

A Complement to the Valiron-Titchmarsh Theorem for Subharmonic Functions

Alexander I. Kheyfits*

Bronx Community College and the Graduate Center of the City University of New York, 2155 University Avenue, Bronx, New York 10453, USA

Received 29 December 2013; Accepted (in revised version) 5 March 2014

Available online 31 March 2014

Abstract. The Valiron-Titchmarsh theorem on asymptotic behavior of entire functions with negative zeros has been recently generalized onto subharmonic functions with the Riesz measure on a half-line in \mathbb{R}^n , $n \geq 3$. Here we extend the Drasin complement to the Valiron-Titchmarsh theorem and show that if u is a subharmonic function of this class and of order $0 < \rho < 1$, then the existence of the limit $\lim_{r \rightarrow \infty} \log u(r) / N(r)$, where $N(r)$ is the integrated counting function of the masses of u , implies the regular asymptotic behavior for both u and its associated measure.

Key Words: Valiron-Titchmarsh theorem, Tauberian theorems for entire functions with negative zeros, Subharmonic functions in \mathbb{R}^n with Riesz masses on a ray, associated Legendre functions on the cut.

AMS Subject Classifications: 31B05, 30D15, 30D35

1 Main result

The well-known Valiron-Titchmarsh Tauberian theorem [6] states that if an entire function $f(z)$ of non-integer order ρ with negative zeros has regular behavior for $z = x > 0$, i.e., there exists the finite limit

$$\lim_{r \rightarrow \infty} r^{-\rho} \log f(r) = h,$$

then its zeros have the density $\lim_{t \rightarrow \infty} t^{-\rho} n(t) = \frac{\sin \pi \rho}{\pi} h$, where n is the counting function of the zeros of f . In turn, this implies that the function f is of completely regular growth in the entire complex plane. For the history of this result and the relevant references see, e.g., [5]. Drasin [1] proved a complementary result.

*Corresponding author. *Email address:* akheyfits@gc.cuny.edu (A. I. Kheyfits)

If f is an entire function of order λ , $0 < \lambda < 1$, with all zeros real and negative, then either one of the conditions

$$\log \frac{M(r)}{n(r)} \rightarrow L > 0, \quad r \rightarrow \infty,$$

or

$$\log \frac{M(r)}{N(r)} \rightarrow L\lambda > 0, \quad r \rightarrow \infty,$$

where $N(r) = \int_0^r t^{-1}n(t)dt$, implies the asymptotic relation as $r \rightarrow \infty$,

$$\log M(r) \sim r^\lambda \psi(r).$$

Here λ is determined by the transcendental equation $L = \pi / \sin(\pi\lambda)$ and ψ is a slowly varying function, that is, $\psi(\sigma r) / \psi(r) \rightarrow 1$ as $r \rightarrow \infty$ for each fixed $\sigma > 0$. The relation $a \sim b$ hereafter means the existence of the limit $\lim_{r \rightarrow \infty} a(r) / b(r) = 1$.

The author [5] has recently generalized the Valiron-Titchmarsh theorem onto subharmonic functions in \mathbb{R}^n , $n \geq 3$. In the present note we complement the results of [5] by extending the Drasin theorem onto the subharmonic functions in \mathbb{R}^n , $n \geq 3$. Introduce in \mathbb{R}^n spherical coordinates $x = (r, \theta)$, $r = |x|$, $\theta = (\theta_1, \dots, \theta_{n-1})$, such that $x_1 = r \cos \theta_1$, $0 \leq \theta_1 \leq \pi$, and $0 \leq \theta_k \leq 2\pi$ for $k = 2, 3, \dots, n-1$.

In the case under consideration, the subharmonic functions can be represented as [4, Eq. (4.5.16)]

$$u(x) = \int_{\mathbb{R}^n} P_n(r, t, \theta_1) d\mu(y) + u_0(x), \tag{1.1}$$

where μ is the Riesz associated mass of u , u_0 is a subharmonic function of smaller growth than u , and the kernel P_n is the modified Weierstrass canonical kernel,

$$P_n(r, t, \theta_1) = rt^{n-2} ((n-1)r^2 \cos \theta_1 + rt[n + (n-2) \cos^2 \theta_1] + (n-1)t^2 \cos \theta_1).$$

Without loss of generality, hereafter we assume $u(0) = u_0 = 0$. Let $n(t) = \mu(\overline{B}_t)$ be the counting function of the associated masses of u , where \overline{B}_t is the closed ball of radius t centered at the origin of \mathbb{R}^n , and $N(r) = (n-2) \int_0^r t^{1-n} n(t) dt$ its average. Now we can state our result.

Theorem 1.1. *Let u be a subharmonic function in \mathbb{R}^n , $n \geq 3$, of order ρ , $0 < \rho < 1$, whose Riesz masses are distributed over the negative x_1 -axis. If the limit*

$$\lim_{r \rightarrow \infty} \frac{u(r)}{n(r)} = \Delta \tag{1.2}$$

exists, then, as $r \rightarrow \infty$,

$$u(x) \sim H(\theta) r^\rho \psi(r) \tag{1.3}$$

and

$$N(r) \sim r^\rho \frac{\psi(r)}{\rho}, \quad (1.4)$$

where ψ is a slowly varying function for $0 < r < \infty$ and the order ρ satisfies the transcendental equation

$$\Delta = \frac{\Gamma(n-1-\rho)}{(n-2)!\Gamma(1-\rho)} \frac{\pi\rho}{\sin(\pi\rho)}, \quad (1.5)$$

Γ is Euler's function.

Moreover, the indicator $H(\theta) = H(\theta_1)$ can be expressed through the associated Legendre spherical functions of the first kind $P_\nu^\mu(\cos\theta_1)$ on the cut as

$$H(\theta_1) = \frac{\pi 2^{\frac{n-3}{2}} \Gamma(\frac{n-1}{2}) \prod_{k=1}^{n-2} (\rho+k) \Delta}{(n-3)! \sin(\pi\rho) (\sin\theta_1)^{\frac{n-3}{2}}} P_{\rho+\frac{n-3}{2}}^{\frac{3-n}{2}}(\cos\theta_1). \quad (1.6)$$

Eq. (1.4) holds good for $\theta_1 = \pi$ as well, since in this case both its sides are equal to $-\infty$.

Remark 1.1. A similar result can be proved if the limit in (1.2) is replaced by the limit $\lim_{r \rightarrow \infty} u(r, \theta) / n(r)$ with any θ such that $0 \leq \theta_1 \leq \pi/2$, however, in this case the transcendental equation (1.3) for the order must be replaced by more cumbersome one, and we do not state this more general result here.

Remark 1.2. Similar results are also valid for tube domains.

Remark 1.3. The precise bounds for the ratio $u(r, \theta) / n(r)$ were obtained by Gol'dberg and Ostrovskii [3]:

For a Weierstrass canonical integral v of noninteger order ρ , $q = E(\rho)$, with Riesz masses on the negative x_1 -half-axis

$$\liminf_{r \rightarrow \infty} \frac{v(r, \theta)}{n(r)} \leq (\rho + n - 2) \int_0^\infty u^{-1-\rho} h_n(u, \theta_1, q) du \leq \limsup_{r \rightarrow \infty} \frac{v(r, \theta)}{n(r)},$$

where θ is any point of the unit sphere with $0 \leq \theta_1 < \pi$. Both upper and lower inequalities here are exact.

Our result leads to the complete description of the functions giving the equalities in the Gol'dberg-Ostrovskii theorem.

Corollary 1.1. The bounds in these inequalities are given by functions (1.1)-(1.2) and only by these functions.

2 Proof of Theorem 1.1

Since we assume $u(0) = 0$, the function u in (1.1) can be written as [4, Eq. (4.5.15)],

$$u(x) = \int_0^\infty \frac{P_n(|x|, t, \theta_1) N(t) dt}{(|x|^2 + 2|x|t \cos \theta_1 + t^2)^{n/2+1}}. \tag{2.1}$$

We use the following special case of the Tauberian theorem of Drasin [1, Theorem 1].

Let the kernel k be positive almost everywhere, and its Laplace transform

$$\mathcal{L}k(s) = \int_{-\infty}^\infty e^{-st} k(t) dt$$

exists if $-\sigma < s < \rho$ for some positive, maybe infinite σ and ρ , whereas $\mathcal{L}k(-\sigma) = \mathcal{L}k(\rho) = \infty$. If $f(t)$, $-\infty < t < \infty$, is a positive increasing function such that its convolution with the kernel $k(t)$ satisfies

$$\int_{-\infty}^\infty k(t-y) f(y) dy = \{L + o(1)\} f(t), \quad t \rightarrow \infty,$$

then

$$f(t) = e^{\lambda t} \psi(t),$$

where for every fixed a , $\psi(t+a) / \psi(t) \rightarrow 1$, $t \rightarrow \infty$. Moreover, $\lambda (\geq 0)$ must satisfy $\mathcal{L}k(\lambda) = L$.

After substituting $t = e^y$ and $r = |x| = e^p$, (1.5) becomes

$$u(e^p, \theta_1) = \int_{-\infty}^\infty k(p-y) f(y) dy,$$

where $f(y) = N(e^y)$ and

$$k(t) = \frac{e^t \{ (n-1) \cos \theta_1 + [n + (n-2) \cos^2 \theta_1] e^t + (n-1) \cos \theta_1 e^{2t} \}}{(1 + 2e^t \cos \theta_1 + e^{2t})^{n/2+1}}.$$

It is clear that $k(t) \geq 0$ for all $0 \leq \theta_1 \leq \pi/2$. Moreover, since $P_n(r, t, \theta_1)$ is a quadratic trinomial with respect to $\cos \theta_1$, elementary considerations show that the upper boundary $\theta_1 \leq \pi/2$ cannot be increased. In particular, we have in the Drasin theorem cited above, $\sigma = n - 1$, $\rho = 1$, and all the other conditions are directly verified. If $\theta_1 = 0$, the kernel is

$$k(t) = (n-1) e^{(n-1)t} (1 + e^t)^{-n},$$

thus its positivity is obvious, the computation of its Laplace transform is straightforward, and the transcendental equation for the order ρ is given by (1.3).

Returning to the variables $r = e^p$ and $t = e^y$, we see that $N(t)$ is positive and non-decreasing. Thus, by the cited Drasin's theorem, the limits (1.3) and (1.4) exist for this ρ ,

$$N(t) = t^\rho \psi(t)$$

and

$$u(r) = H(0)r^\rho \psi(r),$$

where ψ is a slowly varying function. Combining these equations with (2.1) and the results from [5], we straightforwardly derive Eqs. (1.5) and (1.6).

Acknowledgements

This work was done during a sabbatical leave from the City University of New York and reported at the International Conference ICFIDCAA 21 at Nanjing University, Nanjing, China, July 2013. The author is indebted to the organizers of the meeting and especially to Professors Chung-Chun Yang and Liangwen Liao for the invitation to the ICFIDCAA 21 and warm hospitality during the meeting, and to the anonymous referee for the very careful reading of the manuscript.

References

- [1] D. Drasin, Tauberian theorems and slowly varying functions, *Trans. Amer. Math. Soc.*, 133 (1968), 333–356.
- [2] A. Edrei and W. H. G. Fuchs, Tauberian theorems for a class of meromorphic functions with negative zeros and positive poles, pp. 339-358 in *Contemporary Problems of the Theory of Analytic Functions*, "Nauka", Moscow, 1966.
- [3] A. A. Goldberg and I. V. Ostrovskii, On the growth of a subharmonic function with Riesz measure on a ray, *Matematicheskaya Fizika, Analiz, Geometriya*, 11 (2004), 107–113.
- [4] W. K. Hayman and P. B. Kennedy, *Subharmonic Functions*, Vol. 1, Academic Press, London-New York-San Francisco, 1976.
- [5] A. Kheyfits, Valiron-Titchmarsh theorem for subharmonic functions in \mathbb{R}^n with masses on a half-line, *Annales de la Faculté des Sciences de Toulouse*, 23(1) (2014).
- [6] G. Valiron, Sur les fonctions entières d'ordre nul et d'ordre fini et en particulier les fonctions à correspondance régulière, *Annales de la faculté des sciences de Toulouse*, 5(3) (1913), 117–257.