Analysis in Theory and Applications
Anal. Theory Appl., Vol. **33**, No. 1 (2017), pp. 29-45

DOI: 10.4208/ata.2017.v33.n1.4

Entropy Unilateral Solution for Some Noncoercive Nonlinear Parabolic Problems Via a Sequence of Penalized Equations

Ahmed Aberqi[1], J. Bennouna[1,*] and H. Redwane[2]

[1] *Université Sidi Mohammed Ben Abdellah, Département de Mathématiques, Laboratoire LISA, Ecole Nationale des Sciences Appliqués, Fes, Morocco*
[2] *Faculté des Sciences Juridiques, Economiques et Sociales, Université Hassan 1, B.P 784, Settat. Morocco*

Received 22 March 2014; Accepted (in revised version) 16 October 2016

Abstract. We give an existence result of the obstacle parabolic equations

$$\frac{\partial b(x,u)}{\partial t} - div(a(x,t,u,\nabla u)) + div(\phi(x,t,u)) = f \quad \text{in } Q_T,$$

where $b(x,u)$ is bounded function of u, the term $-\mathrm{div}(a(x,t,u,\nabla u))$ is a Leray-Lions type operator and the function ϕ is a nonlinear lower order and satisfy only the growth condition. The second term f belongs to $L^1(Q_T)$. The proof of an existence solution is based on the penalization methods.

Key Words: Obstacle parabolic problems, entropy solutions, penalization methods.

AMS Subject Classifications: 47A15, 46A32, 47D20

1 Introduction

In this paper, we investigate the problem of existence of solutions of the obstacle problems associated to the following nonlinear parabolic problem:

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t} - div(a(x,t,u,\nabla u)) + div(\phi(x,t,u)) = f & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ b(x,u)(t=0) = b(x,u_0(x)) & \text{in } \Omega, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* aberqi_ahmed@yahoo.fr (A. Aberqi), jbennouna@hotmail.com (J. Bennouna), redwane_hicham@yahoo.fr (H. Redwane)

where Ω is a bounded open set of IR^N $(N\geq 2)$, T is a positive real number, and $Q_T=\Omega\times(0,T)$. Let $b:\Omega\times IR\longrightarrow IR$ is a Carathéodory function such that for every $x\in\Omega$, $b(x,\cdot)$ is a strictly increasing C^1-function, the data f and $b(\cdot,u_0)$ in $L^1(Q_T)$ and $L^1(\Omega)$ respectively. The term $-div(a(x,t,u,\nabla u))$ is a Leray-Lions operator defined on $L^p(0,T;W_0^{1,p}(\Omega))$ (see assumptions (3.3a)-(3.3c)). The function $\phi(x,t,u)$ is a Carathéodory assumed to be continuous on u (see assumptions (3.3d)-(3.3e)). Under these assumptions, the above problem does not admit, in general, a weak solution since the fields $a(x,t,u,\nabla u)$ and $\phi(x,t,u)$ does not belongs in $(L^1_{loc}(Q))^N$ in general.

In the case of equation in the classical Sobolev spaces H. Redwane [5] proved the existence of solution of problem (1.1) where $\phi(x,t,u)=0$, and where $div(\phi(x,t,u))=H(x,t,u,\nabla u)$ and $f\in L^1(Q)$ by Y. Akdim et al. [2] in the degenerated Sobolev spaces without the sign condition and the coercivity condition on the term $H(x,t,u,\nabla u)$.

The existence of a solution is shown in [5, 8] with $b(x,u)=u$, using the framework of renormalized solution, and in [7] for the case $-div(a(x,t,u,\nabla u))=-\Delta u$, using the framework of entropy solution.

It is our purpose, in this paper to generalize the result of [2, 7], and we prove the existence of unilateral entropy solution for the problem (1.1) and without the coercivity condition on ϕ. More precisely, this paper deals with the existence of a solution to the obstacle parabolic problem associated to (1.1) in the sense of unilateral entropy solution (see Theorem 3.1).

The aim of this work is to investigate the relationship between the obstacle problem (1.1) and some penalized sequence of approximate equations (3.9). We study the possibility to find a solution of (1.1) (see Theorem 3.1) as limit of a subsequence u_ϵ of solutions of (3.9). The penalized term $\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-$ introduced in (3.9) play a crucial role in the proof of our main result, in particular term allows to prove that the solution of (3.9) belongs in the convex set K_ψ.

The plan of the paper is as follows: in Section 2 we give some preliminaries and basic assumptions. In Section 3 we give the definition of entropy solution of (1.1), and we establish (see Theorem 3.1) the existence of such solution.

2 Preliminaries

Let Ω be a bounded open set of IR^N $(N\geq 2)$, T is a positive real number, and $Q_T=\Omega\times(0,T)$. We need the Sobolev embeddings result.

Lemma 2.1 (Gagliardo-Niremberg). *Let $v\in L^q(0,T;L^q(\Omega))\cap L^\infty(0,T;L^\rho(\Omega))$, with $q\geq 1$ and $\rho\geq 1$. Then $v\in L^\sigma(\Omega)$ with $\sigma=q(\frac{N+\rho}{N})$ and*

$$\int_{Q_T}|v|^\sigma dxdt\leq C\|v\|^{\frac{\rho q}{N}}_{L^\infty(0,T;L^\rho(\Omega))}\int_{Q_T}|\nabla v|^q dxdt.$$

Lemma 2.2 (see [8]). *Assume that Ω is an open set of IR^N of finite measure and $1<p<+\infty$. Let u be a measurable function satisfying $T_k(u)\in L^p(0,T;W_0^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ for every k and such that:*

$$\sup_{t\in(0,T)}\int_\Omega|\nabla T_k(u)|^2dx+\int_{Q_T}|\nabla T_k(u)|^pdxdt\le Mk,\quad\forall k>0,$$

where M is a positive constant. Then

$$\begin{aligned}\||u|^{p-1}\|_{L^{\frac{p(N+1)-N}{N(p-1)},\infty}(Q_T)}&\le CM^{(\frac{p}{N}+1)\frac{N}{N+p'}}|Q_T|^{\frac{1}{p'}(\frac{N}{N+p'})},\\ \||\nabla u|^{p-1}\|_{L^{\frac{p(N+1)-N}{(N+1)(p-1)},\infty}(Q_T)}&\le CM^{\frac{(N+2)(p-1)}{p(N+1)-N}},\end{aligned}$$

where C is a constant depend only on N and p.

3 Assumptions and statements of main results

Throughout this paper, we assume that the following assumptions hold true:

$$b:\Omega\times IR\to IR \text{ is a Carathéodory function such that for every } x\in\Omega, \tag{3.1}$$

$b(x,\cdot)$ is a strictly increasing $\mathcal{C}^1(IR)$-function and $b\in L^\infty(\Omega\times IR)$ with $b(x,0)=0$.

Next, there exists a constant $\lambda>0$ and two functions $A\in L^\infty(\Omega)$ and $B\in L^p(\Omega)$ such that:

$$\lambda\le\frac{\partial b(x,s)}{\partial s}\le A(x)\quad\text{and}\quad\Big|\nabla_x\Big(\frac{\partial b(x,s)}{\partial s}\Big)\Big|\le B(x) \tag{3.2}$$

almost every $x\in\Omega$ and for all $s\in IR$.

For any $k>0$, there exists $h_k\in L^{p'}(Q_T)$ such that

$$|a(x,t,s,\xi)|\le\nu\Big(h_k(x,t)+|\xi|^{p-1}\Big),\quad\forall|s|\le k,\text{ and with }\nu>0, \tag{3.3a}$$

$$a(x,t,s,\xi)\xi\ge\alpha|\xi|^p\quad\text{with }\alpha>0, \tag{3.3b}$$

$$(a(x,t,s,\xi)-a(x,t,s,\eta))(\xi-\eta)>0\quad\text{with }\xi\ne\eta, \tag{3.3c}$$

$$|\phi(x,t,s)|\le c(x,t)|s|^\gamma\quad\text{with }\gamma=\frac{N+2}{N+p}(p-1), \tag{3.3d}$$

$$c(x,t)\in(L^\tau(Q_T))^N\quad\text{with }\tau=\frac{N+p}{p-1}, \tag{3.3e}$$

for almost every $(x,t)\in Q_T$, for every $s\in IR$ and every $\xi,\eta\in IR^N$,

$$f\in L^1(Q_T), \tag{3.4a}$$

$$u_0\in L^1(\Omega)\text{ such that }b(x,u_0)\in L^1(\Omega). \tag{3.4b}$$

Let ψ be a measurable function with values in $\overline{IR}$ such that

$$\psi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q_T) \tag{3.5}$$

and let

$$K_\psi = \Big\{ u \in L^p(0,T;W_0^{1,p}(\Omega))/u \geq \psi \text{ almost every in } Q_T \Big\}.$$

Throughout the paper, T_k denotes the truncation function at height $k \geq 0$:

$$T_k(r) = \max(-k,\min(k,r)), \quad \forall r \in IR.$$

Theorem 3.1. *Assume that* (3.2)-(3.5) *hold true. Then there exists at last one solution u such that $b(x,u) \in L^\infty(0,T;L^1(\Omega))$, $b(x,u)(t=0) = b(x,u_0)$ a.e. in Ω and for all $t \in [0,T]$,*

$$\begin{cases} T_k(u) \in L^p(0,T,W_0^{1,p}(\Omega)),\ u \geq \psi \ \text{ a.e. in } Q_T, \\ \displaystyle\int_0^t \Big\langle \frac{\partial b(x,u)}{\partial s}; T_k(u-\varphi) \Big\rangle ds + \int_{Q_t} a(x,s,u,\nabla u) \nabla T_k(u-\varphi) dxds \\ \qquad \displaystyle - \int_{Q_t} \phi(x,s,u) \nabla T_k(u-\varphi) dxds \leq \int_{Q_t} f T_k(u-\varphi) dxds, \\ \forall k>0 \ \text{ and } \ \forall \varphi \in K_\psi \cap L^\infty(Q) \ \text{ such that } \ \dfrac{\partial \varphi}{\partial t} \in L^{p'}(0,T,W^{-1,p'}(\Omega)), \end{cases} \tag{3.6}$$

where $Q_t = \Omega \times (0,t)$.

Proof. The proof is divided into six steps.
Step 1: Approximate problem and a priori estimates. For each $\epsilon > 0$, we define the following approximations

$$a_\epsilon(x,t,s,\xi) = a(x,t,T_{\frac{1}{\epsilon}}(s),\xi) \quad \text{a.e. } (x,t) \in Q_T,\ \forall s \in IR,\ \forall \xi \in IR^N, \tag{3.7a}$$

$$\phi_\epsilon(x,t,r) = \phi(x,t,T_{\frac{1}{\epsilon}}(r)) \quad \text{a.e. } (x,t) \in Q_T,\ \forall r \in IR, \tag{3.7b}$$

$$f_\epsilon \in L^{p'}(Q_T) \text{ such that } f_\epsilon \to f \text{ strongly in } L^1(Q_T), \tag{3.7c}$$

and

$$u_{0\epsilon} \in \mathcal{C}_0^\infty(\Omega) \text{ such that } b(x,u_{0\epsilon}) \to b(x,u_0) \text{ strongly in } L^1(\Omega). \tag{3.8}$$

Consider the approximate problem:

$$\begin{cases} \dfrac{\partial b(x,u_\epsilon)}{\partial t} - div(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)) + div(\phi_\epsilon(x,t,u_\epsilon)) - \dfrac{1}{\epsilon} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- = f_\epsilon & \text{in } Q_T, \\ u_\epsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ b(x,u_\epsilon)(t=0) = b(x,u_{0\epsilon}) & \text{in } \Omega. \end{cases} \tag{3.9}$$

As a consequence, proving existence of a weak solution $u_\epsilon \in L^p(0,T;W_0^{1,p}(\Omega))$ is an easy task (see e.g., [13]).

Step 2: Let $\tau_1 \in (0,T)$ and t fixed in $(0,\tau_1)$. By choosing $T_h(u_\epsilon - T_\beta(u_\epsilon)) \equiv T_{\beta+h}(u_\epsilon) - T_\beta(u_\epsilon)$ with $\beta \geq \|\psi\|_\infty$ as test function in (3.9), we get

$$\begin{aligned}
&\int_\Omega B_{\beta,h}(x,u_\epsilon(t))dx + \int_{Q_t} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))dxds \\
&\quad -\frac{1}{\epsilon}\int_{Q_t} T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- T_h(u_\epsilon - T_\beta(u_\epsilon))dxds \\
\leq &\int_{Q_t} c(x,t)|u_\epsilon|^\gamma |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|dxds + \int_{Q_t} f_\epsilon T_h(u_\epsilon - T_\beta(u_\epsilon))dxds \\
&\quad + \int_\Omega B_{\beta,h}(x,u_{0\epsilon})dx, \qquad (3.10)
\end{aligned}$$

where

$$B_{\beta,h}(x,r) = \int_0^r T_h(s - T_\beta(s))\frac{\partial b(x,s)}{\partial s}ds.$$

Due to definition of B_k^ϵ we have:

$$\int_\Omega B_{\beta,h}(x,u_\epsilon(t))dx \geq \frac{\lambda}{2}\int_\Omega |T_h(u_\epsilon - T_\beta(u_\epsilon))|^2 dx, \quad \forall h>0, \tag{3.11}$$

and

$$0 \leq \int_\Omega B_{\beta,h}(x,u_{0\epsilon})dx \leq h\int_\Omega |b(x,u_{0\epsilon})|dx = h||b(x,u_{0\epsilon})||_{L^1(\Omega)}, \quad \forall h>0. \tag{3.12}$$

Using (3.11) and (3.12) and since $T_h(u_\epsilon - T_\beta(u_\epsilon)) = u_\epsilon - \beta$ on $\{(x,t)/\beta \leq |u_\epsilon| \leq \beta+h\}$, we obtain

$$\begin{aligned}
&\frac{\lambda}{2}\int_\Omega |T_h(u_\epsilon - T_\beta(u_\epsilon))|^2 dx + \alpha\int_{Q_t} |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|^p dxds \\
&\quad -\frac{1}{\epsilon}\int_{Q_t} T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- T_h(u_\epsilon - T_\beta(u_\epsilon))dxds \\
\leq &\int_{Q_t} c(x,t)|T_h(u_\epsilon - T_\beta(u_\epsilon)) + \beta|^\gamma |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|dsdx \\
&\quad + h(||b(x,u_{0\epsilon})||_{L^1(\Omega)} + ||f_\epsilon||_{L^1(Q_T)}) \\
\leq &C\int_{Q_t} c(x,t)|T_h(u_\epsilon - T_\beta(u_\epsilon))|^\gamma |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|dsdx \\
&\quad + C\int_{Q_t} c(x,t)|\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|dsdx \\
&\qquad + h(||b(x,u_{0\epsilon})||_{L^1(\Omega)} + ||f_\epsilon||_{L^1(Q_T)}), \qquad (3.13)
\end{aligned}$$

where C is a constant which varies from line to line and depends only the data. By

Gagliardo-Niremberg and Young inequalities we deduce

$$
\begin{aligned}
&\int_{Q_t} c(x,t)|T_h(u_\epsilon - T_\beta(u_\epsilon))|^\gamma |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))| dsdx \\
&\quad + \int_{Q_t} c(x,t)|\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))| dsdx \\
&\leq C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})} \sup_{t\in(0,\tau_1)} \int_\Omega |T_h(u_\epsilon - T_\beta(u_\epsilon))|^2 dx \\
&\quad + C\frac{N+2-\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})} \Big(\int_{Q_{\tau_1}} |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|^p dxds\Big)^{(\frac{1}{p}+\frac{N\gamma}{(N+2)p})\frac{N+2}{N+2-\gamma}}.
\end{aligned} \tag{3.14}
$$

Since

$$\gamma = \frac{(N+2)}{N+p}(p-1),$$

and by using (3.13) and (3.14), we can easily see that

$$
\begin{aligned}
&\frac{\lambda}{2}\int_\Omega |T_h(u_\epsilon - T_\beta(u_\epsilon))|^2 dx + \alpha \int_{Q_t} |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|^p dxds \\
&\quad - \frac{1}{\epsilon}\int_{Q_t} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_h(u_\epsilon - T_\beta(u_\epsilon)) dxds \\
&\leq C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})} \sup_{t\in(0,\tau_1)} \int_\Omega |T_h(u_\epsilon - T_\beta(u_\epsilon))|^2 dx \\
&\quad + C\frac{N+2-\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})} \int_{Q_{\tau_1}} |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|^p dxds \\
&\quad + h\ (||b(x,u_{0\epsilon})||_{L^1(\Omega)} + ||f_\epsilon||_{L^1(Q_T)}),
\end{aligned} \tag{3.15}
$$

which is equivalent to

$$
\begin{aligned}
&\Big(\frac{\lambda}{2} - C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big) \sup_{t\in(0,\tau_1)} \int_\Omega |T_h(u_\epsilon - T_\beta(u_\epsilon))|^2 dx \\
&\quad + \Big(\alpha - C\frac{N+2-\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big) \int_{Q_{\tau_1}} |\nabla T_h(u_\epsilon - T_\beta(u_\epsilon))|^p dxds \\
&\quad - \frac{1}{\epsilon}\int_{Q_t} T_{\frac{1}{\epsilon}}(u_\epsilon - \psi)^- T_h(u_\epsilon - T_\beta(u_\epsilon)) dxds \\
&\leq h(||b(x,u_{0\epsilon})||_{L^1(\Omega)} + ||f_\epsilon||_{L^1(Q_T)}).
\end{aligned}
$$

If we choose τ_1 such that

$$\Big(\frac{\lambda}{2} - C\frac{\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big) > 0, \tag{3.16}$$

and

$$\Big(\alpha - C\frac{N+2+\gamma}{N+2}||c(x,t)||_{L^\tau(Q_{\tau_1})}\Big) > 0, \tag{3.17}$$

then, let us denote by C the minimum between (3.16) and (3.17), we obtain

$$\begin{aligned}&\sup_{t\in(0,\tau_1)}\int_\Omega|T_h(u_\epsilon-T_\beta(u_\epsilon))|^2dx+\int_{Q_{\tau_1}}|\nabla T_h(u_\epsilon-T_\beta(u_\epsilon))|^pdxdt\\&\quad-\frac{C}{\epsilon}\int_{Q_{\tau_1}}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-T_h(u_\epsilon-T_\beta(u_\epsilon))dxds\le Ch.\end{aligned}\tag{3.18}$$

It follows that

$$-\int_{Q_{\tau_1}}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-\frac{T_h(u_\epsilon-T_\beta(u_\epsilon))}{h}dxdt\le C,$$

since

$$-\int_{Q_{\tau_1}}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-\frac{T_h(u_\epsilon-T_\beta(u_\epsilon))}{h}dxdt\ge 0,$$

for every $\beta\ge\|\psi\|_\infty$, we deduce by Fatou's lemma as $h\to 0$ that

$$\int_{Q_{\tau_1}}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-\le C.\tag{3.19}$$

Let $\tau_1\in(0,T)$ and t fixed in $(0,\tau_1)$. Using $T_k(u_\epsilon)\chi_{(0,t)}$ as test function in (3.9), we integrate between $(0,\tau_1)$, we obtain with the same techniques used previously that

$$\sup_{t\in(0,\tau_1)}\int_\Omega|T_k(u_\epsilon)|^2dx+\int_{Q_{\tau_1}}|\nabla T_k(u_\epsilon)|^pdxdt\le Ck.\tag{3.20}$$

Then, by (3.20) and Lemma 2.2, we conclude that $T_k(u_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ independently of ϵ and for any $k\ge 0$, so there exists a subsequence still denoted by u_ϵ such that

$$T_k(u_\epsilon)\rightharpoonup\xi_k\quad\text{weakly in }\ L^p(0,T;W_0^{1,p}(\Omega)).\tag{3.21}$$

Let $k>0$ be large enough and B_R be a ball of Ω, we have:

$$\begin{aligned}k\ meas\Big\{\{|u_\epsilon|>k\}\cap B_R\times[0,T]\Big\}&=\int_0^T\int_{\{|u_\epsilon|>k\}\cap B_R}|T_k(u_\epsilon)|dxdt\\&\le\int_0^T\int_{B_R}|T_k(u_\epsilon)|dxdt\\&\le\Big(\int_Q|T_k(u_\epsilon)|^pdxdt\Big)^{\frac{1}{p}}\Big(\int_0^T\int_{B_R}dxdt\Big)^{\frac{1}{p'}}\\&\le TC_R(CMk)^{\frac{1}{p}}.\end{aligned}$$

Which implies that:

$$meas\Big\{\{|u_\epsilon|>k\}\cap B_R\times[0,T]\Big\}\le\frac{c_1}{k^{1-\frac{1}{p}}},\quad\forall k\ge 1,$$

so we have
$$\lim_{k\to+\infty} meas\Big\{\{|u_\epsilon|>k\}\cap B_R\times[0,T]\Big\}=0.$$

We will now use a method in [3] to show that for a subsequence still indexed by u_ϵ and $b(x,u_\epsilon)$ converges almost everywhere in Q_T. For any integer $M\geq 1$, let S_M an increasing function of $C^2(IR)$ and such that $S_M(r)=r$ for $|r|\leq\frac{M}{2}$ and $S_M(r)=M$ for $|r|\geq M$. Remark that $suppS'_M\subset[-M,M]$. We will show in the sequel that for any fixed integer M the sequence
$$B_{S_M}(x,z)=\int_0^z\frac{\partial b(x,s)}{\partial s}S'_M(s)ds,$$
satisfies
$$B_{S_M}(x,u_\epsilon)\ \text{ is bounded in }\ L^p(0,T;W_0^{1,p}(\Omega)), \tag{3.22}$$
and
$$\frac{\partial B_{S_M}(x,u_\epsilon)}{\partial t}\ \text{ is bounded in }\ L^1(Q_T)+L^{p'}(0,T;W^{-1,p'}(\Omega)), \tag{3.23}$$
independently of ϵ. Indeed, we have first
$$\Big|\nabla B_{S_M}(x,u_\epsilon)\Big|\leq\|A\|_{L^\infty(\Omega)}|\nabla T_M(u^\varepsilon)|\|S'_M\|_{L^\infty(IR)}+M\|S'_M\|_{L^\infty(IR)}B(x)\ \text{ a.e. in }\ Q_T. \tag{3.24}$$
As a consequence of (3.20), (3.24) we then obtain (3.22). To show that (3.23) hold true, we multiply the approximate equation by $S'_M(u_\epsilon)$, we get
$$\begin{aligned}\frac{\partial B_{S_M}(x,u_\epsilon)}{\partial t}=&div\Big(a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S'_M(u_\epsilon)\Big)-a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S''_M(u_\epsilon)\nabla u_\epsilon\\&-div\Big(\phi_\epsilon(x,t,u_\epsilon)S'_M(u_\epsilon)\Big)+S''_M(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon\\&+\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-S'_M(u_\epsilon)+f_\epsilon S'_M(u_\epsilon).\end{aligned} \tag{3.25}$$
Each term in the right hand side of (3.25) is bounded in $L^1(Q_T)$ or in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. Actually, since $suppS'_M$ and $suppS''_M$ are both included in $[-M,M]$, u_ϵ may be replaced by $T_M(u_\epsilon)$ in each of these terms. For $0<\epsilon<\frac{1}{M}$, by (3.7b) we obtain
$$\begin{aligned}\Big|\int_{Q_T}\phi_\epsilon(x,t,u_\epsilon)^{p'}(S'_M(u_\epsilon))^{p'}dxdt\Big|\leq&\int_{Q_T}c(x,t)^{p'}|T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'\gamma}|S'_M(u_\epsilon))|^{p'}dxdt\\=&\int_{\{(x,t);|u_\epsilon|\leq k\}}c(x,t)^{p'}|T_M(u_\epsilon)|^{p'\gamma}|S'_M(u_\epsilon)|^{p'}dxdt.\end{aligned}$$
Using Hölder and Gagliardo-Niremberg inequality, in the right hand side integral, we obtain
$$\begin{aligned}&\int_{\{(x,t);|u_\epsilon|\leq k\}}c(x,t)^{p'}|T_M(u_\epsilon)|^{p'\gamma}|S'_M(b(u_\epsilon))|^{p'}dxdt\\\leq&\|S'_M\|_{L^\infty(IR)}||c(x,t)||^{p'}_{L^\tau(Q_T)}\Big(\sup_{t\in(0,T)}(\int_\Omega|T_M(u_\epsilon)|^2)^{\frac{p}{N}}+\int_{Q_T}|\nabla T_M(u_\epsilon)|^pdxdt\Big)\\\leq&c_M,\end{aligned}$$

where c_M is a constant independently of ϵ which will vary from line to line. By (3.3d) we deduce that

$$\begin{aligned}&\Big|\int_{Q_T}\phi_\epsilon(x,t,u_\epsilon)^{p'}(S''_M(u_\epsilon)\nabla u_\epsilon)^{p'}dxdt\Big|\\ \leq&\int_{Q_T}(S''_M(u_\epsilon))^{p'}|c(x,t)|^{p'}|T_{\frac{1}{\epsilon}}(u_\epsilon)|^{p'}|\nabla u_\epsilon|^{p'}dxdt\leq c_M,\end{aligned}\tag{3.26}$$

as a consequence of (3.25), we obtain (3.23).

As mentioned above, from (3.22) and (3.23), we deduce that for a subsequence, still indexed by ϵ, u_ϵ and $b(x,u_\epsilon)$ converges almost everywhere in Q_T, as ϵ goes to zero (see e.g., [4]) to a measurable functions u and $b(x,u)$ respectively. With the fact that $T_k(u_\epsilon)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ then $T_k(u_\epsilon)\rightharpoonup T_k(u)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ for any $k\geq 0$ as ϵ tends to zero. Actually $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

Indeed by using (3.12), (3.13), (3.14), (3.19) and (3.20) we deduce that

$$\int_\Omega B_k(x,u_\epsilon)dx\leq kC+C_1,$$

and passing to the limit-inf as ϵ tends to zero, we obtain that with

$$B_k(x,r)=\int_0^r\frac{\partial b(x,s)}{\partial s}T_k(s)ds.$$

On the other hand, we have

$$\frac{1}{k}\int_\Omega B_k(x,u(\tau))dx\leq C_2$$

for almost any τ in $(0,T)$. Due to the definition of $B_k(x,r)$ and the fact that

$$\frac{1}{k}B_k(x,u)$$

converges pointwise to

$$\int_0^u sg(s)\frac{\partial b(x,s)}{\partial s}ds=|b(x,u)|$$

as k tends to $+\infty$, as a consequence $b(x,u)$ belongs to $L^\infty(0,T;L^1(\Omega))$.

Lemma 3.1. *Let u_ϵ be a solution of the approximate problem* (3.9). *Then*

$$\lim_{n\to+\infty}\limsup_{\epsilon\to 0}\int_{\{n\leq|u_\epsilon|\leq n+1\}}a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon dxdt=0.$$

Proof. Let us now the Lipschitz continuous bounded function $\theta_n(u_\epsilon)\equiv T_{n+1}(u_\epsilon)-T_n(u_\epsilon)$ as a test function in (3.9) to obtain

$$\begin{aligned}&\int_\Omega B_n(x,u_\epsilon)(T)dx+\int_{Q_T}a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\theta_n(u_\epsilon)dxdt-\int_{Q_T}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-\theta_n(u_\epsilon)dxdt\\ \leq&\int_{Q_T}c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\theta_n(u_\epsilon)|dxdt+\int_{Q_T}f_\epsilon\theta_n(u_\epsilon)dxdt+\int_\Omega B_n(x,u_{0\epsilon})dx,\end{aligned}\tag{3.27}$$

where

$$B_n(x,r)=\int_0^r \frac{\partial b(x,s)}{\partial s}\theta_n(s)ds,$$

gives

$$\begin{aligned}&\int_\Omega B_n(x,u_\epsilon)(T)dx+\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\theta_n(u_\epsilon)dxdt-\int_{Q_T}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-\theta_n(u_\epsilon)\\ \le&\int_{Q_T} c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\theta_n(u_\epsilon)|dxdt+\int_\Omega B_n(x,u_{0\epsilon})dx+\int_{Q_T} f_\epsilon\theta_n(u_\epsilon)dxdt.\end{aligned}$$

We have set $\theta_n\ge 0$,

$$\int_\Omega B_n(x,u_\epsilon)(T)dx\ge 0 \quad \text{and} \quad -\int_{Q_T}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-\theta_n(u_\epsilon)dxdt,$$

which indeed a positive function, then for every $0<\epsilon<\frac{1}{n+1}$, we have

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\theta_n(u_\epsilon)=a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\theta_n(u_\epsilon) \ \text{ a.e. in } \ Q_T.$$

As a consequence

$$\begin{aligned}&\int_{Q_T} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla\theta_n(u_\epsilon)dxdt\\ \le&\int_{Q_T} c(x,t)|T_{\frac{1}{\epsilon}}(u_\epsilon)|^\gamma|\nabla\theta_n(u_\epsilon)|dxdt+\int_\Omega B_n(x,u_{0\epsilon})dx+\int_{Q_T} f_\epsilon\theta_n(u_\epsilon)dxdt. \qquad (3.28)\end{aligned}$$

Proceeding as in [1,4,6], we show $\theta_n(u)$ converges to 0 strongly in $L^p(0,T;W_0^{1,p}(\Omega))$, and by passing to the limit in (3.28) as ϵ tends to zero, and n tends to $+\infty$, we obtain

$$\lim_{n\to+\infty}\overline{\lim_{\epsilon\to 0}}\int_{\{n\le|u_\epsilon|\le n+1\}} a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon dxdt=0. \qquad (3.29)$$

Thus, we complete the proof. □

Step 3: This step is devoted to introduce for $k\ge 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the monotonicity method. This kind of regularization has been introduced by R. Landes (see Lemma 6 and Proposition 3, pp. 230, and Proposition 4, pp. 231, in [12]). Let v_0^μ be a sequence of function in $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\|v_0^\mu\|_{L^\infty(\Omega)}\le k$ for all $\mu>0$. and v_0^μ converges to $T_k(u_0)$ a.e. in Ω and $\frac{1}{\mu}\|v_0^\mu\|_{L^p(\Omega)}$ converges to 0. For $k\ge 0$ and $\mu>0$, let us consider the unique solution $(T_k(u))_\mu\in L^\infty(Q_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$ of the monotone problem:

$$\begin{aligned}&\frac{\partial(T_k(u))_\mu}{\partial t}+\mu((T_k(u))_\mu-T_k(u))=0 &&\text{in } D'(Q_T),\\ &(T_k(u))_\mu(t=0)=v_0^\mu &&\text{in } \Omega.\end{aligned}$$

Remark that $(T_k(u))_\mu$ converges to $T_k(u)$ a.e. in Q_T, weakly-$*$ in $L^\infty(Q_T)$ and strongly in $L^p(0,T;W_0^{1,p}(\Omega))$ as μ tends to $+\infty$, and we have

$$||(T_k(u))_\mu||_{L^\infty(Q_T)} \leq \max(||(T_k(u))||_{L^\infty(Q_T)}, ||v_0^\mu||_{L^\infty(\Omega)}) \leq k, \quad \forall \mu,\ k>0.$$

Lemma 3.2 (see H. Redwane [15]). *Let $k \geq 0$ be fixed. Let S be a $C^\infty(IR)-$function such that $S(r)=r$ for $|r|\leq k$, and $suppS'$ is compact. Then*

$$\liminf_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T \Big\langle \frac{\partial b(x,u_\epsilon)}{\partial t}, S'(u_\epsilon)(T_k(u_\epsilon)-(T_k(u))_\mu)\Big\rangle dt \geq 0,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $L^1(\Omega)+W^{-1,p'}(\Omega)$ and $L^\infty(\Omega)\cap W^{1,p}(\Omega)$.

We prove the following lemma which is the critical point in the development of the monotonicity method.

Lemma 3.3. *The subsequence of u_ϵ satisfies for any $k\geq 0$*

$$\limsup_{\epsilon\to 0} \int_{Q_T} a(x,t,u_\epsilon,\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon)dxdt \leq \int_{Q_T}\sigma_k \nabla T_k(u)dxdt.$$

Proof. Let S_n be a sequence of increasing C^∞-function such that

$$S_n(r)=r \text{ for } |r|\leq n,\ supp(S_n')\subset[-(n+1),(n+1)] \text{ and } \|S_n''\|_{L^\infty(IR)}\leq 1 \text{ for any } n\geq 1.$$

We use the sequence $(T_k(u))_\mu$ of approximation of $T_k(u)$, let

$$W_\mu^\epsilon = T_k(u_\epsilon)-(T_k(u))_\mu,$$

and plug the test function $S_n'(u_\epsilon)W_\mu^\epsilon$ in (3.9), we obtain

$$\begin{aligned}
&\int_0^T \Big\langle \frac{\partial b(x,u_\epsilon)}{\partial t}, S_n'(u_\epsilon)W_\mu^\epsilon\Big\rangle dt + \int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S_n'(u_\epsilon)\nabla W_\mu^\epsilon dxdt\\
&+\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S_n''(u_\epsilon)\nabla u_\epsilon \nabla W_\mu^\epsilon dxdt\\
&-\int_{Q_T}\phi_\epsilon(x,t,u_\epsilon)S_n'(u_\epsilon)\nabla W_\mu^\epsilon dxdt - \int_{Q_T}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- S_n'(u_\epsilon)W_\mu^\epsilon dxdt\\
&-\int_{Q_T} S_n''(u_\epsilon)\phi_\epsilon(x,t,u_\epsilon)\nabla u_\epsilon\nabla W_\mu^\epsilon dxdt\\
=&\int_{Q_T} f_\epsilon S_n'(u_\epsilon)W_\mu^\epsilon dxdt. \qquad (3.30)
\end{aligned}$$

Now we pass to the limit in (3.30) as $\epsilon\to 0$, $\mu\to+\infty$ and then $n\to+\infty$ for k real number fixed. In order to perform this task we prove below the following results for any fixed

$k\geq 0$

$$\liminf_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_0^T \Big\langle \frac{\partial b(x,u_\epsilon)}{\partial t}, W_\mu^\epsilon \Big\rangle dt \geq 0 \qquad \text{for any } n\geq k, \tag{3.31a}$$

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_{Q_T}\phi_\epsilon(x,t,u_\epsilon)S_n'(u_\epsilon)\nabla W_\mu^\epsilon dxdt=0 \qquad \text{for any } n\geq 1, \tag{3.31b}$$

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_{Q_T}\phi_\epsilon(x,t,u_\epsilon)S_n''(u_\epsilon)\nabla u_\epsilon\nabla W_\mu^\epsilon dxdt=0 \qquad \text{for any } n\geq 1, \tag{3.31c}$$

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\epsilon\to 0}\int_{Q_T}a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S_n''(u_\epsilon)\nabla u_\epsilon\nabla W_\mu^\epsilon dxdt=0, \tag{3.31d}$$

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_{Q_T}\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- S_n'(u_\epsilon)W_\mu^\epsilon dxdt=0, \tag{3.31e}$$

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_{Q_T}f_\epsilon S_n'(u_\epsilon)W_\mu^\epsilon dxdsdt=0. \tag{3.31f}$$

Proof of (3.31a). The function S_n is increasing and belongs in $C^\infty(IR)$, then we have for $|r|\leq k\leq n$, $S_n(r)=r$, while $suppS_n'$ is compact. In view of the definition of W_μ^ϵ and lemma 3.2 applies with $S=S_n$ for fixed $n\geq k$, as a consequence (3.31a) holds true.

Proof of (3.31b). Let us recall the main properties of W_μ^ϵ. For fixed $\mu>0$: W_μ^ϵ converges to $T_k(u)-(T_k(u))_\mu$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ as $\epsilon\to 0$. Remark that $||W_\mu^\epsilon||_{L^\infty(Q_T)}\leq 2k$ for any $\epsilon,\mu>0$, then we deduce that

$$W_\mu^\epsilon \rightharpoonup T_k(u)-(T_k(u))_\mu \quad \text{a.e in } Q_T \text{ and } L^\infty(Q_T) \text{ weakly } *, \tag{3.32}$$

when $\epsilon\to 0$, one had $suppS_n''\subset[-(n+1),-n]\cup[n,n+1]$ for any fixed $n\geq 1$ and $0<\epsilon<\frac{1}{n+1}$: $\phi_\epsilon(x,t,u_\epsilon)S_n'(u_\epsilon)\nabla W_\mu^\epsilon=\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n'(u_\epsilon)\nabla W_\mu^\epsilon$ a.e. in Q_T, since $suppS'\subset[-(n+1),n+1]$, on the other hand $\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n'(u_\epsilon)$ converges to $\phi(x,t,T_{n+1}(u))S_n'(u)$ a.e. in Q_T, and $|\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n'(u_\epsilon)|\leq c(x,t)(n+1)^\gamma$ for $n\geq 1$, by (3.32) and strongly convergence of $T_k(u_\epsilon)_\mu$ in $L^p(0,T,W_0^{1,p}(\Omega))$ we obtain (3.31b).

Proof of (3.31c). For any fixed $n\geq 1$ and $0<\epsilon<\frac{1}{n+1}$.

$$\phi_\epsilon(x,t,u_\epsilon)S_n''(u_\epsilon)\nabla u_\epsilon W_\mu^\epsilon=\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n''(u_\epsilon)\nabla T_{n+1}(u_\epsilon)W_\mu^\epsilon \quad \text{a.e. in } Q_T,$$

as in the previous step it is possible to pass to the limit for $\epsilon\to 0$ since by (3.32) we have

$$\phi_\epsilon(x,t,T_{n+1}(u_\epsilon))S_n''(u_\epsilon)W_\mu^\epsilon\to\phi(x,t,T_{n+1}(u))S_n''(u)W_\mu \quad \text{a.e. in } Q_T.$$

Since $|\phi(x,t,T_{n+1}(u))S_n''(u)W_\mu|\leq 2k|c(x,t)|(n+1)^\gamma$ a.e. in Q_T and $(T_k(u))_\mu$ converges to 0 in $L^p(0,T;W_0^{1,p}(\Omega))$, we obtain (3.31c).

Proof of (3.31d). In view of the definition of S_n we have

$$\begin{aligned}&\Big|\int_{Q_T}a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)S_n''(u_\epsilon)W_\mu^\epsilon dxdt\Big|\\ \leq& T\|S_n''(u_\epsilon)\|_{L^\infty(IR)}\|W_\mu^\epsilon\|_{L^\infty(Q_T)}\int_{\{n\leq|u_\epsilon|\leq n+1\}}a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla u_\epsilon dxdt\end{aligned}$$

for any $n\geq 1$, any $0<\epsilon<\frac{1}{n+1}$ any $\mu>0$. By (3.29) it is possible to establish (3.31d).
Proof of (3.31f). By (3.7c), the pointwise convergence of u_ϵ and W^ϵ_μ and its boundness it is possible to pass the limit for $\epsilon\to 0$, then for $\mu\to+\infty$ and for any $n\geq 1$:

$$\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_{Q_T} f_\epsilon S'_n(u_\epsilon)(T_k(u_\epsilon)-(T_k(u))_\mu)dxdt=0.$$

Proof of (3.31e). Similar to (3.31f). Now we turn back to the proof of Lemma 3.3. Due to (3.31a)-(3.31f) we can to pass to the limit-sup when μ tends to $+\infty$ and to the limit as n tends to $+\infty$ in (3.30). using the definition of W^ϵ_μ we deduce that for any $k\geq 0$

$$\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\epsilon\to 0}\int_{Q_T} S'_n(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)(\nabla T_k(u_\epsilon)-\nabla(T_k(u)_\mu)dxdt\leq 0.$$

Since

$$S'_n(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)=a(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)$$

for $k\leq\frac{1}{\epsilon}$ and $k\leq n$, using the properties of S'_n the above inequality implies that for $k\leq n$:

$$\begin{aligned}&\limsup_{\epsilon\to 0}\int_{Q_T} a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)dxdt\\ \leq&\lim_{n\to+\infty}\limsup_{\mu\to+\infty}\limsup_{\epsilon\to 0}\int_{Q_T} S'_n(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u)_\mu dxdt.\end{aligned}\tag{3.33}$$

On the other hand, for $\epsilon<\frac{1}{n+1}$,

$$S'_n(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)=S'_n(u_\epsilon)a(x,t,T_{n+1}(u_\epsilon),\nabla T_{n+1}(u_\epsilon))\quad \text{a.e. in } Q_T.$$

Furthermore we have

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\rightharpoonup\sigma_k\quad\text{weakly in } (L^{p'}(Q_T))^N,\tag{3.34}$$

it follows that for a fixed $n\geq 1$: $S'_n(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)$ converges to $S'_n(u_\epsilon)\sigma_{n+1}$ weakly in $L^{p'}(Q_T)$ when ϵ tends to 0. Finally, using the strong convergence of $(T_k(u)_\mu)$ to $T_k(u)$ in $L^p(0,T;W^{1,p}_0(\Omega))$ as μ tends to $+\infty$, we get

$$\begin{aligned}&\lim_{\mu\to+\infty}\lim_{\epsilon\to 0}\int_{Q_T} S'_n(u_\epsilon)a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon)_\mu dxdsdt\\ =&\int_{Q_T} S'_n(u_\epsilon)\sigma_{n+1}\nabla T_k(u)dxdt\end{aligned}\tag{3.35}$$

as soon as $k\leq n$. Now for $k\leq n$ we have

$$a(x,t,T_{n+1}(u_\epsilon),\nabla T_{n+1}(u_\epsilon))\chi_{\{|u_\epsilon|\leq k\}}=a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\chi_{\{|u_\epsilon|\leq k\}}\quad\text{a.e. in } Q_T,$$

which implies that, by the fact that $u_\epsilon \to u$ a.e.Q_T, and (3.34), and by passing to the limit when ϵ tends to 0,

$$\sigma_{n+1}\chi_{|u|\le k} = \sigma_k \chi_{\{|u|\le k\}} \quad \text{a.e. in } \ Q_T - \{|u|=k\} \quad \text{for } \ k\le n. \tag{3.36}$$

Finally, by (3.36) and (3.34) we have for $k\le n$: $\sigma_{n+1}\nabla T_k(u)=\sigma_k \nabla T_k(u)$ a.e. in Q_T. Recalling (3.33), (3.35) the proof of the lemma is complete. □

Step 4: We prove that the weak limit σ_k of $a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))$ can be identified with $a(x,t,T_k(u),\nabla T_k(u))$.

Lemma 3.4. *the subsequence of u_ϵ defined in Step 1 satisfies for any $k\ge 0$*

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{Q_T}\Big(a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,t,T_k(u_\epsilon),\nabla T_k(u))\Big)\Big(\nabla T_k(u_\epsilon)-\nabla T_k(u)\Big)dxdt\\ &=0.\end{aligned} \tag{3.37}$$

Proof. Using (3.3b) we have

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_{Q_T}\Big(a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)) - a(x,t,T_k(u_\epsilon),\nabla T_k(u))\Big)\Big(\nabla T_k(u_\epsilon)-\nabla T_k(u)\Big)dxdt\\ &\ge 0.\end{aligned} \tag{3.38}$$

Furthermore, by (3.3a), the almost everywhere convergence of u_ϵ, we have the sequence $a(x,t,T_k(u_\epsilon),\nabla T_k(u)))$ converges to $a(x,t,T_k(u),\nabla T_k(u)))$ a.e. in Q_T, and

$$|a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon)))| \le \nu[h_k(x,t)+|\nabla T_k(u_\epsilon)|^{p-1}] \quad \text{a.e. in } \ Q_T,$$

uniformly with respect to ϵ. As a consequence

$$a(x,t,T_k(u_\epsilon),\nabla T_k(u))) \to a(x,t,T_k(u),\nabla T_k(u))) \quad \text{strongly in } \ (L^{p'}(Q_T))^N. \tag{3.39}$$

Finally, using the fact that $u_\epsilon \to u$ a.e. in Q_T, (3.34) and (3.39) make it possible to pass to the lim-sup as ϵ tends to 0 in (3.38) and we have (3.37). □

Lemma 3.5. *For fixed $k\ge 0$, we have*

$$\sigma_k = a(x,t,T_k(u),\nabla T_k(u))) \quad \textit{a.e. in } \ Q_T, \tag{3.40}$$

and as ϵ tends to 0

$$a(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)))\nabla T_k(u) \tag{3.41}$$

weakly in $L^1(Q_T)$.

$$\lim_{n\to+\infty}\int_{\{n\le|u|\le n+1\}} a(x,t,u,\nabla u)\nabla u dxdt = 0, \tag{3.42a}$$

$$u\ge\psi \ \textit{a.e. in } \ \Omega. \tag{3.42b}$$

Proof. We observe that for for any $k>0$, any $0<\epsilon<\frac{1}{k}$ and any $\xi\in IR^N$:

$$a_\epsilon(x,t,T_k(u_\epsilon),\xi)=a(x,t,T_k(u_\epsilon),\xi)=a_{\frac{1}{k}}(x,t,T_k(u_\epsilon),\xi)\quad \text{a.e. in } Q_T.$$

Since $T_k(u_\epsilon)$ converges to $T_k(u)$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$, and by (3.37) we obtain

$$\lim_{\epsilon\to 0}\int_{Q_T} a_{\frac{1}{k}}(x,t,T_k(u_\epsilon),\nabla T_k(u_\epsilon))\nabla T_k(u_\epsilon)dxdt=\int_{Q_T}\sigma_k\nabla T_k(u)dxdt. \tag{3.43}$$

Since, for fixed $k>0$, the function $a_{\frac{1}{k}}(x,t,s,\xi)$ is continuous and bounded with respect to s, the usual Minty's argument applies in view of (3.34) and (3.43). It follows that (3.40) holds true. In order to prove (3.43), by (3.3b), (3.37) and proceeding as in [4] it's easy to show (3.41). Taking the limit as ϵ tends to 0 in (3.29) and using (3.41) show that u satisfies (3.42a).

Using the estimate (3.19), we have

$$\int_{Q_T} T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-\leq C\epsilon,$$

by letting ϵ to 0, we obtain

$$\int_{Q_T} T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- dxdt=0,$$

then we conclude that $u\geq\psi$ a.e. in Q_T. □

Step 5: We show that u satisfies (3.6). Let $\varphi\in K_\psi\cap L^\infty(Q_T)$ such that

$$\frac{\partial\varphi}{\partial t}\in L^{p'}(0,T;W^{-1,p'}(\Omega)).$$

Pointwise multiplication of the approximate equation (3.9) by $T_k(u_\epsilon-\varphi)$ and use the integration par parts, we get:

$$\begin{aligned}
&\int_0^t\Big\langle\frac{\partial b(x,u_\epsilon)}{\partial s};T_k(u_\epsilon-\varphi)\Big\rangle ds+\int_{Q_t}a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon-\varphi)dxds\\
&-\int_{Q_t}\phi_\epsilon(x,t,u_\epsilon)T_k(u_\epsilon-\varphi)dxds-\frac{1}{\epsilon}\int_{Q_t}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^-T_k(u_\epsilon-\varphi)dxds\\
=&\int_{Q_t}f_\epsilon T_k(u_\epsilon-\varphi)dxds.
\end{aligned}\tag{3.44}$$

We pass to the limit as in (3.44) ϵ tend to 0. The first term of (3.44) can be written

$$\begin{aligned}
&\int_0^t\Big\langle\frac{\partial b(x,u_\epsilon)}{\partial s};T_k(u_\epsilon-\varphi)\Big\rangle ds\\
=&\int_0^t\Big\langle\frac{\partial\varphi}{\partial t},\frac{\partial b(x,u_\epsilon)}{\partial s}T_k(u_\epsilon-\varphi)\Big\rangle ds\\
&+\int_\Omega\int_0^{u_\epsilon-\varphi}T_k(r)\frac{\partial b(x,r+\varphi)}{\partial r}dsdx-\int_\Omega\int_0^{u_{0\epsilon-\varphi(0)}}T_k(r)\frac{\partial b(x,r+\varphi(0))}{\partial r}drdx.
\end{aligned}$$

In view of (3.4b), (3.5), (3.8) and since u_ϵ converges to u a.e in Q_T, we deduce that

$$\int_0^t \Big\langle \frac{\partial b(x,u_\epsilon)}{\partial s}; T_k(u_\epsilon-\varphi)\Big\rangle ds$$

converges to

$$\int_0^t \Big\langle \frac{\partial b(x,u)}{\partial s}; T_k(u-\varphi)\Big\rangle ds$$

as ϵ tends to zero and for all $t\in(0,T)$.

The term

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon-\varphi)=a(x,t,T_M(u_\epsilon),\nabla T_M(u_\epsilon))\nabla T_k(T_M(u_\epsilon)-\varphi)$$

for $\epsilon\le\frac{1}{M}$, where $M=k+\|\varphi\|_{L^\infty(Q_T)}$, so using Lemma 3.5, we get

$$a_\epsilon(x,t,u_\epsilon,\nabla u_\epsilon)\nabla T_k(u_\epsilon-\varphi)$$

converges to $a(x,t,u,\nabla u)\nabla T_k(u-\varphi)$ weakly in $L^1(Q_T)$.

Furthermore, since

$$\phi_\epsilon(x,t,u_\epsilon)\nabla T_k(u_\epsilon-\varphi)=\phi(x,t,T_M(u_\epsilon))\nabla T_k(T_M(u_\epsilon)-\varphi)$$

a.e. in Q_T, for $\epsilon\le\frac{1}{M}$ and where $M=k+\|\varphi\|_{L^\infty(Q_T)}$. We can easily see that $\phi_\epsilon(x,t,u_\epsilon)\nabla T_k(u_\epsilon-\varphi)$ converges to $\phi(x,t,u)\nabla T_k(u-\varphi)$ weakly in $L^1(Q_T)$.

Finally, the term

$$-\frac{1}{\epsilon}T_{\frac{1}{\epsilon}}(u_\epsilon-\psi)^- T_k(u_\epsilon-\varphi)$$

is positive and we have $f_\epsilon T_k(u_\epsilon-\varphi)$ converges to $fT_k(u-\varphi)$ strongly in $L^1(Q_T)$.

As a consequence of the above convergence result, we are in a position to pass to the limit as ϵ tends to 0 in Eq. (3.44) and to conclude that u satisfies (3.6).

It remains to show that $b(x,u)$ satisfies the initial condition. To this end, firstly remark that, in view of the definition of S'_M (see (3.22)), we have $B_M(x,u_\epsilon)$ is bounded in $L^\infty(Q_T)\cap L^p(0,T;W_0^{1,p}(\Omega))$. Secondly, by (3.23) we show that

$$\frac{\partial B_M(x,u_\epsilon)}{\partial t}$$

is bounded in $L^1(Q)+L^{p'}(0,T;W^{-1,p'}(\Omega))$. As a consequence, an Aubin's type Lemma (see e.g., [16], Corollary 4) implies that $B_M(x,u_\epsilon)$ lies in a compact set of $C^0([0,T];W^{-1,s}(\Omega))$ for any $s<\inf(p',\frac{N}{N-1})$. It follows that, on one hand, $B_M(x,u_\epsilon)(t=0)$ converges to $B_M(x,u)(t=0)$ strongly in $W^{-1,s}(\Omega)$. On the order hand, the smoothness of B_M imply that $B_M(x,u_\epsilon)$ $(t=0)$ converges to $B_M(x,u)(t=0)$ strongly in $L^q(\Omega)$ for all $q<+\infty$, we conclude that $B_M(x,u_\epsilon)(t=0)=B_M(x,u_{0\epsilon})$ converges to $B_M(x,u)(t=0)$ strongly in $L^q(\Omega)$, we obtain $B_M(x,u)(t=0)=B_M(x,u_0)$ a.e. in Ω and for all $M>0$, now letting M to $+\infty$, we conclude that $b(x,u)(t=0)=b(x,u_0)$ a.e. in Ω.

As a conclusion of Step 1 to Step 5, the proof of Theorem 3.1 is complete. $\square$

References

[1] A. Aberqi, J. Bennouna, M. Mekkour and H. Redwane, Existence results for a nonlinear parabolic problem with a lower odrer term, Int. J. Math. Anal., 7(27) (2013), 1323–1340.

[2] Y. Akdim, J. Bennouna and M. Mekkour, Solvability of degenerate parabolic equations without sign condition and three unbounded nonlinearities, Electronic J. Differetial Equations, (2011), 1–25.

[3] D. Blanchard and G. Francfort, A few results on a class of degenerate parabolic equations, Ann. SCM. Norm. Sup. Piss, 8(2) (1991), 213–249.

[4] D. Blanchard and H. Redwane, Renormalized solutions for class of nonlinear evolution problems, J. Math. Pure, 77 (1998), 117–151.

[5] D. Blanchard, F. Murat and H. Redwane, Existence and uniqueness of renormalized solution for fairly general class of non linear parabolic problems, J. Differential Equations, 177 (2001), 331–374.

[6] D. Blanchard and F. Murat, Renormalized solutions of nonlinear parabolic with L^1 data: existence and uniqueness, Proc. Roy. Soc. Edinburgh Sec. A, 127 (1997), 1137–1152.

[7] L. Boccardo, L. Orsina and A. Porretta, Some noncoercive parabolic equations with a lower order terms in divergence form, J. Evol. Equ. 3.3 Dedicated to Philippe Bénilan, 407-418, (2003).

[8] R.-Di Nardo, Nonlinear parabolic equations with a lower order term and L^1 data, Commun. Pure Appl. Ann., 9 (2010), 929–942.

[9] R. Di Nardo, F. Fo and O. Guibé, Existence result for nonlinear parabolic equations with lower order terms. Anal. Appl, Singap., 9(2) (2011), 161–186.

[10] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuala Norm. Sup. Pisa Cl. Sci. (4) (1999), 28.

[11] R.-J. Diperna and P.-L. Lions, On the Cauchy Problem for the Boltzmann Equations: Global existence and weak stability, Ann. Math., 130 (1989), 285–366.

[12] R. Landes, On the existence of weak solutions for quasilinear parabolic initial-boundary problems, Proc. Roy. Soc. Edinburgh Sect. A, 89 (1981), 321–366.

[13] J.-L. Lions, Quelques méthodes de resolution des problémes aux limites non linéaires, Dunod et Gauthier-Villars, Paris, (1969).

[14] A. Porretta, Existence results for nonlinear parabolic equations via strong convergence of trauncations, Ann. Mat. Pura ed Appl., 177 (1999), 143–172.

[15] H. Redwane, Existence of Solution for a class of a parabolic equation with three unbounded nonlinearities, Adv. Dyn. Syt. A2PL, 2 (2007), 241–264.

[16] J. Simon, Compact set in the space $L^p(0,T,B)$, Ann. Math. Pure Appl., 146 (1987), 65–96.