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TRIGONOMETRIC APPROXIMATION IN REFLEXIVE ORLICZ SPACES

Ali Guven
(*Balikesir University, Turkey)*

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Abstract. The Lipschitz classes $\mathrm{Lip}(\alpha,M)\,, 0<\alpha\leq 1$ are defined for Orlicz space generated by the Young function M, and the degree of approximation by matrix transforms of $f\in Lip\,(\alpha,M)$ is estimated by $n^{-\alpha}$.

Key words: *Lipschitz class, matrix transform, modulus of continuity, Nölund transform, Orlicz space*

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1 Introduction and the Main Results

A convex and continuous function $M:[0,\infty)\to[0,\infty)$, for which $M(0)=0$, $M(x)>0$ for $x>0$ and

$$\lim_{x\to 0}\frac{M(x)}{x}=0,\qquad \lim_{x\to\infty}\frac{M(x)}{x}=\infty$$

is called a Young function. The complementary Young function N of M is defined by

$$N(y):=\max\{xy-M(x):x\geq 0\}$$

for $y\geq 0$.

Let M be a Young function. We denote by $\widetilde{L}_M=\widetilde{L}_M([0,2\pi])$ the set of $2\pi-$periodic measurable functions $f:\mathbf{R}\to\mathbf{R}$ such that

$$\int_0^{2\pi} M(|f(x)|)\,\mathrm{d}x<\infty.$$

The linear span of $\widetilde{L}_M$ is denoted by $L_M=L_M([0,2\pi])$. Equipped with the norm

$$\|f\|_M:=\sup\left\{\int_0^{2\pi}|f(x)\,g(x)|\,\mathrm{d}x:\int_0^{2\pi}N(|g(x)|)\,\mathrm{d}x\leq 1\right\},$$

where N is the complementary function of M, L_M becomes a Banach space, called the Orlicz space generated by M.

The Orlicz spaces are known as the generalization of the Lebesgue spaces; in special case, the Orlicz space generated by the Young function $M_p(x) = x^p/p, 1 < p < \infty$, is isometrically isomorphic to the Lebesgue space L_p. More general information about Orlicz spaces can be found in [6], [11] and [12].

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse of the Young function M and let

$$h(t) := \limsup_{x\to\infty} \frac{M^{-1}(x)}{M^{-1}(tx)}, \qquad t > 0.$$

The numbers α_M and β_M defined by

$$\alpha_M := \lim_{t\to\infty} -\frac{\log h(t)}{\log t}, \qquad \beta_M := \lim_{t\to 0^+} -\frac{\log h(t)}{\log t}$$

are called the lower and upper Boyd indices of the Orlicz space L_M, respectively. It is known that the Boyd indices satisfy

$$0 \le \alpha_M \le \beta_M \le 1$$

and

$$\alpha_N + \beta_M = 1, \qquad \alpha_M + \beta_N = 1.$$

The Orlicz space L_M is reflexive if and only if its Boyd indices are nontrivial, that is $0 < \alpha_M \le \beta_M < 1$ (see, for example [5]).

If $1 \le q < 1/\beta_M \le 1/\alpha_M < p \le \infty$, then $L_p \subset L_M \subset L_q$, where the inclusions being continuous, and hence the relation $L_\infty \subset L_M \subset L_1$ holds. We refer to [1] and [2] for a complete discussion of Boyd indices properties.

The modulus of continuity of the function $f \in L_M$ is defined by

$$\omega(f,\delta)_M = \sup_{0<h\le\delta} \|f(\cdot+h) - f\|_M, \qquad \delta > 0.$$

Let $0 < \alpha \le 1$. The Lipschitz class $\mathrm{Lip}(\alpha,M)$ is defined as

$$\mathrm{Lip}(\alpha,M) = \{f \in L_M : \omega(f,\delta)_M = O(\delta^\alpha), \delta > 0\}.$$

Let $f \in L^1$ has the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). \tag{1.7}$$

Denote by $S_n(f)(x)$, $n = 0,1,\cdots$ the nth partial sums of the series (1.7) at the point x, that is,

$$S_n(f)(x) = \sum_{k=0}^{n} u_k(f)(x),$$

where

$$u_0(f)(x) = \frac{a_0}{2}, \quad u_k(f)(x) = a_k \cos kx + b_k \sin kx, \quad k = 1, 2, \cdots .$$

Let (p_n) be a sequence of positive numbers. The Nörlund means of the series (1.7) with respect to the sequence (p_n) are defined by

$$N_n(f)(x) = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k(f)(x), \tag{1.8}$$

where $P_n = \sum_{k=0}^{n} p_k$, and $p_{-1} = P_{-1} := 0$.

If $p_n = 1$ for $n = 0, 1, \cdots$, then $N_n(f)(x)$ coincides with the Cesàro means $\sigma_n(f)(x)$, that is

$$N_n(f)(x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f)(x).$$

The sequence (p_n) is called almost monotone decreasing (increasing) if there exists a constant K, depending only on (p_n), such that $p_n \le K p_m$ $(p_m \le K p_n)$ for $n \ge m$.

In the Lebesgue space L_p, the following results are obtained recently.

Theorem A[3]. *Let $f \in \mathrm{Lip}(\alpha, p)$ and (p_n) be a sequence of positive numbers such that $(n+1) p_n = O(P_n)$. If either*

(i) $p > 1$, $0 < \alpha \le 1$ *and* (p_n) *is monotonic*

or

(ii) $p = 1$, $0 < \alpha < 1$ *and* (p_n) *is non-decreasing,*

then

$$\|f - N_n(f)\|_p = O\left(n^{-\alpha}\right).$$

Theorem B[7]. *Let $f \in \mathrm{Lip}(\alpha, p)$ and (p_n) be a sequence of positive numbers. If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ *and* (p_n) *is almost monotone decreasing,*

(ii) $p > 1$, $0 < \alpha < 1$, (p_n) *is almost monotone increasing and* $(n+1) p_n = O(P_n)$,

(iii) $p > 1$, $\alpha = 1$ *and* $\sum_{k=1}^{n-1} k |p_k - p_{k+1}| = O(P_n)$,

(iv) $p > 1$, $\alpha = 1$ *and* $\sum_{k=0}^{n-1} |p_k - p_{k+1}| = O(P_n/n)$,

(v) $p = 1$, $0 < \alpha < 1$ *and* $\sum_{k=-1}^{n-1} |p_k - p_{k+1}| = O(P_n/n)$

maintains, then

$$\|f - N_n(f)\|_p = O\left(n^{-\alpha}\right).$$

It is clear that Theorem B is more general than Theorem A.

In the paper [8], the authors extended Theorem A to more general classes of triangular matrix methods.

Let $A = (a_{n,k})$ be an infinite lower triangular regular matrix with nonnegative entries and let $s_n^{(A)}$ $(n = 0, 1, \cdots)$ denote the row sums of this matrix, that is $s_n^{(A)} = \sum_{k=0}^{n} a_{n,k}$.

The matrix $A = (a_{n,k})$ is said to have monotone rows if, for each n, $(a_{n,k})$ is either non-increasing or non-decreasing with respect to k, $0 \le k \le n$.

For a given infinite lower triangular regular matrix $A = (a_{n,k})$ with nonnegative entries we consider the matrix transform

$$T_n^{(A)}(f)(x) = \sum_{k=0}^{n} a_{n,k} S_k(f)(x). \tag{1.9}$$

Theorem C[8]**.** *Let* $f \in \mathrm{Lip}(\alpha, p)$, *A has monotone rows and satisfy* $\left| s_n^{(A)} - 1 \right| = O(n^{-\alpha})$. *If one of the conditions*

(i) $p > 1$, $0 < \alpha < 1$ *and* $(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1)$ *where* $r = [n/2]$,

(ii) $p > 1$, $\alpha = 1$ *and* $(n+1)\max\{a_{n,0}, a_{n,r}\} = O(1)$ *where* $r = [n/2]$,

(iii) $p = 1$, $0 < \alpha < 1$ *and* $(n+1)\max\{a_{n,0}, a_{n,n}\} = O(1)$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_p = O\left(n^{-\alpha}\right).$$

For a given positive sequence (p_n), if we consider the lower triangular matrix with entries $a_{n,k} = p_{n-k}/P_n$, then the Nörlund transform (1.8) can be regarded as a matrix transform of the form (1.9). Further, in this case the condition of Theorem A implies that of Theorem C and hence Theorem C is more general than Theorem A (see [8]).

In the present paper we give generalizations of Theorems B and C in reflexive Orlicz spaces.

We say the matrix $A = (a_{n,k})$ has almost monotone increasing (decreasing) rows if there exists a constant K, depending only on A, such that $a_{n,k} \le K a_{n,m}$ $(a_{n,m} \le K a_{n,k})$ for each n and $0 \le k \le m \le n$.

Our main results are the following.

Theorem 1. *Let* L_M be a reflexive Orlicz space, $0 < \alpha < 1$, $f \in \mathrm{Lip}(\alpha, M)$ *and* $A = (a_{n,k})$ *be a lower triangular regular matrix with* $\left| s_n^{(A)} - 1 \right| = O(n^{-\alpha})$. *If one of the conditions*

(i) *A has almost monotone decreasing rows and* $(n+1)\, a_{n,0} = O(1)$,

(ii) *A has almost monotone increasing rows and* $(n+1)\, a_{n,r} = O(1)$ *where* $r := [n/2]$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_M = O\left(n^{-\alpha}\right).$$

Theorem 2. *Let* L_M be a reflexive Orlicz space, $f \in \mathrm{Lip}(1, M)$ *and* $A = (a_{n,k})$ *be a lower triangular regular matrix with* $\left| s_n^{(A)} - 1 \right| = O\left(n^{-1}\right)$. *If one of the conditions*

(i) $\sum_{k=1}^{n-1} |a_{n,k-1} - a_{n,k}| = O\left(n^{-1}\right)$,

(ii) $\sum_{k=1}^{n-1} (n-k)\, |a_{n,k-1} - a_{n,k}| = O(1)$,

holds, then

$$\left\| f - T_n^{(A)}(f) \right\|_M = O\left(n^{-1}\right).$$

Let (p_n) be a sequence of positive numbers, $0<\alpha<1$ and $1<p<\infty$. Consider the lower triangular matrix $A=(a_{n,k})$ with $a_{n,k}=p_{n-k}/P_n$. It is clear that in this case $s_n^{(A)}=1$.

If (p_n) is almost monotone decreasing, then the Nörlund matrix A has almost monotone increasing rows and

$$(n+1)\,a_{n,r}\le(n+1)\,Ka_{n,n}=K\,(n+1)\,\frac{p_0}{P_n}\le 1,$$

where $r=[n/2]$. Thus, A satisfies the condition (ii) of Theorem 1.

If (p_n) is almost monotone increasing and $(n+1)\,p_n=O\,(P_n)$, then A has almost monotone decreasing rows and

$$(n+1)\,a_{n,0}=(n+1)\,\frac{p_n}{P_n}=\frac{1}{P_n}O\,(P_n)=O\,(1)\,.$$

Thus, A satisfies the condition (i) of Theorem 1.

Hence the part (ii) of Theorem 1 is more general than the part (i) of Theorem B and the part (i) of Theorem 1 is more general than that part (ii) of Theorem B even in the case $M\,(x)=x^p/p$, $1<p<\infty$.

Also, it is clear that parts (i) and (ii) of Theorem 1 are more general than corresponding parts of Theorem C.

Now let $p>1$, $\alpha=1$ and $\sum\limits_{k=1}^{n-1}k\,|p_k-p_{k+1}|=O\,(P_n)$. Then,

$$\begin{aligned}\sum_{k=1}^{n-1}(n-k)\,|a_{n,k-1}-a_{n,k}| &= \sum_{k=1}^{n-1}(n-k)\left|\frac{p_{n-k+1}}{P_n}-\frac{p_{n-k}}{P_n}\right|\\ &= \frac{1}{P_n}\sum_{k=1}^{n-1}k\,|p_k-p_{k+1}|=\frac{1}{P_n}O\,(P_n)=O\,(1)\,.\end{aligned}$$

Thus, the Nörlund matrix $A=(p_{n-k}/P_n)$ satisfies the condition (ii) of Theorem 2. Hence, the part (iii) of Theorem B is a special case of the part (ii) of Theorem 2. Similarly, one can easily show that the part (i) of Theorem 2 is more general than the part (iv) of Theorem B even if $M\,(x)=x^p/p$, $1<p<\infty$.

2 Auxiliary Results

Lemma 1. *Let L_M be a reflexive Orlicz space and $0<\alpha\le 1$. Then for every $f\in\mathrm{Lip}\,(\alpha,M)$ the estimate*

$$\|f-S_n\,(f)\|_M=O\left(n^{-\alpha}\right),\qquad n=1,2,\cdots \tag{2.3}$$

holds.

Proof. Let t_n^* $(n=0,1,\cdots)$ be the trigonometric polynomial of best approximation to $f\in\mathrm{Lip}\,(\alpha,M)$, i. e.

$$\|f-t_n^*\|_M=\inf\|f-t\|_M\,,$$

where the infimum is taken over all trigonometric polynomials t of degree at most n.

From Theorem $1^{'}$ of [10] it can be deduced that

$$\|f-t_n^*\|_M = O\left(\omega\left(f,1/n\right)_M\right),$$

and hence

$$\|f-t_n^*\|_M = O\left(n^{-\alpha}\right).$$

By the uniform boundedness of the partial sums $S_n(f)$ in the reflexive Orlicz spaces[13], we get

$$\begin{aligned}\|f-S_n(f)\|_M &\leq \|f-t_n^*\|_M + \|t_n^* - S_n(f)\|_M = \|f-t_n^*\|_M + \|S_n(t_n^*-f)\|_M \\ &= O\left(\|f-t_n^*\|_M\right) = O\left(n^{-\alpha}\right).\end{aligned}$$

Lemma 2. *Let L_M be a reflexive Orlicz space . If $f\in \mathrm{Lip}(1,M)$, then f is absolutely continuous and $f'\in L_M$.*

Proof. Since L_M is reflexive, the Boyd indices satisfy $0<\alpha_M\leq\beta_M<1$. If we choose a number q such that $1<q<1/\beta_M$, then L_M is continuously embedded in the Lebesgue space L_q. Hence we have

$$\|f(\cdot+h)-f\|_q \leq c\,\|f(\cdot+h)-f\|_M$$

for every h with $0<h\leq\delta$, $\delta>0$. This inequality yields

$$\omega(f,\delta)_q \leq c\;\omega(f,\delta)_M.$$

Hence, $f\in\mathrm{Lip}(1,M)$ implies $\omega(f,\delta)_q = O(\delta)$, and this implies that f is absolutely continuous and $f'\in L^q$[4,pp. 51–54].

Since f is absolutely continuous,the relation

$$\frac{f(x+\delta)-f(x)}{\delta}\to f'(x),\quad \delta\to 0^+$$

holds almost everywhere. Hence, by Fatou Lemma, for every g with $\int_0^{2\pi} N(|g(x)|)\,\mathrm{d}x\leq 1$,

$$\begin{aligned}\int_0^{2\pi}|f'(x)|\,|g(x)|\,\mathrm{d}x &= \int_0^{2\pi}\left(\lim_{\delta\to0^+}\frac{|f(x+\delta)-f(x)|}{\delta}\right)|g(x)|\,\mathrm{d}x \\ &\leq \liminf_{\delta\to0^+}\frac{1}{\delta}\int_0^{2\pi}|f(x+\delta)-f(x)|\,|g(x)|\,\mathrm{d}x \\ &\leq \liminf_{\delta\to0^+}\frac{1}{\delta}\|f(\cdot+\delta)-f\|_M \\ &\leq \liminf_{\delta\to0^+}\frac{1}{\delta}\omega(f,\delta)_M = \liminf_{\delta\to0^+}\frac{4}{\delta}\,O(\delta)=O(1),\end{aligned}$$

and this means that $f'\in L_M$.

Lemma 3. *Let L_M be a reflexive Orlicz space and $f \in \mathrm{Lip}(1,M)$. Then for $n = 1,2,\cdots$ the estimate*

$$\|S_n(f) - \sigma_n(f)\|_M = O\left(n^{-1}\right) \tag{2.4}$$

holds.

Proof. By Lemma 2, f is absolutely continuous and $f' \in L_M$. If f has the Fourier series

$$f(x) \sim \sum_{k=0}^{\infty} u_k(f)(x),$$

then the Fourier series of the conjugate function $\widetilde{f'}$ is

$$\widetilde{f'}(x) \sim \sum_{k=1}^{\infty} k u_k(f)(x).$$

On the other hand,

$$S_n(f)(x) - \sigma_n(f)(x) = \sum_{k=1}^{n} \frac{k}{n+1} A_k(f)(x) = \frac{1}{n+1} S_n\left(\widetilde{f'}\right)(x).$$

Considering the boundedness of the partial sums and the conjugation operator in reflexive Orlicz spaces[13] yield (2.4).

In the classical Lebesgue spaces L^p, $1 < p < \infty$, the analogue of Lemma 3 was proved in [9].

Lemma 4. *Let $A = (a_{n,k})$ be an infinite lower triangular matrix and $0 < \alpha < 1$. If one of the conditions*

(i) *A has almost monotone decreasing rows and $(n+1)a_{n,0} = O(1)$,*

(ii) *A has almost monotone increasing rows, $(n+1)a_{n,r} = O(1)$ where $r := [n/2]$,* and $\left|s_n^{(A)} - 1\right| = O\left(n^{-\alpha}\right)$,

holds, then

$$\sum_{k=1}^{n} k^{-\alpha} a_{n,k} = O\left(n^{-\alpha}\right). \tag{2.5}$$

Proof. (i) Since $\sum\limits_{k=1}^{n} k^{-\alpha} = O\left(n^{1-\alpha}\right)$ and $a_{n,k} \le K a_{n,0}$ for $k = 1,\cdots,n$, we get

$$\sum_{k=1}^{n} k^{-\alpha} a_{n,k} \le K a_{n,0} \sum_{k=1}^{n} k^{-\alpha} = O\left(\frac{1}{n+1}\right) O\left(n^{1-\alpha}\right) = O\left(n^{-\alpha}\right).$$

(ii) Since $a_{n,k} \le K a_{n,r}$ for $k = 1,\ldots,r$ and $\left|s_n^{(A)} - 1\right| = O\left(n^{-\alpha}\right)$,

$$\begin{aligned}
\sum_{k=1}^{n} k^{-\alpha} a_{n,k} &= \sum_{k=1}^{r} k^{-\alpha} a_{n,k} + \sum_{k=r+1}^{n} k^{-\alpha} a_{n,k} \\
&\le K a_{n,r} \sum_{k=1}^{r} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=r+1}^{n} a_{n,k} \le K a_{n,r} \sum_{k=1}^{n} k^{-\alpha} + (r+1)^{-\alpha} \sum_{k=0}^{n} a_{n,k} \\
&= O\left(\frac{1}{n+1}\right) O\left(n^{1-\alpha}\right) + O\left(n^{-\alpha}\right) s_n^{(A)} = O\left(n^{-\alpha}\right).
\end{aligned}$$

3 Proofs of the Main Results

Proof of Theorem 1. By the definition of $T_n^{(A)}(f)$, we have

$$\begin{aligned}
T_n^{(A)}(f)(x) - f(x) &= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) - f(x) \\
&= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) - f(x) + s_n^{(A)} f(x) - s_n^{(A)} f(x) \\
&= \sum_{k=0}^{n} a_{n,k} \left(S_k(f)(x) - f(x) \right) + \left(s_n^{(A)} - 1 \right) f(x).
\end{aligned}$$

Hence, by (2.3) and (2.5) we obtain

$$\begin{aligned}
\left\| f - T_n^{(A)}(f) \right\|_M &\leq \sum_{k=1}^{n} a_{n,k} \left\| S_k(f) - f \right\|_M + a_{n,0} \left\| S_0(f) - f \right\|_M + \left| s_n^{(A)} - 1 \right| \left\| f \right\|_M \\
&= \sum_{k=1}^{n} a_{n,k} k^{-\alpha} + O\left(\frac{1}{n+1} \right) + O\left(n^{-\alpha} \right) \\
&= O\left(n^{-\alpha} \right),
\end{aligned}$$

since $\left| s_n^{(A)} - 1 \right| = O(n^{-\alpha})$.

Proof of Theorem 2. By (2.3),

$$\begin{aligned}
\left\| f - T_n^{(A)}(f) \right\|_M &\leq \left\| S_n(f) - T_n^{(A)}(f) \right\|_M + \left\| f - S_n(f) \right\|_M \\
&= \left\| S_n(f) - T_n^{(A)}(f) \right\|_M + O\left(n^{-1} \right).
\end{aligned}$$

Thus, we have to show that

$$\left\| S_n(f) - T_n^{(A)}(f) \right\|_M = O\left(n^{-1} \right). \tag{3.1}$$

Set $A_{n,k} := \sum\limits_{m=k}^{n} a_{n,m}$. Hence,

$$\begin{aligned}
T_n^{(A)}(f)(x) &= \sum_{k=0}^{n} a_{n,k} S_k(f)(x) = \sum_{k=0}^{n} a_{n,k} \left(\sum_{m=0}^{k} u_m(f)(x) \right) \\
&= \sum_{k=0}^{n} \left(\sum_{m=k}^{n} a_{n,m} \right) u_k(f)(x) = \sum_{k=0}^{n} A_{n,k} u_k(f)(x).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
S_n(f)(x) &= \sum_{k=0}^{n} u_k(f)(x) = A_{n,0} \sum_{k=0}^{n} u_k(f)(x) + (1 - A_{n,0}) \sum_{k=0}^{n} u_k(f)(x) \\
&= \sum_{k=0}^{n} A_{n,0} u_k(f)(x) + \left(1 - s_n^{(A)} \right) S_n(f)(x).
\end{aligned}$$

Thus,

$$T_n^{(A)}(f)(x) - S_n(f)(x) = \sum_{k=1}^{n}(A_{n,k} - A_{n,0})\,u_k(f)(x) + \left(s_n^{(A)} - 1\right)S_n(f)(x).$$

By the boundedness of partial sums we get

$$\begin{aligned} \left\|S_n(f) - T_n^{(A)}(f)\right\|_M &\le \left\|\sum_{k=1}^{n}(A_{n,k} - A_{n,0})\,u_k(f)\right\|_M + \left|s_n^{(A)} - 1\right|\|f\|_M \\ &= \left\|\sum_{k=1}^{n}(A_{n,k} - A_{n,0})\,u_k(f)\right\|_M + O\left(n^{-1}\right). \end{aligned} \tag{3.2}$$

Thus, the problem is reduced to proving that

$$\left\|\sum_{k=1}^{n}(A_{n,k} - A_{n,0})\,u_k(f)\right\|_M = O\left(n^{-1}\right). \tag{3.3}$$

If we set

$$b_{n,k} := \frac{A_{n,k} - A_{n,0}}{k}, \qquad k = 1, \cdots, n,$$

Abel transform yields

$$\begin{aligned} \sum_{k=1}^{n}(A_{n,k} - A_{n,0})\,u_k(f) &= \sum_{k=1}^{n} b_{n,k} k u_k(f) \\ &= b_{n,n}\sum_{m=1}^{n} m u_m(f) + \sum_{k=1}^{n-1}(b_{n,k} - b_{n,k+1})\left(\sum_{m=1}^{k} m u_m(f)\right). \end{aligned}$$

Hence,

$$\begin{aligned} \left\|\sum_{k=1}^{n}(A_{n,k} - A_{n,0})\,u_k(f)\right\|_M &\le |b_{n,n}|\left\|\sum_{m=1}^{n} m u_m(f)\right\|_M \\ &\quad + \sum_{k=1}^{n-1}|b_{n,k} - b_{n,k+1}|\left(\left\|\sum_{m=1}^{k} m u_m(f)\right\|_M\right). \end{aligned}$$

Considering (2.4), we have

$$\begin{aligned} \left\|\sum_{m=1}^{n} m u_m(f)\right\|_M &= (n+1)\,\|S_n(f) - \sigma_n(f)\|_M \\ &= (n+1)\,O\left(n^{-1}\right) = O(1). \end{aligned}$$

This and the previous inequality yield

$$\left\|\sum_{k=1}^{n}(A_{n,k} - A_{n,0})\,u_k(f)\right\|_M = O(1)\,|b_{n,n}| + O(1)\sum_{k=1}^{n-1}|b_{n,k} - b_{n,k+1}|. \tag{3.4}$$

Since $\left|s_n^{(A)}-1\right| = O\left(n^{-1}\right)$,

$$\begin{aligned} |b_{n,n}| &= \tfrac{\left|A_{n,n}-A_{n,0}\right|}{n} = \tfrac{\left|a_{n,n}-s_n^{(A)}\right|}{n} \\ &= \tfrac{1}{n}\left(s_n^{(A)}-a_{n,n}\right) \le \tfrac{1}{n}s_n^{(A)} \\ &= \tfrac{1}{n}O(1) = O\left(n^{-1}\right). \end{aligned} \tag{3.5}$$

Therefore, it remains to prove that

$$\sum_{k=1}^{n-1}|b_{n,k}-b_{n,k+1}| = O\left(n^{-1}\right). \tag{3.6}$$

A simple calculation yields

$$b_{n,k}-b_{n,k+1} = \frac{1}{k(k+1)}\left\{(k+1)a_{n,k}-\sum_{m=0}^{k}a_{n,m}\right\}.$$

(i) Let $\sum_{k=1}^{n-1}|a_{n,k-1}-a_{n,k}| = O\left(n^{-1}\right)$.

Let's verify by induction that

$$\left|\sum_{m=0}^{k}a_{n,m}-(k+1)a_{n,k}\right| \le \sum_{m=1}^{k}m\,|a_{n,m-1}-a_{n,m}| \tag{3.7}$$

for $k = 1,\cdots,n$.

If $k=1$, then

$$\left|\sum_{m=0}^{1}a_{n,m}-2a_{n,1}\right| = |a_{n,0}-a_{n,1}|,$$

thus (3.7) holds. Now let us assume that (3.7) is true for $k=\nu$. For $k=\nu+1$,

$$\begin{aligned} \left|\sum_{m=0}^{\nu+1}a_{n,m}-(\nu+2)a_{n,\nu+1}\right| &= \left|\sum_{m=0}^{\nu}a_{n,m}-(\nu+1)a_{n,\nu+1}\right| \\ &\le \left|\sum_{m=0}^{\nu}a_{n,m}-(\nu+1)a_{n,\nu}\right| + |(\nu+1)a_{n,\nu}-(\nu+1)a_{n,\nu+1}| \\ &\le \sum_{m=1}^{\nu}m\,|a_{n,m-1}-a_{n,m}| + (\nu+1)\,|a_{n,\nu}-a_{n,\nu+1}| \\ &= \sum_{m=1}^{\nu+1}m\,|a_{n,m-1}-a_{n,m}|, \end{aligned}$$

and hence (3.7) holds for $k=1,\cdots,n$. Therefore,

$$\begin{aligned}
\sum_{k=1}^{n-1}|b_{n,k}-b_{n,k+1}| &= \sum_{k=1}^{n-1}\left|\frac{1}{k(k+1)}\left\{(k+1)a_{n,k}-\sum_{m=0}^{k}a_{n,m}\right\}\right| \\
&= \sum_{k=1}^{n-1}\frac{1}{k(k+1)}\left|\sum_{m=0}^{k}a_{n,m}-(k+1)a_{n,k}\right| \\
&\leq \sum_{k=1}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}| \\
&= \sum_{m=1}^{n-1}m|a_{n,m-1}-a_{n,m}|\sum_{k=m}^{n-1}\frac{1}{k(k+1)} \\
&\leq \sum_{m=1}^{n-1}m|a_{n,m-1}-a_{n,m}|\sum_{k=m}^{\infty}\frac{1}{k(k+1)} \\
&= \sum_{m=1}^{n-1}|a_{n,m-1}-a_{n,m}| \\
&= O\left(n^{-1}\right).
\end{aligned}$$

(ii) Let

$$\sum_{k=1}^{n-1}(n-k)|a_{n,k-1}-a_{n,k}|=O(1).$$

By (3.7),

$$\begin{aligned}
\sum_{k=1}^{n-1}|b_{n,k}-b_{n,k+1}| &\leq \sum_{k=1}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}| \\
&\leq \sum_{k=1}^{r}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}|+\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}|,
\end{aligned}$$

where $r:=[n/2]$. By Abel transform,

$$\begin{aligned}
\sum_{k=1}^{r}\frac{1}{k(k+1)}\sum_{m=1}^{k}m|a_{n,m-1}-a_{n,m}| &\leq \sum_{k=1}^{r}|a_{n,k-1}-a_{n,k}| \\
&= \sum_{k=1}^{r}\frac{1}{n-k}(n-k)|a_{n,k-1}-a_{n,k}| \\
&\leq \frac{1}{n-r}\sum_{k=1}^{r}(n-k)|a_{n,k-1}-a_{n,k}| \\
&= \frac{1}{n-r}O(1)=O\left(n^{-1}\right).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \\
\leq\ &\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\left\{\sum_{m=1}^{r}m\left|a_{n,m-1}-a_{n,m}\right|+\sum_{m=r}^{k}m\left|a_{n,m-1}-a_{n,m}\right|\right\} \\
=\ &\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{r}m\left|a_{n,m-1}-a_{n,m}\right|+\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=r}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \\
=\ &: I_{n1}+I_{n2}.
\end{aligned}$$

Since $\sum_{k=1}^{r}\left|a_{n,k-1}-a_{n,k}\right|=O\left(n^{-1}\right)$,

$$\begin{aligned}
I_{n1} &\leq \sum_{k=r}^{n-1}\frac{1}{k+1}\sum_{m=1}^{r}\left|a_{n,m-1}-a_{n,m}\right| \\
&= O\left(n^{-1}\right)\sum_{k=r}^{n-1}\frac{1}{k+1} \\
&= O\left(n^{-1}\right)(n-r)\frac{1}{r+1} \\
&= O\left(n^{-1}\right).
\end{aligned}$$

Let's also estimate I_{n2}.

$$\begin{aligned}
I_{n2} &= \sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=r}^{k}m\left|a_{n,m-1}-a_{n,m}\right| \\
&\leq \sum_{k=r}^{n-1}\frac{1}{k+1}\sum_{m=r}^{k}\left|a_{n,m-1}-a_{n,m}\right| \\
&\leq \frac{1}{r+1}\sum_{k=r}^{n-1}\left(\sum_{m=r}^{k}\left|a_{n,m-1}-a_{n,m}\right|\right) \\
&\leq \frac{2}{n}\sum_{k=r}^{n-1}\left(\sum_{m=r}^{k}\left|a_{n,m-1}-a_{n,m}\right|\right) \\
&= \frac{2}{n}\sum_{k=n-r}^{n-1}(n-k)\left|a_{n,k-1}-a_{n,k}\right| \\
&\leq \frac{2}{n}\sum_{k=1}^{n-1}(n-k)\left|a_{n,k-1}-a_{n,k}\right| \\
&= \frac{2}{n}O(1)=O\left(n^{-1}\right).
\end{aligned}$$

Thus

$$\sum_{k=r}^{n-1}\frac{1}{k(k+1)}\sum_{m=1}^{k}m\left|a_{n,m-1}-a_{n,m}\right|=O\left(n^{-1}\right),$$

and hence

$$\sum_{k=1}^{n-1} |b_{n,k} - b_{n,k+1}| = O\left(n^{-1}\right).$$

Therefore, (3.6) is verified both in cases (i) and (ii). Finally, combining (3.1), (3.2), (3.3), (3.4), (3.5) and (3.6) finishes the proof.

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Department of Mathematics
Faculty of Arts and Sciences
Balikesir University, 10145
Balikesir, Turkey

E-mail: ag_guven@yahoo.com