_{ij}(\mathbf{x},t)| \mathrm{d}\mathbf{x} \right]^2 \right\},
\end{aligned}$$

$$\mathcal{I}(m) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta,$$

$$\mathcal{J} = n \sum_{i=1}^n \sum_{j=1}^n \lambda_{ij}^2 + 2 \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}^2 + \left[\sum_{i=1}^n \alpha_{ii} \right]^2 - \left[\sum_{i=1}^n \lambda_{ii} \right]^2,$$

$$\mathcal{S}(\alpha, \delta, \varepsilon, n) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2 |\eta|^{n+\delta}} \mathrm{d}\eta.$$

These explicit representations of the constants $\mathcal{A}(m)$, $\mathcal{B}(m)$, $\mathcal{C}(m)$, $\mathcal{D}(m)$ make it possible to directly or indirectly influence the accuracy and stability of numerical schemes in scientific computations for the n-dimensional incompressible Navier-Stokes equations.

Proof. The main ideas and the main steps are almost the same as those for the n-dimensional incompressible magnetohydrodynamics equations. There are some minor differences (it is simpler) in the details. Let us provide the estimates in the comprehensive analysis. There hold the following estimates.

(1)

$$\begin{aligned}
& t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m \mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \\
\le\ & \frac{7}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \sum_{i=1}^n \sum_{j=1}^n \left\{ \int_{\mathbb{R}^n} |\phi_{ij}(\mathbf{x})| \mathrm{d}\mathbf{x} \right\}^2 \\
& + \frac{7}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \sum_{i=1}^n \sum_{j=1}^n \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\psi_{ij}(\mathbf{x}, t)| \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2 \\
& + \frac{14}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2 \\
& + \frac{7C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
& \cdot \left\{ \sum_{i=1}^n \sum_{j=1}^n \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{m+(3n+2+\delta)/4} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \psi_{ij}(\mathbf{x}, \tau)| \mathrm{d}\mathbf{x} \right]^2 \right\} \\
& + \frac{14C_0}{(2\pi t)^n} \left\{ \int_{\mathbb{R}^n} \frac{[1 - \exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2 |\eta|^{n+\delta}} \mathrm{d}\eta \right\} \\
& \cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1+n/2} \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
& \cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{2m+n+\delta/2} \int_{\mathbb{R}^n} |(-\triangle)^{m+(n-2+\delta)/4} \mathbf{u}(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.
\end{aligned}$$

(2)

$$\begin{aligned}
& t^{2m+1+n/2} \int_{\mathbb{R}^n} |(-\triangle)^m [\mathbf{u}(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \\
\le\ & \frac{8}{(2\pi)^n} \int_{\mathbb{R}^n} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathbf{u}(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \mathrm{d}t \right\}^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{8C_0}{(2\pi t)^n}\left\{\int_{\mathbb{R}^n}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{n+\delta}}\mathrm{d}\eta\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
&\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[\tau^{2m+n+\delta/2}\int_{\mathbb{R}^n}|(-\triangle)^{m+(n-2+\delta)/4}\mathbf{u}(\mathbf{x},\tau)|^2\mathrm{d}\mathbf{x}\right]\right\}.
\end{aligned}
$$

These estimates are true for all positive constants $0<\delta<4$ and $0<\varepsilon<1$.

The other details of the proof are skipped. □

Summary

Consider the n-dimensional incompressible Navier-Stokes equations. Suppose that the initial function and the external force are divergence free and satisfy the following conditions

$$
\begin{aligned}
&\mathbf{u}_0\in C^1(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n),\\
&\mathbf{f}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R}^n)).
\end{aligned}
$$

Suppose that there exist real scalar smooth functions

$$
\phi_{ij}\in C^2(\mathbb{R}^n)\cap L^1(\mathbb{R}^n),\qquad \psi_{ij}\in C^\infty(\mathbb{R}^n\times\mathbb{R}^+)\cap L^1(\mathbb{R}^n\times\mathbb{R}^+),
$$

such that

$$
\sum_{i=1}^n\sum_{j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}\phi_{ij}(\mathbf{x})=0,\qquad \sum_{i=1}^n\sum_{j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}\psi_{ij}(\mathbf{x},t)=0,
$$

and

$$
\begin{aligned}
\mathbf{u}_0(\mathbf{x})&=\left(\sum_{j=1}^n\frac{\partial}{\partial x_j}\phi_{1j}(\mathbf{x}),\sum_{j=1}^n\frac{\partial}{\partial x_j}\phi_{2j}(\mathbf{x}),\cdots,\sum_{j=1}^n\frac{\partial}{\partial x_j}\phi_{nj}(\mathbf{x})\right),\\
\mathbf{f}(\mathbf{x},t)&=\left(\sum_{j=1}^n\frac{\partial}{\partial x_j}\psi_{1j}(\mathbf{x},t),\sum_{j=1}^n\frac{\partial}{\partial x_j}\psi_{2j}(\mathbf{x},t),\cdots,\sum_{j=1}^n\frac{\partial}{\partial x_j}\psi_{nj}(\mathbf{x},t)\right),
\end{aligned}
$$

for all $(\mathbf{x},t)\in\mathbb{R}^n\times\mathbb{R}^+$.

Then there hold the following exact limits

$$
\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n(n+2)}\mathcal{I}(m)\mathcal{J},\\
&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}=\frac{1}{n(n+2)}\mathcal{I}(m)\mathcal{K},
\end{aligned}
$$

for all real constants $m\ge 0$.

In these limits,

$$
\mathcal{I}(m)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,
$$

$$\mathcal{J} = n\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^2 + \left[\sum_{i=1}^{n}\alpha_{ii}\right]^2 - \left[\sum_{i=1}^{n}\lambda_{ii}\right]^2,$$
$$\mathcal{K} = n\sum_{i=1}^{n}\sum_{j=1}^{n}\rho_{ij}^2 - \left[\sum_{i=1}^{n}\rho_{ii}\right]^2,$$

where

$$\begin{aligned}
\alpha_{ij} &= \int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\lambda_{ij} &= \int_{\mathbb{R}^n}\phi_{ij}(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^n}\psi_{ij}(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,\\
\rho_{ij} &= \int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned}$$

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Additionally, there hold the following primary decay estimates with sharp rates

$$\sup_{t>T}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty,$$

$$\sup_{t>T}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty,$$

for all real constants $m \geq 0$, where T is a sufficiently large positive constant.

Furthermore, there hold the following improved decay estimates with sharp rates for all order derivatives of the global weak solutions

$$\begin{aligned}
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \leq \mathcal{A}(\alpha,\delta,\varepsilon,m,n) + \mathcal{B}(\alpha,\delta,\varepsilon,m,n)t^{-n},\\
&t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m[\mathbf{u}(\mathbf{x},t)-\mathbf{v}(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x} \leq \mathcal{C}(\alpha,\delta,\varepsilon,m,n) + \mathcal{D}(\alpha,\delta,\varepsilon,m,n)t^{-n},
\end{aligned}$$

for all $m \geq 0$ and for all sufficiently large t.

Remarks and Open Problems

Remark 4.1. The exact limits depend on the integrals of ϕ_{ij}, the integrals of ψ_{ij} and the integrals of the nonlinear functions u_iu_j. However, they are independent of

(1) the integrals of any order derivatives of the functions ϕ_{ij},

(2) the integrals of any order derivatives of the functions ψ_{ij},

(3) the integrals of any order derivatives of the functions u_iu_j,

for all $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, n$.

Remark 4.2. The exact limits are increasing functions of the order m of derivatives. The exact limits are decreasing functions of the diffusion coefficient α.

Remark 4.3. The exact limits for the n-dimensional incompressible Navier-Stokes equations reduce to the exact limits of the global smooth solution of the corresponding linear equation, if the nonlinear functions are dropped.

Remark 4.4. The primary decay estimates with sharp rates for all order derivatives of the global weak solutions of the incompressible Navier-Stokes equations are true for all time, if there exists a global smooth solution; the primary decay estimates are true for all sufficiently large time, if there exists a global weak solution.

Remark 4.5. For the n-dimensional incompressible Navier-Stokes equations, recall that there holds the elementary decay estimate with a sharp rate

$$\sup_{t>0}\left\{(1+t)^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty.$$

Therefore, there exist the following integrals

$$\int_0^\infty\int_{\mathbb{R}^n}u_i(\mathbf{x},t)u_j(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t<\infty,$$

for all $i=1,2,3,\cdots,n$ and $j=1,2,3,\cdots,n$.

Open Problem 4.1 Consider the n-dimensional incompressible Navier-Stokes equations again. Are the following estimates

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\ \le&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\end{aligned}$$

true, for all real constants $m\ge 0$?

Alternatively, are the following estimates

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\\ \ge&\lim_{t\to\infty}\left\{t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}\end{aligned}$$

true, for all real constants $m\ge 0$?

Open Problem 4.2 Consider the Navier-Stokes equations again. The following problem is motivated by the first open problem. Are the following estimates

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\le t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$$

true, for all real constants $m\ge 0$ and for all $t>0$?

Alternatively, are the following estimates

$$t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\ge t^{2m+1+n/2}\int_{\mathbb{R}^n}|(-\triangle)^m\mathbf{v}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$$

true, for all real constants $m\ge 0$ and for all $t>0$?

These are important open problems closely related to the existence of the global smooth solution of the n-dimensional incompressible Navier-Stokes equations.

5. The two-dimensional incompressible dissipative quasi-geostrophic equation

The Mathematical Model Equations and Known Related Results

Consider the Cauchy problem for the two-dimensional incompressible dissipative quasi-geostrophic equation

$$\frac{\partial}{\partial t}u + \alpha(-\triangle)^{\rho}u + J(u, (-\triangle)^{-1/2}u) = f(\mathbf{x}, t), \tag{5.1}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}). \tag{5.2}$$

In this problem, $\alpha > 0$ and $\rho > 0$ are positive constants, $\mathbf{x} = (x, y)$, $u = u(\mathbf{x}, t)$ represents the temperature of the fluid, $(-\triangle)^{-1/2}u$ is called the stream function. The Jacobian determinant is defined by

$$J(u, (-\triangle)^{-1/2}u) \stackrel{\text{def}}{=} \frac{\partial}{\partial x}u\frac{\partial}{\partial y}(-\triangle)^{-1/2}u - \frac{\partial}{\partial y}u\frac{\partial}{\partial x}(-\triangle)^{-1/2}u.$$

The model equation is called subcritical if $\rho > 1/2$, critical if $\rho = 1/2$, and supercritical if $\rho < 1/2$.

The linear operators

$$\frac{\partial}{\partial x}(-\triangle)^{-1/2} \qquad \text{and} \qquad \frac{\partial}{\partial y}(-\triangle)^{-1/2}$$

represent the standard Riesz transformations in $\mathbb{R}^2$. The vector field

$$\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \mathbf{F}(\mathbf{x}, t) \stackrel{\text{def}}{=} \begin{pmatrix} -\frac{\partial}{\partial y}(-\triangle)^{-1/2}u \\ +\frac{\partial}{\partial x}(-\triangle)^{-1/2}u \end{pmatrix}$$

represents the velocity of the fluid, for each fixed time $t > 0$. The fluid is incompressible because $\nabla \cdot \mathbf{F} = 0$. The model is the dimensionally correct analogue of the three-dimensional incompressible Navier-Stokes equations, if $\rho = 1/2$. It is derived from a general quasi-geostrophic equation in the special case of constant potential vorticity and buoyancy frequency. It is a model in geophysical fluid dynamics, it may arise in meteorology and oceanography. Therefore, it is of great interest in applied mathematics. In particular, the critical dissipative quasi-geostrophic equation is a very important model for the investigation of the existence of the global smooth solution of the three-dimensional incompressible Navier-Stokes equations.

Also consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v + \alpha(-\triangle)^{\rho}v = f(\mathbf{x}, t), \tag{5.3}$$

$$v(\mathbf{x}, t) = u_0(\mathbf{x}). \tag{5.4}$$

Here $u_0 = u_0(\mathbf{x})$ represents the initial function and $f = f(\mathbf{x}, t)$ represents the external force. Note that the initial functions for the nonlinear problem and the linear problem are the same, the external forces for both problems are the same as well. There exists a unique global smooth solution to the Cauchy problem for the linear equation, under appropriate conditions on u_0 and f.

It is well known that there exists a global smooth solution

$$u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+),$$

if $1/2 \leq \rho \leq 1$ and if the initial function and the external force satisfy the conditions

$$u_0 \in C^1(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2),$$
$$f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)).$$

There exists a global weak solution

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)), \qquad (-\triangle)^{\rho/2} u \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^2)).$$

if $0 < \rho < 1/2$ and if the initial function and the external force satisfy the following conditions

$$u_0 \in L^2(\mathbb{R}^2), \qquad f \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).$$

Moreover, there holds the following elementary uniform energy estimate for the global weak solutions

$$\left\{ \int_{\mathbb{R}^2} |u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t \int_{\mathbb{R}^2} |(-\triangle)^{\rho/2} u(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \mathrm{d}\tau \right\}^{1/2}$$
$$\leq \left\{ \int_{\mathbb{R}^2} |u_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} + \int_0^\infty \left\{ \int_{\mathbb{R}^2} |f(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} \mathrm{d}t.$$

Additionally, the global weak solutions become small enough and sufficiently smooth after a long time. That is, there exists a sufficiently large positive constant T, such that

$$\sup_{t>T} \left\{ \int_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty,$$

for all positive constants $m > 0$. Furthermore, there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0} \left\{ t^{1/\rho} \int_{\mathbb{R}^2} |u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty.$$

The Main Purposes

For the two-dimensional incompressible dissipative quasi-geostrophic equation, we will consider two cases for the initial function and the external force.

Case 1: Let the initial function and the external force satisfy the conditions

$$u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2),$$
$$f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)),$$

such that

$$\int_{\mathbb{R}^2} u_0(\mathbf{x}) \mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^2} f(\mathbf{x}, t) \mathrm{d}\mathbf{x} \mathrm{d}t \neq 0.$$

Case 2: Let the initial function and the external force satisfy the conditions

$$u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2),$$
$$f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).$$

Suppose that there exist real scalar smooth functions

$$\phi_1 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), \ \psi_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+),$$
$$\phi_2 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2), \ \psi_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+),$$

such that

$$u_0(\mathbf{x}) = \frac{\partial}{\partial x}\phi_1(\mathbf{x}) + \frac{\partial}{\partial y}\phi_2(\mathbf{x}), \qquad f(\mathbf{x},t) = \frac{\partial}{\partial x}\psi_1(\mathbf{x},t) + \frac{\partial}{\partial y}\psi_2(\mathbf{x},t),$$

for all $(\mathbf{x},t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

For Case 1, we will accomplish the following exact limits

$$\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},$$
$$\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m [u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}.$$

For Case 2, we will accomplish the exact limits

$$\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\},$$
$$\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m [u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\}.$$

Moreover, we will establish the following primary decay estimates with sharp rates

$$\sup_{t>T}\left\{t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty,$$

for Case 1, and

$$\sup_{t>T}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty,$$

for Case 2, for all positive constants $m > 0$, where T is a sufficiently large positive constant.

Furthermore, we will accomplish the following improved decay estimates with sharp rates for all order derivatives of the global weak solutions:

$$t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$$

$$\leq \mathcal{A}_1(\alpha,\delta,\varepsilon,\rho,m) + \mathcal{B}_1(\alpha,\delta,\varepsilon,\rho,m)t^{-(2-2\rho)/\rho},$$
$$t^{(2m+2)/\rho}\int_{\mathbb{R}^2} |(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x}$$
$$\leq \mathcal{C}_1(\alpha,\delta,\varepsilon,\rho,m) + \mathcal{D}_1(\alpha,\delta,\varepsilon,\rho,m)t^{-(2-2\rho)/\rho},$$

for Case 1, and

$$t^{(2m+2)/\rho}\int_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}$$
$$\leq \mathcal{A}_2(\alpha,\delta,\varepsilon,\rho,m) + \mathcal{B}_2(\alpha,\delta,\varepsilon,\rho,m)t^{-(4-2\rho)/\rho},$$
$$t^{(2m+2)/\rho}\int_{\mathbb{R}^2} |(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x}$$
$$\leq \mathcal{C}_2(\alpha,\delta,\varepsilon,\rho,m) + \mathcal{D}_2(\alpha,\delta,\varepsilon,\rho,m)t^{-(4-2\rho)/\rho},$$

for Case 2. These estimates are true for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all $m \geq 0$ and for all sufficiently large t, where the positive constants

$$\mathcal{A}_k = \mathcal{A}_k(\alpha,\delta,\varepsilon,\rho,m),\ \ \mathcal{B}_k = \mathcal{B}_k(\alpha,\delta,\varepsilon,\rho,m)$$
$$\mathcal{C}_k = \mathcal{C}_k(\alpha,\delta,\varepsilon,\rho,m),\ \ \mathcal{D}_k = \mathcal{D}_k(\alpha,\delta,\varepsilon,\rho,m)$$

will be specified shortly, for $k = 1, 2$.

The Main Difficulties

The most difficult technical problems in the mathematical analysis are the control of the following integrals

$$\int_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] \left[\sum_{k=1}^{2} \eta_k \widehat{\psi}_k(t^{-1/(2\rho)}\eta,\tau)\right] \mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$
$$\int_{\mathbb{R}^2} |\eta|^{4m} \left| \int_{(1-\varepsilon)t}^{t} \exp\left[-\alpha|\eta|^{2\rho}(1-\frac{\tau}{t})\right] t^{1/(2\rho)} \widehat{\mathcal{N}(u)}(t^{-1/(2\rho)}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta,$$

for all real constants $m \geq 0$, where $\mathcal{N}(u) = J(u, (-\triangle)^{-1/2}u)$.

We are able to use a few novel ideas to establish the optimal estimates for these integrals. In particular, we are able to use the following singular integral

$$\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta$$

and the primary decay estimates with sharp rates to obtain the best possible estimates for these integrals.

First of all, we will apply iteration technique to get the primary decay estimates with sharp sates for all order derivatives of the global weak solutions. Secondly, we will couple together various ideas, methods and techniques to accomplish the exact limits and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions.

The Mathematical Assumptions

We make the following mathematical assumptions for the two-dimensional incompressible dissipative quasi-geostrophic equation. Let $\alpha > 0$, $0 < \delta < 2\rho$, $0 < \varepsilon < 1$ and $0 < \rho < 1$ be positive constants. Let $m \geq 0$ be a real constant.

(A1) Suppose that the initial function u_0 and the external force f satisfy the following assumptions

$$\begin{aligned} &u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \\ &f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)), \end{aligned}$$

such that

$$\int_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^2} f(\mathbf{x}, t)\mathrm{d}\mathbf{x}\mathrm{d}t \neq 0.$$

Suppose that there exists the following limit

$$\lim_{t\to\infty} \left\{ t^{(m+1)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^m f(\mathbf{x}, t)|\mathrm{d}\mathbf{x} \right\}^2,$$

for all real constants $m \geq 0$.

Here is a slightly weaker condition than the above condition

$$\sup_{t>0} \left\{ t^{(m+1)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^m f(\mathbf{x}, t)|\mathrm{d}\mathbf{x} \right\}^2 < \infty,$$

for all real constants $m \geq 0$.

(A2) Suppose that the initial function and the external force satisfy the following conditions

$$\begin{aligned} &u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \\ &f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)). \end{aligned}$$

Suppose that there exist real scalar smooth functions

$$\begin{aligned} &\phi_1 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+), \\ &\phi_2 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+), \end{aligned}$$

such that

$$u_0(\mathbf{x}) = \frac{\partial}{\partial x}\phi_1(\mathbf{x}) + \frac{\partial}{\partial y}\phi_2(\mathbf{x}), \qquad f(\mathbf{x}, t) = \frac{\partial}{\partial x}\psi_1(\mathbf{x}, t) + \frac{\partial}{\partial y}\psi_2(\mathbf{x}, t),$$

for all $(\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}^+$.

(A3) Suppose that there exist the following limits

$$\lim_{t\to\infty} \left\{ \sum_{k=1}^{2} \left[t^{(m+2)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^m \psi_k(\mathbf{x}, t)|\mathrm{d}\mathbf{x} \right]^2 \right\} \stackrel{\text{def}}{=} \mathrm{F}_4(m),$$

for all real constants $m \geq 0$.

Here are some slightly weaker conditions than the above condition:

$$\sup_{t>0} \left\{ \sum_{k=1}^{2} \left[t^{(m+2)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^m \psi_k(\mathbf{x}, t)|\mathrm{d}\mathbf{x} \right]^2 \right\} < \infty,$$

for all real constants $m \geq 0$.

(A4) Suppose that there exists a unique global smooth solution

$$u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+),$$

if $1/2 \leq \rho \leq 1$, and if the initial function and the external force satisfy

$$\begin{aligned} &u_0 \in C^1(\mathbb{R}^2) \cap H^{2m}(\mathbb{R}^2), \\ &f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R}^2)). \end{aligned}$$

Suppose that there exists a global weak solution

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)), \qquad (-\triangle)^{\rho/2} u \in L^2(\mathbb{R}^+, L^2(\mathbb{R}^2)),$$

if $0 < \rho < 1/2$, and if the initial function and the external force satisfy

$$u_0 \in L^2(\mathbb{R}^2), \qquad f \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).$$

Suppose that there holds the following uniform energy estimate

$$\begin{aligned} &\left\{ \int_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} + 2\alpha \int_0^t \int_{\mathbb{R}^2} |(-\triangle)^{\rho/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \mathrm{d}\tau \right\}^{1/2} \\ &\leq \left\{ \int_{\mathbb{R}^2} |u_0(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} + \int_0^\infty \left\{ \int_{\mathbb{R}^2} |f(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\}^{1/2} \mathrm{d}t. \end{aligned}$$

(A5) Suppose that the global weak solutions become small enough and sufficiently smooth after a long time. That is, there exists a sufficiently large positive constant T, such that the global weak solutions of the Cauchy problem for the two-dimensional incompressible dissipative quasi-geostrophic equation satisfy

$$\sup_{t>T} \left\{ \int_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty, \qquad \int_T^\infty \int_{\mathbb{R}^2} |(-\triangle)^{m+1/2} u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \mathrm{d}t < \infty,$$

for all positive constants $m > 0$.

(A6) Suppose that there holds the following representation for the Fourier transformation of the global weak solutions

$$\begin{aligned} \widehat{u}(\xi,t) = \exp(-\alpha|\xi|^{2\rho}t)\widehat{u}_0(\xi) &+ \int_0^t \exp[-\alpha|\xi|^{2\rho}(t-\tau)]\widehat{f}(\xi,\tau)\mathrm{d}\tau \\ &- \int_0^t \exp[-\alpha|\xi|^{2\rho}(t-\tau)]\widehat{\mathcal{N}(u)}(\xi,\tau)\mathrm{d}\tau, \end{aligned}$$

for all $(\xi,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, where $\mathcal{N}(u) = J(u, (-\triangle)^{-1/2}u)$.

(A7) Suppose that there holds the following elementary decay estimate with a sharp rate

$$\sup_{t>0} \left\{ t^{1/\rho} \int_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x} \right\} < \infty.$$

For the existence of the global smooth solution, the existence of the global weak solution and the elementary decay estimate of the global weak solutions, see

Constantin, Cordoba and Wu [6], Constantin, Iyer and Wu [7], Constantin and Wu [8], Dabkowski [9], Dong [10], Dong and Pavlovic [11], Ferreira, Niche and Planas [14], Ju [18]-[19], Kiselev, Nazarov and Volberg [21], Maria E. Schonbek and Tomas P. Schonbek [36]. The assumptions (A1), (A2), (A3), (A4), (A5), (A6) and (A7) are made based on these results.

Suppose that the initial function, the external force and the global weak solutions satisfy the conditions (A1)-(A7), dropping either condition (A1) or (A2). To make the statements of the main results simple and clear, let us define the following notations

$$\mathcal{I}(m) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} |\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,$$

$$\mathcal{J}_1 = \left\{\int_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathcal{J}_2 = \sum_{k=1}^2\left\{\int_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty\int_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2$$

$$+\sum_{k=1}^2\left\{\frac{1}{(2\pi)^2}\int_0^\infty\int_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2,$$

$$\mathcal{K} = \sum_{k=1}^2\left\{\frac{1}{(2\pi)^2}\int_0^\infty\int_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2.$$

The Main Results

There are two parts in the main results.

Part 1: The exact limits for all order derivatives of the global weak solutions of the two-dimensional incompressible dissipative quasi-geostrophic equation.

Theorem 5.1. *There hold the following exact limits for all order derivatives of the global weak solutions of the two-dimensional incompressible dissipative quasi-geostrophic equation:*

$$\lim_{t\to\infty}\left\{t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} = \mathcal{I}(m-1/2)\mathcal{J}_1,$$

$$\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\} = \frac{1}{2}\mathcal{I}(m)\mathcal{K},$$

for all constants $m \geq 0$, for Case 1; and

$$\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} = \frac{1}{2}\mathcal{I}(m)\mathcal{J}_2,$$

$$\lim_{t\to\infty}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right\} = \frac{1}{2}\mathcal{I}(m)\mathcal{K},$$

for all constants $m \geq 0$, for Case 2.

Note that the decay rates are different in the two cases.

Theorem 5.2. *The ratio of the exact limits of the global weak solutions of the nonlinear equation is the same as the ratio of the exact limits of the global smooth solution of the linear problem. For Case 1, there hold the following results*

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+\rho}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\frac{(2m+1)(2m+\rho+1)}{(2\alpha\rho)^2},\\
&\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\frac{(2m+1)(2m+\rho+1)(2m+2\rho+1)(2m+3\rho+1)}{(2\alpha\rho)^4},
\end{aligned}
$$

for all real constants $m\geq 0$.

For Case 2, there hold the following results

$$
\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+\rho}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\frac{(2m+2)(2m+\rho+2)}{(2\alpha\rho)^2},\\
&\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
/&\left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m}u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}\\
=&\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}
\end{aligned}
$$

$$\Big/ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$= \frac{(2m+2)(2m+\rho+2)(2m+2\rho+2)(2m+3\rho+2)}{(2\alpha\rho)^4},$$

and

$$\left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+\rho}[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\Big/ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$= \left\{\lim_{t\to\infty}\left[t^{(2m+2\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\Big/ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$= \frac{(2m+2)(2m+\rho+2)}{(2\alpha\rho)^2},$$
$$\left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\Big/ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$= \left\{\lim_{t\to\infty}\left[t^{(2m+4\rho+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^{m+2\rho}v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$\Big/ \left\{\lim_{t\to\infty}\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m v(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right]\right\}$$
$$= \frac{(2m+2)(2m+\rho+2)(2m+2\rho+2)(2m+3\rho+2)}{(2\alpha\rho)^2},$$

for all real constants $m \geq 0$.

Part 2: The primary decay estimates and the improved decay estimates with sharp rates for all order derivatives of the global weak solutions of the two-dimensional incompressible dissipative quasi-geostrophic equation.

Theorem 5.3. *There hold the following primary decay estimates with sharp rates for all order derivatives of the global weak solutions:*

$$\sup_{t>T}\left\{t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty,$$

for Case 1 and for all $m \geq 0$; and

$$\sup_{t>T}\left\{t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty,$$

for Case 2 and for all $m \geq 0$.

In the primary decay estimates with sharp rates, the decay rates for the two cases are different, T is a sufficiently large positive constant.

Theorem 5.4. *There hold the following improved decay estimates with sharp rates for all order derivatives of the global weak solutions:*

$$\begin{aligned} & t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \\ \leq\ & \mathcal{A}_1(\alpha,\delta,\varepsilon,\rho,m)+\mathcal{B}_1(\alpha,\delta,\varepsilon,\rho,m)t^{-(2-2\rho)/\rho}, \\ & t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x} \\ \leq\ & \mathcal{C}_1(\alpha,\delta,\varepsilon,\rho,m)+\mathcal{D}_1(\alpha,\delta,\varepsilon,\rho,m)t^{-(2-2\rho)/\rho}, \end{aligned}$$

for Case 1; and

$$\begin{aligned} & t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x} \\ \leq\ & \mathcal{A}_2(\alpha,\delta,\varepsilon,\rho,m)+\mathcal{B}_2(\alpha,\delta,\varepsilon,\rho,m)t^{-(4-2\rho)/\rho}, \\ & t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m[u(\mathbf{x},t)-v(\mathbf{x},t)]|^2\mathrm{d}\mathbf{x} \\ \leq\ & \mathcal{C}_2(\alpha,\delta,\varepsilon,\rho,m)+\mathcal{D}_2(\alpha,\delta,\varepsilon,\rho,m)t^{-(4-2\rho)/\rho}, \end{aligned}$$

for Case 2. These estimates are true for all positive constants $0<\delta<2\rho$ and $0<\varepsilon<1$, for all constants $m\geq 0$ and for all sufficiently large t.

In Theorem 5.4, for Case 1, the positive constants are given by

$$\begin{aligned} \mathcal{A}_1 &= \mathcal{A}_1(\alpha,\delta,\varepsilon,\rho,m)=5\mathcal{I}(m-1/2)\left\{\mathrm{E}_1(u_0)+\frac{\mathrm{E}_2(f)}{\varepsilon^{(2m+1)/\rho}}\right\}+5\mathcal{I}(m)\frac{\mathrm{E}_3(u)}{\varepsilon^{(2m+2)/\rho}}T^{-1/\rho}, \\ \mathcal{B}_1 &= \mathcal{B}_1(\alpha,\delta,\varepsilon,\rho,m) \\ &= 5C_0(m)\mathcal{S}(\alpha,\delta,\varepsilon,\rho)\left\{2\mathrm{E}_4(m-\rho+(1+\delta)/2)+4\mathcal{I}(-1/2)\mathcal{I}(m-\rho+(1+\delta)/2)\mathcal{J}_1^2\right\}, \\ \mathcal{C}_1 &= \mathcal{C}_1(\alpha,\delta,\varepsilon,\rho,m)=2\mathcal{I}(m)\frac{\mathrm{E}_3(u)}{\varepsilon^{(2m+2)/\rho}}, \\ \mathcal{D}_1 &= \mathcal{D}_1(\alpha,\delta,\varepsilon,\rho,m)=8C_0(m)\mathcal{S}(\alpha,\delta,\varepsilon,\rho)\left\{\mathcal{I}(-1/2)\mathcal{I}(m-\rho+(1+\delta)/2)\mathcal{J}_1^2\right\}, \end{aligned}$$

where

$$\begin{aligned} \mathrm{E}_1(u_0) &= \left\{\int_{\mathbb{R}^2}|u_0(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2, \\ \mathrm{E}_2(f) &= \left\{\int_0^\infty\int_{\mathbb{R}^2}|f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2, \\ \mathrm{E}_3(u) &= \left\{\int_0^\infty\int_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2, \\ \mathrm{E}_4(m) &= \lim_{t\to\infty}\left\{t^{(m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m f(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right\}^2, \\ \mathcal{I}(m) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta, \\ \mathcal{J}_1 &= \left\{\int_{\mathbb{R}^2}u_0(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^2}f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2, \\ \mathcal{J}_2 &= \sum_{k=1}^2\left\{\int_{\mathbb{R}^2}\phi_k(\mathbf{x})\mathrm{d}\mathbf{x}+\int_0^\infty\int_{\mathbb{R}^2}\psi_k(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2 \end{aligned}$$

$$+\sum_{k=1}^{2}\left\{\frac{1}{(2\pi)^2}\int_0^\infty\int_{\mathbb{R}^2}\frac{\lambda_k}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t\right\}^2,$$

$$\mathcal{S}(\alpha,\delta,\varepsilon,\rho)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta,$$

$$\mathcal{S}(\alpha,\delta,\varepsilon,\rho)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta.$$

For Case 2, the positive constants are given by

$$\mathcal{A}_2=5\mathcal{I}(m)\left\{\mathrm{F}_1(u_0)+\frac{\mathrm{F}_2(f)+\mathrm{F}_3(u)}{\varepsilon^{(2m+2)/\rho}}\right\},$$

$$\mathcal{B}_2=5C_0(m)\mathcal{S}(\alpha,\delta,\varepsilon,\rho)\left\{2\mathrm{F}_4(m+1-\rho+\delta/2)+\mathcal{I}(0)\mathcal{I}(m+1-\rho+\delta/2)\mathcal{J}_2^2\right\},$$

$$\mathcal{C}_2=2\mathcal{I}(m)\frac{\mathrm{F}_3(u)}{\varepsilon^{(2m+2)/\rho}},$$

$$\mathcal{D}_2=2C_0(m)\mathcal{S}(\alpha,\delta,\varepsilon,\rho)\left\{\mathcal{I}(0)\mathcal{I}(m+1-\rho+\delta/2)\mathcal{J}_2^2\right\},$$

where

$$\mathrm{F}_1(u_0)=\left\{\int_{\mathbb{R}^2}|\phi_1(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2+\left\{\int_{\mathbb{R}^2}|\phi_2(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2,$$

$$\mathrm{F}_2(f)=\left\{\int_0^\infty\int_{\mathbb{R}^2}|\psi_1(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2+\left\{\int_0^\infty\int_{\mathbb{R}^2}|\psi_2(\mathbf{x},t)|\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{F}_3(u)=\left\{\int_0^\infty\int_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t\right\}^2,$$

$$\mathrm{F}_4(m)=\lim_{t\to\infty}\left\{\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m\psi_1(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\}$$

$$+\lim_{t\to\infty}\left\{\left[t^{(2m+2)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m\psi_2(\mathbf{x},t)|\mathrm{d}\mathbf{x}\right]^2\right\},$$

$$\mathcal{I}(m)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}|\eta|^{4m+2}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta,$$

$$\mathcal{S}(\alpha,\delta,\varepsilon,\rho)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta,$$

$$\mathcal{S}(\alpha,\delta,\varepsilon,\rho)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}}\mathrm{d}\eta.$$

Proof. The main ideas and the main steps are almost the same as those for the n-dimensional incompressible magnetohydrodynamics equations. Let us just provide the estimates in the comprehensive analysis.

Lemma 5.1. *There hold the following estimates for Case 1.*
(1)

$$t^{(2m+1)/\rho}\int_{\mathbb{R}^2}|(-\triangle)^m u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}$$

$$\leq\frac{5}{(2\pi)^2}\int_{\mathbb{R}^2}|\eta|^{4m}\exp(-2\alpha|\eta|^{2\rho})\mathrm{d}\eta\left\{\int_{\mathbb{R}^2}|u_0(\mathbf{x})|\mathrm{d}\mathbf{x}\right\}^2$$

$$
\begin{aligned}
&+ \frac{5}{(2\pi)^2} \int_{\mathbb{R}^2} |\eta|^{4m} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^2} |f(\mathbf{x}, t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
&+ \frac{5t^{-1/\rho}}{(2\pi)^2} \int_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^2} |u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
&+ \frac{5C_0}{(2\pi)^2} t^{-(2-2\rho)/\rho} \left\{ \int_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{(2m+3+\delta-2\rho)/(2\rho)} \int_{\mathbb{R}^2} |(-\triangle)^{m-\rho+(1+\delta)/2} f(\mathbf{x}, \tau)| \mathrm{d}\mathbf{x} \right]^2 \right\} \\
&+ \frac{5C_0}{(2\pi)^2} t^{-(3-2\rho)/\rho} \left\{ \int_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1/\rho} \int_{\mathbb{R}^2} |u(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{(2m+3+\delta-2\rho)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
\end{aligned}
$$

(2)

$$
\begin{aligned}
& t^{(2m+2)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x}, t) - v(\mathbf{x}, t)]|^2 \mathrm{d}\mathbf{x} \\
&\le \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^2} |u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\} \\
&+ \frac{2C_0}{(2\pi)^2} t^{(2-2\rho)/\rho} \left\{ \int_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{1/\rho} \int_{\mathbb{R}^2} |u(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[\tau^{(2m+3+\delta-2\rho)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x}, \tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.
\end{aligned}
$$

These estimates are true for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all constants $m \ge 0$ and for all sufficiently large t.

Lemma 5.2. *There hold the following estimates for Case 2.*

$$
\begin{aligned}
& t^{(2m+2)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^m u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x} \\
&\le \frac{5}{(2\pi)^2} \int_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \sum_{k=1}^2 \left[\int_{\mathbb{R}^2} |\phi_k(\mathbf{x})| \mathrm{d}\mathbf{x} \right]^2 \right\} \\
&+ \frac{5}{(2\pi)^2} \int_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \sum_{k=1}^2 \left[\int_0^\infty \int_{\mathbb{R}^2} |\psi_k(\mathbf{x}, t)| \mathrm{d}\mathbf{x}\mathrm{d}t \right]^2 \right\} \\
&+ \frac{5}{(2\pi)^2} \int_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^2} |u(\mathbf{x}, t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
&+ \frac{5C_0}{(2\pi)^2} t^{-(4-2\rho)/\rho} \left\{ \int_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left\{ \sum_{k=1}^{2} \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{(2m+6+\delta-2\rho)/(2\rho)} \int_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2}\psi_k(\mathbf{x},\tau)|\mathrm{d}\mathbf{x} \right]^2 \right\} \\
&+ \frac{5C_0}{(2\pi)^2} t^{(4-2\rho)/\rho} \left\{ \int_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \mathrm{d}\eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2/\rho} \int_{\mathbb{R}^2} |u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{(2m+4+\delta-2\rho)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\},
\end{aligned}
$$

(2)

$$
\begin{aligned}
& t^{(2m+2)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^m [u(\mathbf{x},t) - v(\mathbf{x},t)]|^2 \mathrm{d}\mathbf{x} \\
&\le \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} |\eta|^{4m+2} \exp(-2\alpha|\eta|^{2\rho}) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}^2} |u(\mathbf{x},t)|^2 \mathrm{d}\mathbf{x}\mathrm{d}t \right\}^2 \\
&+ \frac{2C_0}{(2\pi)^2} t^{-(4-2\rho)/\rho} \left\{ \int_{\mathbb{R}^2} \frac{[1-\exp(-\alpha\varepsilon|\eta|^{2\rho})]^2}{\alpha^2|\eta|^{2+2\delta}} \eta \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{2/\rho} \int_{\mathbb{R}^2} |u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\} \\
&\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[\tau^{(2m+4+\delta-2\rho)/\rho} \int_{\mathbb{R}^2} |(-\triangle)^{m+1-\rho+\delta/2} u(\mathbf{x},\tau)|^2 \mathrm{d}\mathbf{x} \right] \right\}.
\end{aligned}
$$

These estimates are true for all positive constants $0 < \delta < 2\rho$ and $0 < \varepsilon < 1$, for all constants $m \ge 0$ and for all sufficiently large t.

The other details in the proof are omitted. □

Summary

Consider the two-dimensional incompressible dissipative quasi-geostrophic equation. Suppose that the initial function and the external force satisfy the conditions

$$
\begin{aligned}
& u_0 \in C^1(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2), \\
& f \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R}^2)).
\end{aligned}
$$

Suppose that there exist real scalar smooth functions*

$$
\begin{aligned}
& \phi_1 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_1 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+), \\
& \phi_2 \in C^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2),\ \ \psi_2 \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^2 \times \mathbb{R}^+),
\end{aligned}
$$

such that

$$
u_0(\mathbf{x}) = \frac{\partial}{\partial x}\phi_1(\mathbf{x}) + \frac{\partial}{\partial y}\phi_2(\mathbf{x}), \qquad f(\mathbf{x},t) = \frac{\partial}{\partial x}\psi_1(\mathbf{x},t) + \frac{\partial}{\partial y}\psi_2(\mathbf{x},t).
$$

*For simplicity, we highlight the results of Case 2 but skip the results of Case 1.

Note that

$$\int_{\mathbb{R}^2} u_0(\mathbf{x})\mathrm{d}\mathbf{x} + \int_0^\infty \int_{\mathbb{R}^2} f(\mathbf{x},t)\mathrm{d}\mathbf{x}\mathrm{d}t = 0.$$

Remarks and Open Problems

Remark 5.1. The exact limits depend on the integrals of ϕ_{1ij} and ϕ_{2ij}, the integrals of ψ_{1ij} and ψ_{2ij}, and the integrals of the nonlinear function u^2. However, they are independent of

(1) the integrals of any order derivatives of the functions ϕ_{1ij} and ϕ_{1ij},

(2) the integrals of any order derivatives of the functions ψ_{1ij} and ψ_{2ij},

(3) the integrals of any order derivatives of the function u^2.

Remark 5.2. The exact limits are increasing functions of the order m of derivatives. The exact limits are decreasing functions of the diffusion coefficient α.

Remark 5.3. The exact limits for the two-dimensional dissipative quasi-geostrophic equation reduce to the exact limits of the global smooth solutions of the corresponding linear equation, if the nonlinear function is dropped.

Remark 5.4. The primary decay estimates with sharp rates for all order derivatives of the global weak solutions of the dissipative quasi-geostrophic equation are true for all time, if there exists a global smooth solution; the decay estimates are true for all sufficiently large time, if there exists a global weak solution.

Remark 5.5. For the two-dimensional incompressible dissipative quasi-geostrophic equation, recall that $0 < \rho < 1$ and there holds the elementary decay estimate

$$\sup_{t>0}\left\{(1+t)^{1/\rho}\int_{\mathbb{R}^2}|u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\} < \infty.$$

Therefore, there exist the following integrals

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^2} |u(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t < \infty,\\
&\int_0^\infty \int_{\mathbb{R}^2} \frac{\lambda_1}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t < \infty,\\
&\int_0^\infty \int_{\mathbb{R}^2} \frac{\lambda_2}{|\lambda|}|\widehat{u}(\lambda,t)|^2\mathrm{d}\lambda\mathrm{d}t < \infty.
\end{aligned}$$

Open problem: For the following slightly more general equation

$$\begin{aligned}
&\frac{\partial}{\partial t}u + \alpha(-\triangle)^\rho u + J(u,(-\triangle)^{-\sigma}u) = f(\mathbf{x},t),\\
&u(\mathbf{x},0) = u_0(\mathbf{x}),
\end{aligned}$$

where $0 < \sigma < 1$ and $0 < \rho < 1$ are positive constants, can we use the same ideas and methods in this paper to obtain very similar results?

6. Conclusion and remarks

6.1. Summary

Consider the n-dimensional incompressible magnetohydrodynamics equations, the n-dimensional incompressible Navier-Stokes equations and the two-dimensional incompressible dissipative quasi-geostrophic equation. For each of these equations, if the initial function and the external force are small, or if the spatial dimension is low, or if the order ρ of the dissipation $(-\triangle)^\rho$ is high, then there exists a unique global smooth solution $\mathbf{u} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. Otherwise, there exists a global weak solution, under simple assumptions on the initial functions and the external forces. It is well known that after a long time, the global weak solutions become small enough and sufficiently smooth, even though the initial function and the external force may be large. The elementary decay estimates with sharp rates for the global weak solutions have been established by coupling together basic uniform energy estimates and the Fourier splitting method.

By coupling together the existence of the global weak solutions on $(0, \infty)$, the existence of the local smooth solution on (T, ∞), the elementary decay estimates with sharp rates for all $t > 0$, and classical ideas (such as the Parseval's identity, a few simple properties of the Fourier transformation, some change of variables, the representations of the Fourier transformations of the global weak solutions, the time interval decomposition $[0, t] = [0, (1-\varepsilon)t] \cup [(1-\varepsilon)t, t]$, we have accomplished the exact limits for all order derivatives of the global weak solutions of the incompressible fluid dynamics equations. We have also established the improved decay estimates with sharp rates for all order derivatives of the global weak solutions.

6.2. Remarks

Remark 6.1. The primary decay estimates with sharp rates for all integer order derivatives of the global weak solutions of the Navier-Stokes equations have been established before. Even though the ideas and methods in the mathematical analysis of this paper are elementary, the conditions on the initial functions and the external forces are weaker, and the results are true for all order derivatives, including all fractional order derivatives. Therefore, the results obtained here are stronger than before.

Remark 6.2. The existence of the global weak solutions, which are also local smooth solutions on (T, ∞), and the elementary decay estimates with sharp rates for many similar incompressible fluid dynamics equations have been established before. By using the ideas and methods developed in this paper, we can accomplish the exact limits for all order derivatives of the global weak solutions.

6.3. Open Problems

Open Problem 6.1 Consider the nonlinear Korteweg-de Vries-Burgers equation

$$
\begin{aligned}
&\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u + \frac{\partial^3}{\partial x^3}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}\phi(u) = f(x,t), \\
&u(x,0) = u_0(x),
\end{aligned}
$$

the nonlinear Benjamin-Bona-Mahony-Burgers equation

$$\begin{aligned}
&\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u - \frac{\partial^3}{\partial x^2 \partial t}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}\phi(u) = f(x,t),\\
&u(x,0) = u_0(x),
\end{aligned}$$

the Benjamin-Ono-Burgers equation

$$\begin{aligned}
&\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u - \alpha\frac{\partial^2}{\partial x^2}u + H\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}\phi(u) = f(x,t),\\
&u(x,0) = u_0(x),
\end{aligned}$$

the two-dimensional nonlinear singular system of differential equations arising from geostrophics

$$\begin{aligned}
&\frac{\partial}{\partial t}[\gamma(\psi_1 - \psi_2) - \triangle\psi_1] + \alpha(-\triangle)^\rho\psi_1 + \beta\frac{\partial}{\partial x}\psi_1 + J(\psi_1, \gamma(\psi_1 - \psi_2) - \triangle\psi_1) = f_1(x,y,t),\\
&\frac{\partial}{\partial t}[\gamma\delta(\psi_2 - \psi_1) - \triangle\psi_2] + \alpha(-\triangle)^\rho\psi_2 + \beta\frac{\partial}{\partial x}\psi_2 + J(\psi_2, \gamma\delta(\psi_2 - \psi_1) - \triangle\psi_2) = f_2(x,y,t),\\
&\psi_1(x,y,0) = \psi_{01}(x,y), \qquad \psi_2(x,y,0) = \psi_{02}(x,y),
\end{aligned}$$

and the n-dimensional Landau-Lifschitz system of differential equations with a Gilbert damping

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{Z} = \alpha(\mathbf{Z} \times \triangle\mathbf{Z}) + \beta[\mathbf{Z} \times (\mathbf{Z} \times \triangle\mathbf{Z})],\\
&\mathbf{Z}(\mathbf{x},0) = \mathbf{Z}_0(\mathbf{x}).
\end{aligned}$$

In these equations, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ and $\rho > 0$ are positive constants. The function $\phi = \phi(u)$ satisfies the following conditions

$$|\phi(u)| \leq C(|u|^2 + |u|^5),$$

for some positive constant $C > 0$ and for all $u \in \mathbb{R}$. H represents the Hilbert operator, which is defined by

$$H\phi(x) = \frac{1}{\pi} \text{ Principal Value } \int_{\mathbb{R}} \frac{\phi(y)}{x-y}\mathrm{d}y,$$

for all $\phi \in L^2(\mathbb{R})$. The Jacobian determinants are defined by

$$\begin{aligned}
J(\psi_1, \gamma(\psi_1 - \psi_2) - \triangle\psi_1) &= \frac{\partial}{\partial x}\psi_1\frac{\partial}{\partial y}[\gamma(\psi_1 - \psi_2) - \triangle\psi_1]\\
&\quad - \frac{\partial}{\partial y}\psi_1\frac{\partial}{\partial x}[\gamma(\psi_1 - \psi_2) - \triangle\psi_1],\\
J(\psi_2, \gamma\delta(\psi_2 - \psi_1) - \triangle\psi_2) &= \frac{\partial}{\partial x}\psi_2\frac{\partial}{\partial y}[\gamma\delta(\psi_2 - \psi_1) - \triangle\psi_2]\\
&\quad - \frac{\partial}{\partial y}\psi_2\frac{\partial}{\partial x}[\gamma\delta(\psi_2 - \psi_1) - \triangle\psi_2],
\end{aligned}$$

for all functions $\psi_1, \psi_2 \in C^3(\mathbb{R}^2)$. The notation $\mathbf{X} \times \mathbf{Y}$ represents the usual cross product of real vectors $\mathbf{X} \in \mathbb{R}^3$ and $\mathbf{Y} \in \mathbb{R}^3$. $\mathbf{Z} = \mathbf{Z}(\mathbf{x},t) \in \mathbb{R}^3$ represents a real vector valued function of $(\mathbf{x}, t)$.

The existence of a global smooth solution or a global weak solution is well known for each Cauchy problem, under certain reasonable conditions. Moreover, there hold some elementary uniform energy estimates. The global weak solution is also a local smooth solution on some unbounded interval (T,∞), where T is a sufficiently large positive constant.

Can we establish the elementary decay estimate with a sharp rate for the global smooth solution (or the global weak solution) of each equation?

Can we accomplish the exact limits for all order derivatives of the global weak solutions of the above equations, if there holds an elementary decay estimate with a sharp rate? These are very interesting open problems in applied mathematics. The decay estimates with sharp rates and the exact limits for all order derivatives of the global weak solutions to the Landau-Lifschitz system of equations have been open for a long time.

Open Problem 6.2 Can we make use of the exact limits to accomplish improved decay estimates with sharp rates for all order derivatives of the global weak solutions? The key point is that we may use the improved decay estimates with sharp rates for all order derivatives to directly or indirectly influence the stability and accuracy of important schemes in numerical simulations.

Open Problem 6.3 Consider the incompressible magnetohydrodynamics Rayleigh-Benard equations

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{u}-\frac{1}{\mathrm{RE}}\triangle\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}-(\mathbf{A}\cdot\nabla)\mathbf{A}+\nabla P=\mathbf{e}_nT+\mathbf{f}(\mathbf{x},t),\\
&\frac{\partial}{\partial t}\mathbf{A}-\frac{1}{\mathrm{RM}}\triangle\mathbf{A}+(\mathbf{u}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{u}=\mathbf{g}(\mathbf{x},t),\\
&\frac{\partial}{\partial t}T-\alpha_3\triangle T+(\mathbf{u}\cdot\nabla)T=h(\mathbf{x},t),\\
&\nabla\cdot\mathbf{u}=0\qquad\nabla\cdot\mathbf{f}=0\qquad\nabla\cdot\mathbf{A}=0\qquad\nabla\cdot\mathbf{g}=0,\\
&\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}),\qquad\mathbf{A}(\mathbf{x},0)=\mathbf{A}_0(\mathbf{x}),\qquad T(\mathbf{x},0)=T_0(\mathbf{x}),\\
&\nabla\cdot\mathbf{u}_0=0,\qquad\nabla\cdot\mathbf{A}_0=0.
\end{aligned}$$

In this system, $\mathbf{u}_0=\mathbf{u}_0(\mathbf{x})$, $\mathbf{A}_0=\mathbf{A}_0(\mathbf{x})$ and $T_0=T_0(\mathbf{x})$ represent initial functions, $\mathbf{f}=\mathbf{f}(\mathbf{x},t)$, $\mathbf{g}=\mathbf{g}(\mathbf{x},t)$ and $h=h(\mathbf{x},t)$ represent external forces. Moreover, $\mathrm{RE}>0$, $\mathrm{RM}>0$ and $\alpha_3>0$ are positive constants, and $\mathbf{e}_n=(0,0,\cdots,0,1)$ is a unit vector (the last component is equal to one and all other components are zero).

The elementary uniform energy estimate

$$\sup_{t>0}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}+\sup_{t>0}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}+\sup_{t>0}\int_{\mathbb{R}^n}|T(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}<\infty,$$

and

$$\int_0^\infty\int_{\mathbb{R}^n}|\nabla\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^n}|\nabla\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^n}|\nabla T(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\mathrm{d}t<\infty,$$

have been open.

The following elementary decay estimates

$$\begin{aligned}
&\sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{u}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,\\
&\sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|\mathbf{A}(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,
\end{aligned}$$

$$\sup_{t>0}\left\{t^{1+n/2}\int_{\mathbb{R}^n}|T(\mathbf{x},t)|^2\mathrm{d}\mathbf{x}\right\}<\infty,$$

have been open.

6.4. A technical lemma

There exists a positive constant $C = C(m,n) > 0$, such that

$$\begin{aligned}
&\left\{\sum_{i=1}^{n}\int_{\mathbb{R}^n}|(-\triangle)^m[\phi_i(\mathbf{x})\psi_i(\mathbf{x})]|\mathrm{d}\mathbf{x}\right\}^2\\
&\le C(m,n)\left\{\sum_{i=1}^{n}\int_{\mathbb{R}^n}|\phi_i(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right\}\left\{\sum_{i=1}^{n}\int_{\mathbb{R}^n}|(-\triangle)^m\psi_i(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right\}\\
&+C(m,n)\left\{\sum_{i=1}^{n}\int_{\mathbb{R}^n}|(-\triangle)^m\phi_i(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right\}\left\{\sum_{i=1}^{n}\int_{\mathbb{R}^n}|\psi_i(\mathbf{x})|^2\mathrm{d}\mathbf{x}\right\},
\end{aligned}$$

for all real vector valued functions $(\phi_1,\phi_2,\phi_3,\cdots,\phi_n)\in H^{2m}(\mathbb{R}^n)$ and $(\psi_1,\psi_2,\psi_3,\cdots,\psi_n)\in H^{2m}(\mathbb{R}^n)$.

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