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CONVERGENCE OF DERIVATIVES OF GENERALIZED BERNSTEIN OPERATORS

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Abstract. In the present paper, we obtain estimations of convergence rate derivatives of the q-Bernstein polynomials $B_n(f,q_n;x)$ approximating to $f'(x)$ as $n\to\infty$, which is a generalization of that relating the classical case $q_n=1$. On the other hand, we study the convergence properties of derivatives of the limit q-Bernstein operators $B_\infty(f,q;x)$ as $q\to 1^-$.

Key words: *limit q-Bernstein operators, derivative of q-Bernstein polynomial, convergence, rate*

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1 Introduction

For an integer $r\geq 0$, let $C^r[0,1]$ be the class of all functions $f(x)$ which have the r-th continuous derivatives on $[0,1]$, where $C^0[0,1]=C[0,1]$ is the usual class of all continuous functions on $[0,1]$ with the supremum norm$\|.\|$.

Let $q>0$. For any $n=0,1,2,\cdots$, the q-integer $[n]_q$ is defined as

$$[n]_q=1+q+\cdots+q^{n-1},\qquad n=1,2,\cdots,\quad [0]_q=0$$

and the q-factorial $[n]_q!$ as

$$[n]_q!=[1]_q[2]_q\cdots[n]_q,\qquad n=1,2,\cdots,\quad [0]_q!=1;$$

For the integers n,k, $n\geq k\geq 0$, the q-binomial, or the Gaussian coefficient is defined as

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}.$$

In 1997, Phillips proposed the generalized Bernstein polynomials (see [7]), or the q-Bernstein polynomials $f(x)\in C[0,1]$,

$$B_n(f,q;x)=\sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)p_{n,k}(q;x),\qquad n=1,2,\ldots,\tag{1.1}$$

where

$$p_{n,k}(q;x)=\begin{bmatrix}n\\k\end{bmatrix}_q x^k\prod_{s=0}^{n-1-k}(1-q^sx),\qquad k=0,1,2,\cdots n. \tag{1.2}$$

(From here on, an empty product denotes 1, and an empty sum denotes 0).

When $q=1$, $B_n(f,q;x)$ reduce to the classical Berntein polynomials

$$B_n(f,1;x)=B_n(f,x)=\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\begin{bmatrix}n\\k\end{bmatrix}x^k(1-x)^{n-k},\qquad n=1,2\ldots,$$

where $\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]=\left[\begin{smallmatrix}n\\k\end{smallmatrix}\right]_1$ is the classical binomial coefficient

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{n!}{k!(n-k)!}.$$

For $f\in C[0,1], t>0$, we define the modulus of continuity $\omega(f,t)$ and second modulus of smoothness $\omega_2(f,t)$ as follows:

$$\begin{aligned}\omega(f,t)&=\sup_{|x-y|\le t,x,y\in[0,1]}|f(x)-f(y)|,\\ \omega_2(f,t)&=\sup_{0<h\le t}\sup_{x\in[0,1-2h)}|f(x+2h)-2f(x+h)+f(x)|.\end{aligned}$$

In the sequel, $C,C_1,C_{2,},\cdots$ denote positive constants (difference at different occurrences).

Recently, it is found that q-Bernstein polynomials possess many remarkable properties (see[3,4,6-10,13]), which make them an area of intensive research.

In [3], Il'inskii and Ostrovska proved that for $f(x)\in C[0,1]$, $B_n(f,q;x)$ converge to $B_\infty(f,q;x)$ as $n\to\infty$ uniformly with respect to $x\in[0,1]$, and $q\in[\alpha,1]$, $0<\alpha<1$, where

$$B_\infty(f,1;x)=f(x),\qquad x\in[0,1],$$

and for $q\in(0,1)$,

$$B_\infty(f,q;x)=\begin{cases}\sum\limits_{k=0}^{\infty}f(1-q^k)p_{\infty,k}(q;x), & 0\le x<1,\\ f(1), & x=1,\end{cases} \tag{1.3}$$

which we call the limit q-Bernstein operators (see [5]), where

$$p_{\infty,k}(q;x)=\frac{x^k}{(1-q)^k[k]_q!}\prod_{s=0}^{\infty}(1-q^sx),\qquad k=0,1,\cdots. \tag{1.4}$$

The limit q-Bernstein operator is a positive shape-preserving linear operator approximating continuous functions on [0,1] as $q\to1^-$. A large number of results relating to various properties of these operators have been obtained (see [3,5,12]).

In [11], for the derivative of classical Bernstein polynomials, L.Xie obtained the following result.

Theorem A. *If $f(x) \in C^1[0,1]$, then there exists a positive C such that*

$$\| B_n^{'}(f) - f^{'} \| \leq C\left[\omega\left(f^{'},\frac{1}{n}\right) + \omega_2\left(f^{'},\frac{1}{\sqrt{n}}\right) + \frac{\|f^{'}\|}{n}\right] \tag{1.5}$$

In this paper, we study the convergence properties of derivatives of $B_n(f,q_n;x)$ for $0 < q_n \leq 1$ as $q_n \to 1$. It's shown that these properties are the extension of those in the classical case $q_n = 1$. We also investigate the convergence of the derivatives of the limit q-Bernstein operators $B_\infty(f,q;x)$ as $q \to 1^-$. Our main results are as follows.

Theorem 1. *Let a sequence q_n satisfy $1-\frac{1}{n^3} \leq q_n < 1$. If $f(x) \in C^1[0,1]$, then*

$$\|B_n^{'}(f,q_n) - f^{'}\| \leq C\left[\omega_2\left(f^{'},\frac{1}{\sqrt{n}}\right) + \omega\left(f^{'},\frac{1}{n}\right) + \frac{\|f^{'}\|}{n}\right]. \tag{1.6}$$

Theorem 2. *Let $0 < q < 1$, $f(x) \in C^2[0,1]$. There exists a positive constant C such that*

$$\|B_\infty^{'}(f,q) - f^{'}\| \leq C\left[\omega\left(f^{''},\sqrt{1-q}\right) + \|f^{''}\|(1-q)\right]. \tag{1.7}$$

2 Auxiliary Results

Lemma 1.[2] *For $f \in C[0,1]$, Then*

$$C^{-1}\omega_2(f,t) \leq K_2\left(f,t^2\right) \leq C\omega_2(f,t) \tag{2.1}$$

for some constants $C > 0$. Here K-functional $K_2\left(f,t^2\right)$ is given by

$$K_2\left(f,t^2\right) = \inf\left\{\|f-g\| + t^2\|g''\|, g \in C^2[0,1]\right\}. \tag{2.2}$$

Lemma 2.[9] *Let $e_i = x^i, i = 0,1,2,\cdots$. Then*

$$B_n(e_0,q;x) = 1, B_n(e_1,q;x) = x, B_n(e_2,q;x) = x^2 + \frac{x(1-x)}{[n]_q}, \tag{2.3}$$

$$B_n(e_3,q;x) = x^2 + \frac{x(1-x)}{[n]_q} - \frac{x(1-x)}{[n]_q}\left(1-\frac{1}{[n]_q}\right)\left(1+q^2[n-2]_q x\right). \tag{2.4}$$

Lemma 3. *Let $0 < q < 1$, $0 \leq k \leq n$, Then for $x \in [0,1]$ the following estimate holds:*

$$J \leq \frac{2n^2(1-q)}{q^{n-1}}, \tag{2.5}$$

where

$$J = [n]_q \sum_{k=0}^{n-1} p_{n-1,k}(q;x)\left|\frac{k+1}{[k+1]_q} - \frac{1-q^{n-k-1}x}{[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^s x}\right|.$$

Proof. Note that

$$p_{n-1,k}(q;x) = \frac{1-x}{q^k(1-q^{n-k-1}x)} p_{n-1,k}(q;qx).$$

Thus

$$\begin{aligned} J &\leq n\sum_{k=0}^{n-1} p_{n-1,k}(q;qx)\frac{1-x}{q^k(1-q^{n-k-1}x)}\left(\left|\frac{k+1}{[k+1]_q}-1\right|+\left|1-\frac{1-q^{n-k-1}x}{[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^sx}\right|\right)\\ &=: J_1+J_2. \end{aligned}$$

Through direct calculation, we get

$$\begin{aligned} J_1 &= n\sum_{k=0}^{n-1} p_{n-1,k}(q;qx)\frac{1-x}{q^k(1-q^{n-k-1}x)}\left|\frac{(1-q)([1]_q+[2]_q+\cdots+[k]_q)}{[k+1]_q}\right|\\ &\leq n\sum_{k=0}^{n-1} p_{n-1,k}(q;qx)\frac{k(1-q)(1-x)}{q^k(1-q^{n-k-1}x)}\\ &\leq \frac{n^2(1-q)}{q^{n-1}}. \end{aligned} \tag{2.6}$$

Next we see that

$$\begin{aligned} &\frac{1-x}{q^k(1-q^{n-k-1}x)}\left|1-\frac{1-q^{n-k-1}x}{[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^sx}\right|\\ =\;& \frac{1-x}{q^k[n-k]_q}\left|\frac{[n-k]_q}{1-q^{n-k-1}x}-\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^sx}\right|\\ =\;& \frac{1-x}{q^k[n-k]_q}\left|\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^{n-k-1}x}-\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^sx}\right|\\ =\;& \frac{1-x}{q^k[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^sx(q^s-q^{n-k-1})}{(1-q^{n-k-1}x)(1-q^sx)}\\ \leq\;& \frac{x}{q^k[n-k]_q(1-q^{n-k-1}x)}\sum_{s=0}^{n-k-1}(1-q^{n-k-1})q^s\\ \leq\;& \frac{x(1-q^{n-k-1})}{q^k(1-q^{n-k-1}x)}. \end{aligned}$$

Thus we obtain the inequality

$$\begin{aligned} J_2 &\leq n\sum_{k=0}^{n-1} p_{n-1,k}(q;qx)\frac{x(1-q^{n-k-1})}{q^k(1-q^{n-k-1}x)}\\ &= n\sum_{k=0}^{n-2}\frac{[n-1]_q}{[n-1-k]_q}\cdot\frac{x(1-q^{n-k-1})}{q^k}p_{n-2,k}(q;qx)\\ &\leq \frac{n^2(1-q)}{q^{n-2}}\sum_{k=0}^{n-2}\frac{p_{n-2,k}(q;qx)}{[n-k-1]_q}\\ &\leq \frac{n^2(1-q)}{q^{n-1}}. \end{aligned} \tag{2.7}$$

From (2.6) and (2.7) , we obtain (2.5). This completes the proof.

Let $q \in (0,1]$, s be a nonnegative integer. We denote

$$T_{\infty,s}(q,x) = \sum_{k=0}^{\infty}(1-q^k-x)^s \frac{x^k}{\prod_{m=0}^{k-1}(1-q^{m+1})}\prod_{m=0}^{\infty}(1-q^m x). \tag{2.8}$$

Note first that the expression (2.8) may be expressed as

$$T_{\infty,s}(q,x) = \sum_{i=0}^{s}\binom{s}{i}(-x)^{s-i}B_\infty(e_i,q;x). \tag{2.9}$$

In particular,

$$T_{\infty,0}(q,x)=1,\ \ T_{\infty,1}(q,x)=0,\ \ T_{\infty,2}(q,x)=x(1-x)(1-q). \tag{2.10}$$

Further, for $s=3$ and $s=4$, we get

$$T_{\infty,3}(q,x) = -(1-q)^2(q+2)x^2(1-x)+(1-q)^2x(1-x) \tag{2.11}$$

and

$$\begin{aligned} T_{\infty,4}(q,x) = {} & (1-q)^2(q^4+2q^3+3q^2-3)[x(1-x)]^2 \\ & +(1-q)^3\left[(q^3+2q^2+2q)x+1\right]x(1-x). \end{aligned} \tag{2.12}$$

By virtue of (2.11) and (2.12), for $x \in [0,1]$, we have

$$\begin{aligned} T_{\infty,3}(q,x) &\le \frac{1}{4}(1-q)^2, \\ 0 \le T_{\infty,4}(q,x) &\le \frac{15}{2}x(1-x)(1-q)^2. \end{aligned} \tag{2.13}$$

Further,

$$T'_{\infty,4}(q,x) \le \frac{47}{4}(1-q)^2. \tag{2.14}$$

We set

$$K_i := \prod_{s=0}^{\infty}(1-q^s x)\sum_{k=1}^{\infty}\frac{(1-q^k-x)^{2i}}{\prod_{s=0}^{k-1}(1-q^{s+1})}kx^{k-1}, \qquad i=1,2.$$

Then we have the following lemma.

Lemma 4. *Let* $0<q<1$*, then for* $x\in[0,1)$*, we have*

$$K_1 \le 2, \quad K_2 \le \frac{81}{4}(1-q). \tag{2.15}$$

Proof. First we estimate K_1. Since

$$T_{\infty,2}(q,x) = \prod_{s=0}^{\infty}(1-q^s x)\sum_{k=0}^{\infty}\frac{(1-q^k-x)^2}{\prod_{s=0}^{k-1}(1-q^{s+1})}x^k, \tag{2.16}$$

We now use the Leibniz rule to differentiate (2.16), giving

$$\begin{aligned} K_1 &= T'_{\infty,2}(q,x)+\sum_{s=0}^{\infty}\frac{q^s}{1-q^sx}\prod_{s=0}^{\infty}(1-q^sx)\sum_{k=0}^{\infty}\frac{(1-q^k-x)^2}{\prod_{s=0}^{k-1}(1-q^{s+1})}x^k \\ &\quad+\prod_{s=0}^{\infty}(1-q^sx)\sum_{k=0}^{\infty}\frac{2(1-q^k-x)}{\prod_{s=0}^{k-1}(1-q^{s+1})}x^k. \end{aligned}$$

By (2.10) we know that

$$K_1=(1-2x)(1-q)+\sum_{s=0}^{\infty}\frac{q^s}{1-q^sx}T_{\infty,2}(q,x)\le 2.$$

Similarly, we get

$$K_2=T'_{\infty,4}(q,x)+\sum_{s=0}^{\infty}\frac{q^s}{1-q^sx}T_{\infty,4}(q,x)+4T_{\infty,3}(q,x).$$

From (2.13) and (2.14), we conclude that

$$K_2\le\frac{47}{4}(1-q)^2+\frac{15}{2}\sum_{s=0}^{\infty}\frac{q^sx(1-x)(1-q)^2}{1-q^sx}+(1-q)^2\le\frac{81}{4}(1-q).$$

3 Proof of the Main Result

Proof of Theorem 1. Firstly, differentiating $B_n(f,q;x)$, we see easily that

$$B'_n(f,q;x) = [n]_q\sum_{k=0}^{n-1}p_{n-1,k}(q;x)\left[f\left(\frac{[k+1]_q}{[n]_q}\right)\frac{k+1}{[k+1]_q}-f\left(\frac{[k]_q}{[n]_q}\right)\frac{(1-q^{n-k-1}x)}{[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^sx}\right].$$

Using (2.3), we get

$$\begin{aligned} B'_n(f,q;x) &= [n]_q\sum_{k=0}^{n-1}p_{n-1,k}(q;x)\frac{k+1}{[k+1]_q}\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}}f'(t)\mathrm{d}t \\ &\quad+[n]_q\sum_{k=0}^{n-1}p_{n-1,k}(q;x)\left(\frac{k+1}{[k+1]_q}-\frac{1-q^{n-k-1}x}{[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^sx}\right)\int_x^{\frac{[k]_q}{[n]_q}}f'(t)\mathrm{d}t \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} B'_n(f,q;x)-f'(x) &= [n]_q\sum_{k=0}^{n-1}p_{n-1,k}(q;x)\frac{k+1}{[k+1]_q}\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}}[f'(t)-f'(x)]\mathrm{d}t+[n]_q\sum_{k=0}^{n-1} \\ & p_{n-1,k}(q;x)\left(\frac{k+1}{[k+1]_q}-\frac{1-q^{n-k-1}x}{[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^sx}\right)\int_x^{\frac{[k]_q}{[n]_q}}[f'(t)-f'(x)]\mathrm{d}t. \end{aligned} \tag{3.2}$$

From (2.1) and (2.2), there exists a positive constant C and $g_n(x) \in C^2[0,1]$, such that

$$\|f' - g_n\| + \frac{1}{n}\|g_n''\| \leq C\omega_2\left(f', \frac{1}{\sqrt{n}}\right). \tag{3.3}$$

Let $G_n(t) = \int_0^t g_n(u)\mathrm{d}u$,

$$\left|B_n'(f,q;x) - f'(x)\right| \leq \left|f'(x) - G_n'(x)\right| + \left|G_n'(x) - B_n'(G_n,q;x)\right| + \left|B_n'(f - G_n,q;x)\right|. \tag{3.4}$$

We set

$$I_1 := \left|B_n'(f - G_n,q;x)\right|, \qquad I_2 := \left|G_n'(x) - B_n'(G_n,q;x)\right|$$

First we estimate I_1. Using Lemma 3 and (3.1), we get

$$\begin{aligned} I_1 &\leq \|f' - G_n'\|\left[\sum_{k=0}^{n-1} p_{n-1,k}(q;x)\frac{q^k(k+1)}{[k+1]_q} + J\right] \\ &\leq \|f' - g_n\|\left[1 + \frac{2n^2(1-q)}{q^{n-1}}\right]. \end{aligned} \tag{3.5}$$

Next we estimate I_2. Using Taylor's formula, we have

$$g_n(t) = g_n(x) + g_n'(x)(t-x) + \int_x^t g_n''(u)(t-u)\mathrm{d}u, \qquad t,x \in [0,1]. \tag{3.6}$$

In view of (3.2) and (3.6), we have

$$\begin{aligned} I_2 &\leq \left|g_n'(x)B_n'\left(\frac{(t-x)^2}{2},q;x\right)\right| + \left|[n]_q\sum_{k=0}^{n-1} p_{n-1,k}(q;x)\frac{k+1}{[k+1]_q}\int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}}\left(\int_x^t g_n''(u)(t-u)\mathrm{d}u\right)\mathrm{d}t\right. \\ &\quad \left. + [n]_q\sum_{k=0}^{n-1} p_{n-1,k}(q;x)\left(\frac{k+1}{[k+1]_q} - \frac{1-q^{n-k-1}x}{[n-k]_q}\sum_{s=0}^{n-k-1}\frac{q^s}{1-q^s x}\right)\int_x^{\frac{[k]_q}{[n]_q}}\left(\int_x^t g_n''(u)(t-u)\mathrm{d}u\right)\mathrm{d}t\right| \\ &=: I_{21} + I_{22}. \end{aligned}$$

Using (3.6), for $t = \frac{1}{2}, x + \frac{1-2x}{2[n]_q}$, and Lemma 2, we have

$$\begin{aligned} I_{21} &= \frac{1}{2}\left|\frac{1}{[n]_q}g_n'(x)\left(\frac{1}{2} - x\right) + g_n'(x)\frac{1-2x}{2[n]_q}\right| \\ &\leq \frac{1}{2}\left[\frac{2\|g_n\| + \|g_n''\|}{[n]_q} + \omega\left(g_n, \frac{1}{[n]_q}\right) + \frac{\|g_n''\|}{[n]_q}\right] \\ &\leq \omega\left(g_n, \frac{1}{[n]_q}\right) + \frac{\|g_n''\|}{[n]_q} + \frac{\|g_n\|}{[n]_q}. \end{aligned} \tag{3.7}$$

Regarding I_{22}, we obtain

$$\begin{aligned} I_{22} &\leq \|g_n''\| \Bigg[[n]_q \sum_{k=0}^{n-1} p_{n-1,k}(q;x) \frac{k+1}{[k+1]_q} \int_{\frac{[k]_q}{[n]_q}}^{\frac{[k+1]_q}{[n]_q}} (t-x)^2 \mathrm{d}t \\ &\quad + [n]_q \sum_{k=0}^{n-1} p_{n-1,k}(q;x) \left| \frac{k+1}{[k+1]_q} - \frac{1-q^{n-k-1}x}{[n-k]_q} \sum_{s=0}^{n-k-1} \frac{q^s}{1-q^s x} \right| \cdot \left| \int_x^{\frac{[k]_q}{[n]_q}} (t-x)^2 \mathrm{d}t \right| \Bigg] \\ &\leq \|g_n''\| \left[B_n'\left(\frac{(t-x)^3}{3}, q; x \right) + 2J \right]. \end{aligned}$$

From (2.4) we get

$$B_n'\left(\frac{(t-x)^3}{3}, q; x \right) \leq \frac{1}{[n]_q}. \tag{3.8}$$

Using Lemma 3 and (3.8), we derive

$$I_{22} \leq \|g_n''\| \left[\frac{1}{[n]_q} + \frac{4n^2(1-q)}{q^{n-1}} \right]. \tag{3.9}$$

It follows from (3.7) and (3.9) that

$$\begin{aligned} I_2 &\leq \omega\left(g_n, \frac{1}{[n]_q} \right) + \|g_n''\| \left[\frac{2}{[n]_q} + \frac{4n^2(1-q)}{q^{n-1}} \right] + \frac{\|g_n\|}{[n]_q} \\ &\leq \omega\left(f' - g_n, \frac{1}{[n]_q} \right) + \omega\left(f', \frac{1}{[n]_q} \right) + \|g_n''\| \left[\frac{2}{[n]_q} + \frac{4n^2(1-q)}{q^{n-1}} \right] + \frac{\|f' - g_n\| + \|f'\|}{[n]_q} \\ &\leq 3\|f' - g_n\| + \omega\left(f', \frac{1}{[n]_q} \right) + \|g_n''\| \left[\frac{2}{[n]_q} + \frac{4n^2(1-q)}{q^{n-1}} \right] + \frac{\|f'\|}{[n]_q}. \end{aligned} \tag{3.10}$$

Applying (3.4), (3.5) and (3.10), we have

$$\begin{aligned} |B_n'(f,q;x) - f'(x)| &\leq \|f' - g_n\| \left[5 + \frac{2n^2(1-q)}{q^{n-1}} \right] \\ &\quad + \omega\left(f', \frac{1}{[n]_q} \right) + \|g_n''\| \left[\frac{2}{[n]_q} + \frac{4n^2(1-q)}{q^{n-1}} \right] + \frac{\|f'\|}{[n]_q}. \end{aligned} \tag{3.11}$$

Setting $q = q_n \in \left(1 - \frac{1}{n^3}, 1\right)$, then we have $q_n^{n-1} \geq e^{-\frac{1}{3}}$, and

$$\frac{n^2(1-q_n)}{q_n^{n-1}} \leq \frac{e^{\frac{1}{3}}}{n}, \quad \frac{1}{[n]_{q_n}} < \frac{e^{\frac{1}{3}}}{n}. \tag{3.12}$$

Using (3.3), (3.11) and (3.12), we have (1.6). This completes the proof of Theorem 1.

Proof of Theorem 2. Setting

$$\delta_k(x) = \frac{f(1-q^k) - f(x) - f'(x)(1-q^k-x) - \frac{1}{2}f''(x)(1-q^k-x)^2}{(1-q^k-x)^2}. \tag{3.13}$$

Clearly,

$$\delta_k(x)=\frac{\int_x^{1-q^k}\left[f''(t)-f''(x)\right](1-q^k-t)dt}{(1-q^k-x)^2}.$$

Then the following estimate holds

$$|\delta_k(x)|\le\frac{1}{2}\omega\left(f'',|1-q^k-x|\right). \tag{3.14}$$

Since for $x\in(0,1)$,

$$B'_\infty(f,q;x)=\left[\prod_{s=0}^{\infty}(1-q^sx)\right]'\sum_{k=0}^{\infty}\frac{f(1-q^k)x^k}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}+\prod_{s=0}^{\infty}(1-q^sx)\sum_{k=1}^{\infty}\frac{f(1-q^k)}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}kx^{k-1},$$

where

$$\left[\prod_{s=0}^{\infty}(1-q^sx)\right]'=-\sum_{s=0}^{\infty}\frac{q^s}{1-q^sx}\prod_{s=0}^{\infty}(1-q^sx),$$

then, we can write that

$$\begin{aligned}|B'_\infty(f,q;x)-f'(x)| &= \left[\prod_{s=0}^{\infty}(1-q^sx)\right]'\sum_{k=0}^{\infty}\frac{f(1-q^k)-f(x)-f'(x)(1-q^k-x)}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}x^k\\ &+ \prod_{s=0}^{\infty}(1-q^sx)\sum_{k=1}^{\infty}\frac{f(1-q^k)-f(x)-f'(x)(1-q^k-x)}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}kx^{k-1}.\end{aligned}$$

From (3.13) and (3.14) we obtain

$$\begin{aligned}|B'_\infty(f,q;x)-f'(x)| &= \left[\prod_{s=0}^{\infty}(1-q^sx)\right]'\sum_{k=0}^{\infty}\frac{\delta_k(x)(1-q^k-x)^2+\frac{1}{2}f''(x)(1-q^k-x)^2}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}x^k\\ &+ \prod_{s=0}^{\infty}(1-q^sx)\sum_{k=1}^{\infty}\frac{\delta_k(x)(1-q^k-x)^2+\frac{1}{2}f''(x)(1-q^k-x)^2}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}kx^{k-1}\\ &\le \frac{1}{2}|f''(x)|(1-q)+\frac{1}{2}\left|[\prod_{s=0}^{\infty}(1-q^sx)]'\right|\sum_{k=0}^{\infty}\frac{\omega\left(f'',|1-q^k-x|\right)x^k}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}\\ &+ \frac{1}{2}\prod_{s=0}^{\infty}(1-q^sx)\sum_{k=1}^{\infty}\frac{\omega\left(f'',|1-q^k-x|\right)(1-q^k-x)^2}{\prod\limits_{s=0}^{k-1}(1-q^{s+1})}kx^{k-1}\\ &= \frac{1}{2}\left|f''(x)\right|(1-q)+I_1+I_2.\end{aligned} \tag{3.15}$$

For $x \in [0,1]$, we estimate I_1 by splitting I_1 into two terms

$$I_{11} = \frac{1}{2}\sum_{s=0}^{\infty}\frac{q^s}{1-q^s x}\sum_{|1-q^k-x|<\sqrt{1-q}}\frac{\omega\left(f'',|1-q^k-x|\right)(1-q^k-x)^2x^k}{\prod_{s=0}^{k-1}(1-q^{s+1})}\prod_{s=0}^{\infty}(1-q^s x),$$

$$I_{12} = \frac{1}{2}\sum_{s=0}^{\infty}\frac{q^s}{1-q^s x}\sum_{|1-q^k-x|\geq\sqrt{1-q}}\frac{\omega\left(f'',|1-q^k-x|\right)(1-q^k-x)^2x^k}{\prod_{s=0}^{k-1}(1-q^{s+1})}\prod_{s=0}^{\infty}(1-q^s x).$$

Using (2.10) and (2.13), we have

$$I_{11} \leq \frac{1}{2}\omega\left(f'',\sqrt{1-q}\right) \tag{3.16}$$

and

$$\begin{aligned}
I_{12} &\leq \frac{1}{2}\sum_{s=0}^{\infty}\frac{q^s}{1-q^s x}\sum_{|1-q^k-x|\geq\sqrt{1-q}}\frac{\omega\left(f'',\sqrt{1-q}\right)\left(1+\frac{|1-q^k-x|}{\sqrt{1-q}}\right)(1-q^k-x)^2x^k}{\prod_{s=0}^{k-1}(1-q^{s+1})}\prod_{s=0}^{\infty}(1-q^s x)\\
&\leq \frac{1}{2}\omega\left(f'',\sqrt{1-q}\right)+\frac{1}{2(1-q)}\omega\left(f'',\sqrt{1-q}\right)\sum_{s=0}^{\infty}\frac{q^s}{1-q^s x}T_{\infty,4}(q,x)\\
&\leq \frac{17}{4}\omega\left(f'',\sqrt{1-q}\right). \qquad (3.17)
\end{aligned}$$

Combining (3.16) and (3.17), we obtain

$$I_1 \leq I_{11}+I_{12} \leq \frac{19}{4}\omega\left(f'',\sqrt{1-q}\right). \tag{3.18}$$

Similarly, we estimate I_2. Using Lemma 4,

$$\begin{aligned}
I_2 &\leq \frac{1}{2}\prod_{s=0}^{\infty}(1-q^s x)\left(\sum_{|1-q^k-x|<\sqrt{1-q}}+\sum_{|1-q^k-x|\geq\sqrt{1-q}}\right)\frac{\omega\left(f'',|1-q^k-x|\right)(1-q^k-x)^2kx^{k-1}}{\prod_{s=0}^{k-1}(1-q^{s+1})}\\
&\leq \frac{K_1}{2}\omega\left(f'',\sqrt{1-q}\right)+\frac{1}{2}\omega\left(f'',\sqrt{1-q}\right)\left(K_1+\frac{K_2}{1-q}\right)\\
&\leq \frac{97}{8}\omega\left(f'',\sqrt{1-q}\right). \qquad (3.19)
\end{aligned}$$

From (3.18) and (3.19), we conclude that

$$\begin{aligned}
\|B'_\infty f - f'\| &\leq \frac{\|f''\|}{2}(1-q)+I_1+I_2\\
&\leq \frac{\|f''\|}{2}(1-q)+\frac{135}{8}\omega\left(f'',\sqrt{1-q}\right)\\
&\leq C\left[\|f''\|(1-q)+\omega\left(f'',\sqrt{1-q}\right)\right].
\end{aligned}$$

The proof of Theorem 2 is finished.

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