JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **37**, No. 1, pp. 47-58

doi: 10.4208/jpde.v37.n1.3
March 2024

On the Viability of Solutions to Conformable Stochastic Differential Equations

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Received 11 December 2021; Accepted 16 December 2022

Abstract. The viability of the conformable stochastic differential equations is studied. Some necessary and sufficient conditions in terms of the distance function to K are given. In addition, when the boundary of K is sufficiently smooth, our necessary and sufficient conditions can reduce to two relations just on the boundary of K. Lastly, an example is given to illustrate our main results.

AMS Subject Classifications: 60H10, 93E03

Chinese Library Classifications: O211.63

Key Words: Viability; conformable derivatives; conformable stochastic differential equation.

1 Introduction

Fractional derivative is as old as calculus. It is the natural generalization of the ordinary calculus involving derivatives and integrals of noninteger order. For the last few decades, fractional calculus has attracted much attention due to its powerful and widely used tool for better modelling and control of processes in various fields of science, physics, finance, engineering and optimal problem, see [1–3]. Nowadays, there are several definitions of fractional derivatives and integrals such as Riemann-Liouville, Grunwald-Letnikov, Caputo, Weyl [4, 5], Caputo-Fabrizio [6] and Atangana-Baleanu [7]. The most popular definitions are the Riemann-Liouville and Caputo definitions. All definitions of fractional derivatives satisfy the property of linearity. However, almost all fractional derivatives lack the properties of the product rule, quotient rule, chain rule, Rolle¡¯s theorem, mean value theorem and composition rule and so on. Due to the special characteristics of the fractional derivative, the compatibility of the stochastic integral and fractional integral encounters many difficulties and limitations.

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To avoid these difficulties, conformable fractional derivative was proposed in Khalil et al. [8]. It has attracted the interest of researchers, as it seems to satisfy all the requirements of the standard derivative. Also, the computing using this new derivative is much easier than using other definitions of fractional derivative. Therefore, there is a large number of works carried out using this new definition and its generalization. The details of the basic theory are reported in [9,10] and the application are reported in [11,12]. The conformable stochastic differential equations were proposed in [13]. It generalized the classical stochastic differential equation and improved the fractional stochastic differential equation. And since the conformable fractional derivative has no special nonlocal characteristic for the fractional derivative, we can directly express the solution of the equation and calculate the numerical solution, and estimate the error of the asymptotic solution. The Itô formula was established and the existence, uniqueness, continuous dependence and the stability of solutions to the conformable stochastic differential equations were studied in [13,14]. Existence and Stability of Solutions to Neutral Conformable Stochastic Functional Differential Equations were studied in [15].

Given a closed convex set $K\subset\mathbb{R}^n$ and a family X of n-dimensional stochastic processes, one is often interested in the viability of the set K with respect to the family X, that is, for each starting point $x\in K$ the process stays in K. Viability of stochastic systems is an important tool and method to study the comparison theorem and attractor of solutions of stochastic systems. It has important applications in the study of asymptotic stability of stochastic differential equations and synchronous control of systems. The first stochastic viability results can be found in Friedman [16] and Doss [17]. Since then, the viability of the classical stochastic differential equation has been studied extensively. One can refer to the results in [18–32], etc. Up to now, to the best of the author's knowledge, the viability of the conformable stochastic differential equations has not been studied in the literature.

In this paper, we will consider the viability of the following conformable stochastic differential equations

$$\begin{cases} D_\rho^\alpha X(t)=b(X(t),t)+\sigma(X(t),t)\dfrac{\mathrm{d}W(t)}{\mathrm{d}t}, \qquad \rho\in(0,1],\ t\in[\alpha,\infty),\\ X(\alpha)=X_\alpha, \end{cases} \tag{1.1}$$

where D_ρ^α is conformable derivative, $b:\mathbb{R}^n\times[\alpha,\alpha+h]\to\mathbb{R}^n$, and $\sigma:\mathbb{R}^n\times[\alpha,\alpha+h]\to\mathbb{R}^{m\times n}$, $W(t)$ is a standard Wiener process on a complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P})$.

The rest of this paper is organized as follows. In Section 2, we introduce some necessary notations and preliminaries. In Section 3, we devote to discussing the necessary and sufficient conditions of the viability of Eq. (1.1), and give some remarks and corollaries. An example is given to illustrate our main results in the final Section.

2 Preliminary

In this section we explain the general notation, introduce some definitions and lemmas that appear in this paper.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $|\cdot|$ be the Euclidean norm. B is the closed unit ball in $\mathbb{R}^n$. Consider an open subset $S\subset\mathbb{R}^n$ and a finite-dimensional space H and denote by $C^{2,1}(S,H)$ the set of functions from S into H that are continuously differentiable up to order 2 and such that the 2th derivatives are Hölder continuous with exponent 1, by $C^{1,1}_{loc}(S,H)$ the set of continuously differential functions from S into H whose derivative is locally Lipschitiz on S. $\mathbb{L}^2_n[\alpha,\alpha+h]$ is the space of n-dimensional 2th integrable functions defined on $[\alpha,\alpha+h]$. Let m and n be positive integers and denote by $\mathbb{M}^{n\times m}$ the space of $n\times m$ matrices A equipped with the trace norm $\|A\|=\sqrt{\text{trace}(AA^*)}$, where A^* is the transpose of the matrix A.

Let K be a closed convex subset of $\mathbb{R}^n$, ∂K its boundary, $\mathring{K}$ its interior and K^c its complement. Define $d_K(x):=\min_{k\in K}|x-k|$ the distance from $x\in\mathbb{R}^n$ to K, and $P_K(x)$ denotes the projection of x onto K, i.e., $d_K(x)=|x-P_K(x)|$. Furthermore $d_K(x)$ and $P_K(x)$ are continuous. Consider the square of the distance $\varphi(x):=d_K^2(x)$. It is well known (see, e.g., [33]) that $\varphi(x)$ is C^1 and

$$\varphi'(x)=2(x-P_K(x)),\qquad \forall x\in\mathbb{R}^n.$$

Furthermore $x\to x-P_K(x)$ is 1-Lipschitz and for each $x\in\mathbb{R}^n$, where $P'_K(x)$ exists, we have

$$\varphi''(x)=2(Id-P'_K(x)).\tag{2.1}$$

Moreover,

$$|\varphi''(x)|\le 2.\tag{2.2}$$

Consequently, $d_K=\varphi^{\frac{1}{2}}\in C^{1,1}_{loc}(K^c,\mathbb{R})$ and for all $x\notin\partial K$ we have

$$d'_K(x)=\frac{x-P_K(x)}{|x-P_K(x)|}.\tag{2.3}$$

Definition 2.1 (Viability). *A closed convex set $K\subset\mathbb{R}^n$ is said to be viable with respect to the conformable stochastic differential equation* (1.1) *if, for each $\xi\in K$, there exists at least a solution of the conformable stochastic differential equation* (1.1) *such that $X(\alpha)=\xi$, we have $X(t)\in K$ for all $t\ge\alpha$, almost surely.*

Definition 2.2 ([9]). *The conformable derivative with low index ρ of a function $f:[\alpha,\infty)\to\mathbb{R}$ is defined as*

$$D^\alpha_\rho f(x)=\lim_{\varepsilon\to 0}\frac{f(x+\varepsilon(x-\alpha)^{1-\rho})-f(x)}{\varepsilon},\quad x>\alpha,\ 0<\rho\le 1.$$

Remark 2.1. Fix $0<\rho\le 1$ and $x>\alpha$. A function $f:[\alpha,\infty)\to\mathbb{R}$ has a conformable derivative $D^\alpha_\rho f(x)$ if and only if it is differentiable at x and $D^\alpha_\rho f(x)=(x-\alpha)^{1-\rho}f'(x)$ holds. Obviously, $D^\alpha_1 f(x)=f'(x)$.

Definition 2.3 ([13]). *We say that an $\mathbb{R}^n$-valued stochastic process $X(\cdot)$ is a solution of* (1.1), *if $X(\cdot)$ is continuous and $\mathcal{F}_t$-adapted and*

$$X(t)=X_\alpha+\int_\alpha^t b(X(s),s)(s-\alpha)^{\rho-1}\mathrm{d}s+\int_\alpha^t \sigma(X(s),s)(s-\alpha)^{\rho-1}\mathrm{d}W(s),\quad t\in[\alpha,\alpha+h]. \tag{2.4}$$

Next, we introduce the Itô's formula of the conformable version, which is the basic result for discussing conformable stochastic differential equations.

Theorem 2.1 ([13]). *Let $0<h<\infty$, $X(t)$, $t\in[\alpha,\alpha+h]$, $h>0$ be an Itô process for*

$$D_\rho^\alpha X(t)=f(t)+g(t)\frac{\mathrm{d}W(t)}{\mathrm{d}t},\quad \rho\in(0,1],$$

$Y(\cdot):=Y(X(\cdot),\cdot)\in C^{2,1}(\mathbb{R}^n\times[\alpha,\alpha+h],\mathbb{R}^n)$*. Then, $Y(t)$, $t\in[\alpha,\alpha+h]$ is an Itô process given by*

$$\begin{aligned}\mathrm{d}Y(t)=&\frac{\partial Y(X(t),t)}{\partial t}\mathrm{d}t+\frac{\partial Y(X(t),t)}{\partial X}f(t)(t-\alpha)^{\rho-1}\mathrm{d}t+\frac{\partial Y(X(t),t)}{\partial X}g(t)(t-\alpha)^{\rho-1}\mathrm{d}W(t)\\&+\frac{1}{2}\frac{\partial Y^2(X(t),t)}{\partial X^2}g^2(t)(t-\alpha)^{2\rho-2}\mathrm{d}t.\end{aligned}$$

3 Stochastic viability

In this section, we state and prove our main results. We first introduce the following assumptions.

(H1) For all $t\in[\alpha,\alpha+h]$, $X,Y\in\mathbb{R}^n$, there exists a constant $L>0$ such that the functions b and σ satisfy

$$|b(X,t)-b(Y,t)|\le L|X-Y|,\quad |\sigma(X,t)-\sigma(Y,t)|\le L|X-Y|.$$

(H2) For all $t\in[\alpha,\alpha+h]$, $X\in\mathbb{R}^n$, there exists a constant $L>0$ such that the functions b and σ satisfy

$$|b(X,t)|\le L|1+X|,\quad |\sigma(X,t)|\le L|1+X|.$$

(H3) $E(|X_\alpha|^2)<+\infty$ and X_α is independent of $W^+(0)$.

Theorem 3.1 ([13], Theorem 4.3). *Suppose that* (H1), (H2) *and* (H3) *hold. Then Eq.* (1.1) *has a unique solution $X(\cdot):=X(X_\alpha,\cdot)\in\mathbb{L}_n^2[\alpha,\alpha+h]$ provided that $\rho\in(\frac{1}{2},1]$ and the solution $X(X_\alpha,\cdot)$ depends continuously on X_α.*

Next, we discuss the viability of Eq. (1.1). We denote by L the differential operator associated to b, σ and defined on the set of functions $V:\mathbb{R}^n\to\mathbb{R}$ by

$$LV(x):=V'(x)b(x,t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(x,t)\sigma^*(x,t)V''(x)(t-\alpha)^{2\rho-2}].$$

We also introduce the differential operator L_K on $C^{1,1}_{loc}(K^c,\mathbb{R})$ by: for every $V\in C^{1,1}_{loc}(K^c,\mathbb{R})$ and all $x\in K^c$, where the second derivative $V''(x)$ does exist,

$$L_K V(x):=V'(x)b(P_K(x),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)V''(x)(t-\alpha)^{2\rho-2}].$$

So, setting $V(x)=\varphi(x)=d_K^2(x)$ and taking into account that

$$V'(x)=2d_K(x)d_K'(x),\qquad V''(x)=2d_K(x)d_K''(x)+2d_K'(x)\otimes d_K'(x),$$

it results that if φ is twice differential at x, then

$$\begin{aligned}L_K\varphi(x)=&\varphi'(x)b(P_K(x),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)\varphi''(x)(t-\alpha)^{2\rho-2}]\\=&2d_K(x)d_K'(x)b(P_K(x),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)(2d_K(x)d_K''(x)\\&+2d_K'(x)\otimes d_K'(x))(t-\alpha)^{2\rho-2}]\\=&2d_K(x)L_Kd_K(x)+|\sigma^*(P_K(x),t)d_K'(x)(t-\alpha)^{\rho-1}|^2.\end{aligned}\tag{3.1}$$

Theorem 3.2. *A closed convex set K is viable for Eq.* (1.1) *if and only if for a.e. $x\in K^c$ the following conditions hold*

$$\sigma^*(P_K(x),t)d_K'(x)=0,\quad L_Kd_K(x)\le 0.\tag{3.2}$$

Proof. We prove the necessity first. Give a useful inequality,

$$d_K(x-P_K(x)+y)\le d_K(x),\qquad \forall x\in K^c,\ \forall y\in K,\tag{3.3}$$

which is derived as follows:

$$d_K(x-P_K(x)+y)=\min_{z\in K}|x-P_K(x)+y-z|\le|x-P_K(x)|=d_K(x).$$

For all $x\in K^c$, let us consider now a weak solution to Eq. (1.1) starting at $P_K(x)$, and let $X(P_K(x),t)$ be its continuous version. Then for all $t\ge\alpha$, $X(P_K(x),t)\in K$ a.s. and owing to (3.3)

$$\varphi(x+X(P_K(x),t)-P_K(x))\le|x-P_K(x)|^2=\varphi(x),\qquad \text{a.s..}$$

Now, by Itô's formula of the conformable version we have for any $t\ge\alpha$,

$$\begin{aligned}&\mathrm{d}\varphi(x+X(P_K(x),t)-P_K(x))=\varphi'(x+X(P_K(x),t)-P_K(x))\mathrm{d}X(P_K(x),t)\\&\quad+\frac{1}{2}Tr[\sigma(X(P_K(x),t),t)\sigma^*(X(P_K(x),t),t)\varphi''(x+X(P_K(x),t)-P_K(x))(t-\alpha)^{2\rho-2}]\mathrm{d}t\\&=\varphi'(x+X(P_K(x),t)-P_K(x))b(X(P_K(x),t),t)(t-\alpha)^{\rho-1}\mathrm{d}t\\&\quad+\varphi'(x+X(P_K(x),t)-P_K(x))\sigma(X(P_K(x),t),t)(t-\alpha)^{\rho-1}\mathrm{d}W(t)\\&\quad+\frac{1}{2}Tr[\sigma(X(P_K(x),t),t)\sigma^*(X(P_K(x),t),t)\varphi''(x+X(P_K(x),t)-P_K(x))(t-\alpha)^{2\rho-2}]\mathrm{d}t.\end{aligned}$$

Hence, integrating between α and t, we can obtain

$$\begin{aligned}&\varphi(x+X(P_K(x),t)-P_K(x))-\varphi(x)\\=&\int_\alpha^t\{\varphi'(x+X(P_K(x),s)-P_K(x))b(X(P_K(x),s),s)(s-\alpha)^{\rho-1}\\&\quad+\frac{1}{2}Tr[\sigma(X(P_K(x),s),s)\sigma^*(X(P_K(x),s),s)\varphi''(x+X(P_K(x),s)-P_K(x))(s-\alpha)^{2\rho-2}]\}\mathrm{d}s\\&\quad+\int_\alpha^t\varphi'(x+X(P_K(x),s)-P_K(x))\sigma(X(P_K(x),s),s)(s-\alpha)^{\rho-1}\mathrm{d}W(s)\\\le&0.\end{aligned}$$

Taking expectation, we can get

$$\begin{aligned}&\mathbb{E}[\varphi(x+X(P_K(x),t)-P_K(x))-\varphi(x)]\\=&\mathbb{E}\Big[\int_\alpha^t\{\varphi'(x+X(P_K(x),s)-P_K(x))b(X(P_K(x),s),s)(s-\alpha)^{\rho-1}\\&\quad+\frac{1}{2}Tr[\sigma(X(P_K(x),s),s)\sigma^*(X(P_K(x),s),s)\varphi''(x+X(P_K(x),s)-P_K(x))(s-\alpha)^{2\rho-2}]\}\mathrm{d}s\Big]\\\le&0.\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{1}{t-\alpha}\mathbb{E}\Big[\int_\alpha^t\{\varphi'(x+X(P_K(x),s)-P_K(x))b(X(P_K(x),s),s)(s-\alpha)^{\rho-1}\\&\quad+\frac{1}{2}Tr[\sigma(X(P_K(x),s),s)\sigma^*(X(P_K(x),s),s)\varphi''(x+X(P_K(x),s)-P_K(x))(s-\alpha)^{2\rho-2}]\}\mathrm{d}s\Big]\\\le&0,\end{aligned}$$

which, letting $t\to\alpha$, yields $L_K\varphi(x)\le 0$. Since $d'_K(x)$ is normal to K at $P_K(x)$, by [24] we know that $\sigma^*(P_K(x),t)d'_K(x)=0$. Therefore, by (3.1), we can get

$$L_K\varphi(x)=2d_K(x)L_Kd_K(x)+|\sigma^*(P_K(x),t)d'_K(x))(t-\alpha)^{\rho-1}|^2=2d_K(x)L_Kd_K(x)\le 0.$$

Since for all $x\in K^c$, $d_K(x)>0$. The necessity of (3.2) follows.

We prove the sufficiency secondly. Let us consider the following conformable stochastic differential equation

$$\begin{cases}D_\rho^\alpha X(t)=b(P_K(X(t)),t)+\sigma(P_K(X(t)),t)\dfrac{\mathrm{d}W(t)}{\mathrm{d}t}, & \rho\in(0,1],\ t\in[\alpha,\infty),\\ X(\alpha)=X_\alpha\in K.\end{cases}\tag{3.4}$$

Notice that K is viable for Eq. (1.1) if and only if it is viable for Eq. (3.4). So, we can consider Eq. (3.4) from now on and denote its generic solution by $X(X_\alpha,t)$.

Let $x\in K$, we consider a solution $X(x,t)$ of Eq. (3.4) starting at x, let τ_K be the first exit time of $X(x,t)$ from K, i.e. $\tau_K:=\inf\{t\geq\alpha: X(x,t)\notin K\}$.

Applying Itô's formula of the conformable version with the stopping time, for every $t\geq\alpha$, we get

$$\begin{aligned}
&\varphi(X(x,t\wedge\tau_K))\\
=&\int_\alpha^{t\wedge\tau_K}\{\varphi'(X(x,s))b(P_K(X(x,s)),s)(s-\alpha)^{\rho-1}\\
&\quad+\frac{1}{2}Tr[\sigma(P_K(X(x,s)),s)\sigma^*(P_K(X(x,s)),s)\varphi''(X(x,s))(s-\alpha)^{2\rho-2}]\}\mathrm{d}s\\
&\quad+\int_\alpha^{t\wedge\tau_K}\varphi'(X(x,s))\sigma(P_K(X(x,s)),s)(s-\alpha)^{\rho-1}\mathrm{d}W(s)\\
=&\int_\alpha^{t\wedge\tau_K}L_K\varphi(X(x,s))ds+\int_\alpha^{t\wedge\tau_K}\varphi'(X(x,s))\sigma(P_K(X(x,s)),s)(s-\alpha)^{\rho-1}\mathrm{d}W(s)\\
=&\int_\alpha^{t}\mathcal{X}_{\tau_K\geq s}L_K\varphi(X(x,s))ds+\int_\alpha^{t}\mathcal{X}_{\tau_K\geq s}\varphi'(X(x,s))\sigma(P_K(X(x,s)),s)(s-\alpha)^{\rho-1}\mathrm{d}W(s).
\end{aligned}$$

Hence, taking expectation, according (3.1) and (3.2), for every $t\geq\alpha$, we can get

$$E[\varphi(X(x,t\wedge\tau_K))]=E\Big[\int_\alpha^t\mathcal{X}_{\tau_K\geq s}L_K\varphi(X(x,s))\mathrm{d}s\Big]\leq 0.$$

This implies $X(x,t\wedge\tau_K)\in K$ a.s. for every $t\geq\alpha$, so that

$$P(\tau_K<\infty)=\lim_{i\to\infty}P(\tau_K\leq i)=0,$$

so, $\tau_K\to\infty$ a.s. and therefore $X(x,t)\in K$ a.s.. The proof is complete. □

Remark 3.1. The condition in the Theorem 3.2,

$$L_Kd_K(x)=d_K'(x)b(P_K(x),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)]d_K''(x)(t-\alpha)^{2\rho-2}.$$

While, in the Theorem 2.2 of [24],

$$L_Kd_K(x)=d_K'(x)b(P_K(x),t)+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)]d_K''(x).$$

For $\rho=1$, the Theorem 3.2 reduces to the Theorem 2.2 in [24]. So our Theorem 3.2 generalizes the Theorem 2.2 in [24].

Obviously, the viability conditions of Theorem 3.2 can be localized.

Theorem 3.3. *The closed convex set K is viable for the Eq.* (1.1) *if and only if for some $\varepsilon>0$ and almost all $x\in(K+\varepsilon B)\setminus K$ the conditions* (3.2) *hold.*

Actually, when the boundary of K is sufficiently smooth, we can characterize viability imposing a simpler set of conditions just on ∂K. Recall that the oriented distance $\bar{d}$ to the boundary ∂K of K is defined by

$$\bar{d}(x):=d_K(x)-d_{\mathbb{R}^n\backslash\mathring{K}}(x),\quad \forall x\in\mathbb{R}^n.$$

Theorem 3.4. *Assume that for some $\varepsilon>0$, $\bar{d}\in C^2$ on $\partial K+\varepsilon B$. Then K is viable for Eq.* (1.1) *if and only if for all $x\in\partial K$,*

$$\sigma^*(x,t)\bar{d}'(x)=0,\qquad L_K\bar{d}(x)\le 0. \tag{3.5}$$

Proof. Assume conditions (3.2) hold, i.e., $\forall y\in K^c$,

$$\sigma^*(P_K(y),t)d_K'(y)=0,\qquad L_Kd_K(y)\le 0.$$

Notice that $\forall y\in K^c$, $\bar{d}(y)=d_K(y)$, we can get

$$\sigma^*(P_K(y),t)\bar{d}'(y)=0,\qquad L_K\bar{d}(y)\le 0.$$

For $\forall x\in\partial K$, letting y converge to x, we can obtain

$$\sigma^*(x,t)\bar{d}'(x)=0$$

and

$$L_Kd_K(y)=d_K'(y)b(P_K(y),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(y),t)\sigma^*(P_K(y),t)d_K''(y)](t-\alpha)^{2\rho-2}$$

converges to

$$\bar{d}'(x)b(P_K(x),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)\bar{d}''(x)](t-\alpha)^{2\rho-2}=L_K\bar{d}(x).$$

So,

$$L_K\bar{d}(x)\le 0.$$

To prove (3.5) imply (3.2) on $\partial K+\varepsilon B$, we first show that for all $y\in\partial K+\varepsilon B$, $\bar{d}'(P_K(y))=d_K'(y)$ and $\bar{d}''(P_K(y))\ge d_K''(y)$ hold. Which were proved in [24]. We notice in [25] that for all $y\in\partial K+\varepsilon B$, $\sigma^*(P_K(y),t)\bar{d}'(y)=0$ holds. Then

$$\begin{aligned}
&L_K\varphi(y)\\
=&\varphi'(y)b(P_K(y),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(y),t)\sigma^*(P_K(y),t)\varphi''(y)(t-\alpha)^{2\rho-2}]\\
=&2d_K(y)d_K'(y)b(P_K(y),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(y),t)\sigma^*(P_K(y),t)d_K''(y)(t-\alpha)^{2\rho-2}]\\
\le&2d_K(y)\bar{d}'(P_K(y))b(P_K(y),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(y),t)\sigma^*(P_K(y),t)\bar{d}''(P_K(y))(t-\alpha)^{2\rho-2}]\\
=&2d_K(y)L_K\bar{d}(P_K(y))\le 0.
\end{aligned}$$

By [25] we know that $\sigma^*(P_K(y),t)d_K'(y)=0$. Therefore, by (3.1), we can get

$$L_K\varphi(y)=2d_K(y)L_Kd_K(y).$$

So, $L_Kd_K(y)\leq 0$. We obtain (3.2). $\square$

Remark 3.2. The condition in the Theorem 3.4,

$$L_K\bar{d}(x)=\bar{d}'(x)b(P_K(x),t)(t-\alpha)^{\rho-1}+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)\bar{d}''(x)](t-\alpha)^{2\rho-2}.$$

While, in the Theorem 2.8 of [24],

$$L_K\bar{d}(x)=\bar{d}'(x)b(P_K(x),t)+\frac{1}{2}Tr[\sigma(P_K(x),t)\sigma^*(P_K(x),t)\bar{d}''(x)].$$

For $\rho=1$, the Theorem 3.4 reduces to the Theorem 2.8 in [24]. So our Theorem 3.4 generalizes the Theorem 2.8 in [24].

4 Example

In this section, we give an example to illustrate our main results.

Example 4.1. Let $K:=\{z\in\mathbb{R}^n:|z|\leq R\}$. Consider the necessary and sufficient conditions on the viability of K under Eq. (1.1).

Let $K^C:=\{z\in\mathbb{R}^n:|z|>R\}$. For $x\in K^C$, we have $P_K(x)=R\frac{x}{|x|}$ and $P_K'(x)=\frac{R}{|x|}Id-\frac{R}{|x|^3}(x\otimes x)$, then

$$d_K(x)=|x-P_K(x)|=\left|x-R\frac{x}{|x|}\right|=|x|-R,$$
$$d_K'(x)=\frac{x}{|x|},\qquad d_K''(x)=\frac{Id}{|x|}-\frac{1}{|x|^3}(x\otimes x).$$

For all $x\in K^C$,

$$\sigma^*(P_K(x),t)d_K'(x)=\sigma^*\left(R\frac{x}{|x|},t\right)\frac{x}{|x|}=0,$$

i.e.,

$$\sigma^*\left(R\frac{x}{|x|},t\right)x=0, \tag{4.1}$$

and

$$L_Kd_K(x)=b\Big(\frac{Rx}{|x|},t\Big)\frac{x}{|x|}(t-\alpha)^{\rho-1}+\frac{1}{2}Tr\Big[\sigma\Big(\frac{Rx}{|x|},t\Big)\sigma^*\Big(\frac{Rx}{|x|},t\Big)\Big(\frac{Id}{|x|}-\frac{x\otimes x}{|x|^3}\Big)\Big](t-\alpha)^{2\rho-2}\leq 0.$$

i.e.,

$$b\Big(\frac{Rx}{|x|},t\Big)x+\frac{1}{2}Tr\Big[\sigma\Big(\frac{Rx}{|x|},t\Big)\sigma^*\Big(\frac{Rx}{|x|},t\Big)\Big](t-\alpha)^{\rho-1}\leq 0. \tag{4.2}$$

So, the necessary and sufficient conditions on the viability of K under Eq. (1.1) become: (4.1) and (4.2) hold for all $x\in K^C$.

Letting x converge to $y=R\frac{x}{|x|}$, from (4.1) and (4.2) we can obtain that: for all $y\in\partial K^C$,

$$\sigma^*(y,t)y=0, \tag{4.3}$$

$$b\Big(y,t\Big)y+\frac{1}{2}Tr\Big[\sigma\Big(y,t\Big)\sigma^*\Big(y,t\Big)\Big](t-\alpha)^{\rho-1}\le 0. \tag{4.4}$$

Conversely, if (4.3) and (4.4) hold, then for any $x\in K^C$, $x=\frac{|x|}{R}R\frac{x}{|x|}=\frac{|x|}{R}y$, where we have set $y=R\frac{x}{|x|}$, and therefore

$$b\Big(\frac{Rx}{|x|},t\Big)x+\frac{1}{2}\frac{|x|}{R}Tr\Big[\sigma\Big(\frac{Rx}{|x|},t\Big)\sigma^*\Big(\frac{Rx}{|x|},t\Big)\Big](t-\alpha)^{\rho-1}\le 0.$$

Since $Tr\Big[\sigma\Big(R\frac{x}{|x|},t\Big)\sigma^*\Big(R\frac{x}{|x|},t\Big)\Big]\ge 0$ and $|x|>R$, for all $x\in K^C$, we can get

$$\frac{|x|}{R}Tr\Big[\sigma\Big(R\frac{x}{|x|},t\Big)\sigma^*\Big(R\frac{x}{|x|},t\Big)\Big]\ge Tr\Big[\sigma\Big(R\frac{x}{|x|},t\Big)\sigma^*\Big(R\frac{x}{|x|},t\Big)\Big].$$

So, (4.3) and (4.4) are equivalent to (4.1) and (4.2).

So, the necessary and sufficient conditions on the viability of K under Eq. (1.1) are also equivalent to: (4.3) and (4.4) hold for all $y\in\partial K^C$. Notice that these conditions concern only the boundary of K.

Acknowledgement

This research is partially supported by the Natural Science Foundation of Hubei Province (No. 2021CFB543) and the NNSF of China (No. 11901058).

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