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# Accuracy Enhancement of Discontinuous Galerkin Method for Hyperbolic Systems

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**Abstract.** We study the enhancement of accuracy, by means of the convolution post-processing technique, for discontinuous Galerkin(DG) approximations to hyperbolic problems. Previous investigations have focused on the superconvergence obtained by this technique for elliptic, time-dependent hyperbolic and convection-diffusion problems. In this paper, we demonstrate that it is possible to extend this post-processing technique to the hyperbolic problems written as the Friedrichs' systems by using an upwind-like DG method. We prove that the $L_2$-error of the DG solution is of order $k+1/2$, and further the post-processed DG solution is of order $2k+1$ if $Q_k$-polynomials are used. The key element of our analysis is to derive the $(2k+1)$-order negative norm error estimate. Numerical experiments are provided to illustrate the theoretical analysis.

**AMS subject classifications**: 65N30, 65M60

**Key words**: Discontinuous Galerkin method, hyperbolic problem, accuracy enhancement, post-processing, negative norm error estimate.

## 1. Introduction

In this paper, we consider an upwind-like DG method for solving the hyperbolic problem written as the Friedrichs' systems [7],

$$\sum_{i=1}^{d} A_i\partial_i \mathbf{u} + B\mathbf{u} = \mathbf{f}, \text{ in } \Omega, \quad (M - D_n)\mathbf{u} = \mathbf{0}, \text{ on } \partial\Omega, \tag{1.1}$$

where $\Omega$ is a bounded domain in $R^d$, matrix $D_n = \sum_{i=1}^{d} A_i n_i$, $n = (n_1, \cdots, n_d)^T$ is the outward unit normal vector. Our main aim is to show that it is possible to enhance the accuracy of this DG approximation by using the convolution post-processing technique. This post-processing technique was originally introduced by Bramble and Schatz [1]

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using continuous finite element methods for elliptic equations and later was developed by Cockburn, Luskin, Shu and S$\ddot{u}$li [2] using the DG method for time-dependent hyperbolic equations, and further by Ji, Xu and Ryan [9] using the LDG method for time-dependent convection-diffusion equations. The post-processing technique is carried out by a convolution operation applied to the finite element solution. The key of this technique is to derive a superconvergence order (for example, the $O(h^{2k+1})$-order) error estimate in the negative norm for the finite element solution. By the post-processing, the order of error in the $L_2$-norm can be enhanced up to the order of error in the negative norm. Some other post-processing techniques [10, 11, 20, 21] also have been proposed in enhancing the accuracy of the finite element solutions, but they do not possess such high accuracy.

DG methods for solving problem (1.1) basically can be classified as both the numerical flux method and the penalty method, see [3, 5, 6, 13, 19, 22] and the references therein. In the numerical flux method, the key element is to choose the numerical trace $D_n\widehat{\mathbf{u}}$ properly in the weak form of problem (1.1):

$$-\int_K \mathbf{u}\cdot\sum_{i=1}^d A_i\partial_i\mathbf{v}dx+\int_K(B-\sum_{i=1}^d\partial_iA_i)\mathbf{u}\cdot\mathbf{v}dx+\int_{\partial K}D_n\widehat{\mathbf{u}}\cdot\mathbf{v}ds=\int_K\mathbf{f}\cdot\mathbf{v}dx, \quad (1.2)$$

where $K$ is the element. In the traditional upwind scheme (see [8, 13, 19]), the numerical trace is defined by first splitting matrix $D_n=\sum A_in_i$ into the symmetric form $D_n=A^++A^-$ with $A^+\geq 0$ (positive semi-definite) and $A^-\leq 0$ (negative semi-definite), and then setting the numerical trace $D_n\widehat{\mathbf{u}}|_{\partial K}=A^+\mathbf{u}^++A^-\mathbf{u}^-$, where $\mathbf{u}^+$ and $\mathbf{u}^-$ are the traces of $\mathbf{u}$ on $\partial K$ from the interior and exterior of $K$, respectively. In this paper, we will present an upwind-like DG scheme which is slightly different from the traditional one. We first decompose each $A_i$ into $A_i=A_i^++A_i^-$, and then define the numerical trace by setting $D_n\widehat{\mathbf{u}}=\sum_{i=1}^dA_i^+n_i\widehat{\mathbf{u}}+\sum_{i=1}^dA_i^-n_i\widehat{\mathbf{u}}$, and $A_i^{\pm}n_i\widehat{\mathbf{u}}=A_i^{\pm}n_i\mathbf{u}^+(A_i^{\pm}n_i\mathbf{u}^-)$ if $A_i^{\pm}n_i\geq 0(A_i^{\pm}n_i\leq 0)$. The advantage of our method is that the splitting can be implemented only once before the triangulation is made, while in the traditional method, since matrix $D_n$ depends on the boundary normal vector $n|_{\partial K}$, then for each element $K$ and each face $\mathcal{F}_K\subset\partial K$, we always need to split $D_n|_{\mathcal{F}_K}=A^++A^-$. Therefor, such splitting is very consuming in practical computation. More importantly, for this DG method, we can derive the $(2k+1)$-order error estimates in the negative norm. It should be point out that Cockburn et al. in [2] (also see [9, 14]) have established a framework to prove the negative norm error estimates for DG methods applied to time-dependent hyperbolic problems, but their analysis is very relied on the time-dependent structure of the problem and is not available to time-independent hyperbolic problem (1.1). In this paper, by means of the a priori error estimate in a mesh-dependent norm and the dual argument technique, we derive the desired negative norm error estimate which allows us to enhance the accuracy of DG solution from $(k+1/2)$-order to $(2k+1)$-order in the $L_2$-norm by using the convolution post-processing technique.

Throughout this paper, let $\Omega$ be a bounded open polyhedral domain in $R^d$, $d\geq 2$. For any open subset $\mathcal{D}\subset\Omega$ and integers $m\geq 0$, we denote by $H^m(\mathcal{D})$ the usual Sobolev spaces equipped with norm $\|\cdot\|_{m,\mathcal{D}}$ and semi-norm $|\cdot|_{m,\mathcal{D}}$, and denote by $(\cdot,\cdot)_{\mathcal{D}}$ and $\|\cdot\|_{0,\mathcal{D}}$ the standard inner product and norm in the space $H^0(\mathcal{D}) = L_2(\mathcal{D})$. When $\mathcal{D} = \Omega$, we omit the index $\mathcal{D}$. We will use letter $C$ to represent a generic positive constant, independent of the mesh size $h$.

The plan of this paper is as follows. In Section 2, the upwind-like DG method is given and the stability of this method is discussed. Section 3 is devoted to the error analysis in the mesh-dependent norm and the negative norm, respectively. In Section 4, we discuss the convolution post-precessing technique and give the (2k+1)-order superconvergence for the post-processed DG solution on uniform rectangular meshes. Further, some numerical experiments are provided to illustrate our theoretical analysis. In Section 5, some conclusions are given.

## 2. Problem and its DG approximation

Consider the following first-order hyperbolic system:

$$\mathcal{L}\mathbf{u} \equiv \boldsymbol{A}\cdot\nabla\mathbf{u} + B\mathbf{u} = \mathbf{f}, \quad x \in \Omega\,, \tag{2.1}$$

$$(M - D_n)\mathbf{u} = \mathbf{0}, \qquad x \in \partial\Omega. \tag{2.2}$$

Here, $\boldsymbol{A} = (A_1, \cdots, A_d)^T$ is a vector matrix function, $\boldsymbol{A}\cdot\nabla\mathbf{u} = \sum_{i=1}^d A_i\partial_i\mathbf{u}$, $A_i$, $B$ and $M$ are some given $m\times m$ matrices, $A_i \in [W^1_\infty(\Omega)]^{m\times m}$, $B$, $M \in [L_\infty(\Omega)]^{m\times m}$, $D_n = \boldsymbol{A}\cdot n = \sum_{i=1}^d A_i n_i$, $n(x) = (n_1, \cdots, n_d)^T$ is the outward unit normal vector at the point $x \in \partial\Omega$, $\mathbf{u} = (u_1, \cdots, u_m)^T$ and $\mathbf{f} = (f_1, \cdots, f_m)^T$ with $f_i \in L_2(\Omega)$ are $m$-dimensional vector functions. We assume that problem (2.1)-(2.2) is a positive and symmetric hyperbolic system (Friedrichs' system [7]), namely,

$$A_i = A_i^T, \quad i = 1, \cdots, d, \qquad x \in \Omega, \tag{2.3}$$

$$B + B^T - div\boldsymbol{A} \geq 2\sigma_0 I, \qquad x \in \Omega, \tag{2.4}$$

$$M + M^T \geq 0, \qquad x \in \partial\Omega, \tag{2.5}$$

$$Ker(M - D_n) + Ker(M + D_n) = R^m, \quad x \in \partial\Omega, \tag{2.6}$$

where constant $\sigma_0 > 0$, $div\mathbf{A} = \partial_1 A_1 + \cdots + \partial_d A_d$, $Ker(M \pm D_n)$ is the kernel space of $M \pm D_n$, and by using the expression $A \geq 0 (\leq 0)$ we imply that the matrix $A$ is positive (negative) semi-definite. In this paper, we assume that problem (2.1)-(2.2) has a unique solution which is smooth enough for our demonstration requirement.

Problem (2.1)-(2.2) can describe many important physics processes. An example of such Friedrichs' system is as follows.

**Maxwell's equations**. Let $\sigma$ and $\mu$ be two positive functions in $L_\infty(\Omega)$ uniformly bounded away from zero. Consider the following Maxwell's equations in $R^3$

$$\begin{aligned} &\mu H + \nabla\times E = h, && x \in \Omega, \\ &\sigma E - \nabla\times H = g, && x \in \Omega, \\ &E \times n = 0, && x \in \partial\Omega, \end{aligned}$$

where $H$ and $E$ are three-dimensional vector functions. This problem can be cast into the form of Friedrichs' system by setting $\mathbf{u} = (H, E)^T$,

$$A_i = \begin{pmatrix} O & Q_i \\ Q_i^T & O \end{pmatrix} \ (i = 1, 2, 3), \quad B = \begin{pmatrix} \mu I & O \\ O & \sigma I \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} h \\ g \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and choosing the boundary matrix

$$M = \begin{pmatrix} O & -R \\ R^T & R^T R \end{pmatrix}, \quad R = \sum_{i=1}^{3} Q_i n_i = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}.$$

The conditions (2.3)-(2.6) can be verified directly.

In this example, although the boundary matrix $M$ should be determined by the boundary value condition of the problem, it is not unique. Here we have chosen the boundary matrix $M$ carefully such that it also satisfies our requirement for the error analysis, see (2.23).

Now let us introduce the DG method for solving problem (2.1)-(2.2). Let $\mathcal{T}_h = \bigcup\{K\}$ be a shape-regular triangulation of domain $\Omega$ parameterized by mesh size $h = max\, h_K$ so that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h}\{\,\overline{K}\,\}$, where $h_K$ is the diameter of element $K$. We say that the triangulation $\mathcal{T}_h$ is shape-regular, if the elements of $\mathcal{T}_h$ are affine and there exists a positive constant $\gamma$, independent of $K \in \mathcal{T}_h$, such that

$$h_K/\rho_K \le \gamma, \ \ \forall\, K \in \mathcal{T}_h,$$

where $\rho_K$ denotes the diameter of the biggest ball included in $K$. The triangulation we consider may have hanging nodes and elements of various shapes since no interelement continuity is required. Such triangulation allows us to implement the finite element adaptive computations more efficiently by the local mesh refinement/coarsening strategy.

To triangulation $\mathcal{T}_h$, we associate the finite-dimensional space $S_h^k$,

$$S_h^k = \{v \in L_2(\Omega) : \ v|_K \in S_k(K), \ \forall K \in \mathcal{T}_h\}, \tag{2.7}$$

where $S_k(K)$ is the local finite element space which at least includes $P_k(K)$. Typically, $S_k(K)$ is the space $P_k(K)$ of polynomials of degree at most $k$ on $K$, or the space $Q_k(K)$ of polynomials of degree at most $k$ in each variable on $K$.

We denote by $\mathcal{E}_h^0 = \bigcup\{\partial K \setminus \partial\Omega : \ K \in \mathcal{T}_h\}$ the union of all element boundaries that are not contained in $\partial\Omega$. Denote the piecewise smooth function space on $\mathcal{T}_h$ by

$$H^s(\mathcal{T}_h) = \{v \in L_2(\Omega) : \, v|_K \in H^s(K), \ \forall\, K \in \mathcal{T}_h\}, \quad \|v\|^2_{H^s(\mathcal{T}_h)} = \sum_{K\in\mathcal{T}_h} \|v\|^2_{s,K}, \ \ s \ge 1.$$

In order to cope with the discontinuity of function across element interfaces, we introduce the jump of function $\phi \in H^1(\mathcal{T}_h)$ on $\partial K$ by $[\phi] = \phi^+ - \phi^-$, where $\phi^+$ and $\phi^-$ are the traces of $\phi$ on $\partial K$ from the interior and exterior of $K$, respectively. Sometimes, for convenience, we will denote $\phi^+$ by $\phi$ on $\partial K$. We will also use the notations

$$(u,v)_h = \sum_{K\in\mathcal{T}_h}(u,v)_K = \sum_{K\in\mathcal{T}_h}\int_K u\,v\,dx, \quad < u,v >_{\mathcal{B}} = \sum_{K\in\mathcal{T}_h}\int_{\partial K\cap\mathcal{B}} uv\,ds,$$

where $\mathcal{B}$ is a subset of $\mathcal{E}_h^0 \bigcup \partial\Omega$, and we set $\|u\|_h^2 = (u,u)_h$.

Introduce the adjoint operator of $\mathcal{L}$,

$$\mathcal{L}^* = -\boldsymbol{A}\cdot\nabla + B^T - div\boldsymbol{A}.$$

By using integration by parts, we have

$$\int_K \mathcal{L}\mathbf{u}\cdot\mathbf{v}dx = \int_K \mathbf{u}\cdot\mathcal{L}^*\mathbf{v}dx + \int_{\partial K} D_n\mathbf{u}\cdot\mathbf{v}ds, \ \ \forall\, K\in\mathcal{T}_h. \tag{2.8}$$

Let $\mathbf{u} \in [H^1(\Omega)]^m$ be the solution of problem (2.1)-(2.2), from (2.8) we see that $\mathbf{u}$ satisfies the following weak form

$$(\mathbf{u},\mathcal{L}^*\mathbf{v})_h + \frac{1}{2} < (M+D_n)\mathbf{u},\mathbf{v} >_{\partial\Omega} + < D_n\mathbf{u},\mathbf{v} >_{\mathcal{E}_h^0} = (\mathbf{f},\mathbf{v}), \ \ \forall\,\mathbf{v}\in[H^1(\mathcal{T}_h)]^m. \tag{2.9}$$

Motivated by this weak formula, we define the DG approximation of problem (2.1)-(2.2) by finding $\mathbf{u}_h \in S_h \equiv [S_h^k]^m$ such that

$$a(\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h), \ \ \forall\,\mathbf{v}_h\in S_h, \tag{2.10}$$

where the bilinear form

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathcal{L}^*\mathbf{v})_h + \frac{1}{2} < (M+D_n)\mathbf{u},\mathbf{v} >_{\partial\Omega} + < D_n\widehat{\mathbf{u}},\mathbf{v} >_{\mathcal{E}_h^0}, \tag{2.11}$$

and $D_n\widehat{\mathbf{u}}$ is the numerical trace which is defined elaborately as follows. We first decompose the symmetric matrix $A_i$ $(1\le i\le d)$ into a sum $A_i = A_i^+ + A_i^-$ of two symmetric matrices, where $A_i^+ \ge 0$ and $A_i^- \le 0$. A classical way to obtain such a splitting is to write

$$\begin{aligned}
&A_i = Q_i(\Lambda_i^+ + \Lambda_i^-)Q_i^T = Q_i\Lambda_i^+Q_i^T + Q\Lambda_i^-Q_i^T = A_i^+ + A_i^-,\\
&\Lambda_i^+ = \text{diag}(\max(\lambda_1,0),\cdots,\max(\lambda_m,0)), \quad \Lambda_i^- = \text{diag}(\min(\lambda_1,0),\cdots,\min(\lambda_m,0)),
\end{aligned}$$

where $Q_i$ is an orthogonal matrix of eigenvectors of $A_i$ and $\{\lambda_i\}$ are the eigenvalues of matrix $A_i$. In general the decomposition $A_i = A_i^+ + A_i^-$ is not unique, but we adopt the rule that if $A_i$ itself is semi-definite, then we always take one of the summands $A_i^\pm$ to be zero. Now we define the numerical trace $D_n\widehat{\mathbf{u}}$ on $\mathcal{E}_h^0$ by

$$\begin{aligned}
&D_n\widehat{\mathbf{u}} = \sum_{i=1}^d A_in_i\widehat{\mathbf{u}} = \sum_{i=1}^d(A_i^+n_i\widehat{\mathbf{u}} + A_i^-n_i\widehat{\mathbf{u}}),\\
&G\widehat{\mathbf{u}} = \left\{\begin{array}{ll} G\mathbf{u}^+, & \text{if } \ G\ge 0, \ x\in\partial K,\\ G\mathbf{u}^-, & \text{if } \ G\le 0, \ x\in\partial K, \ G = A_i^\pm n_i,\end{array}\right.
\end{aligned} \tag{2.12}$$

where $n=(n_1,\cdots,n_d)^T$ represents the outward unit normal vector on the boundaries concerned. Note that when $n_i\geq 0$ (or $n_i\leq 0$), we have $A_i^+n_i\geq 0$ and $A_i^-n_i\leq 0$ (or $A_i^+n_i\leq 0$ and $A_i^-n_i\geq 0$), so $A_i^{\pm}n_i\widehat{\mathbf{u}}$ can be regarded as the upwind value of vector function $A_i^{\pm}n_i\mathbf{u}$ on $\partial K$, and the discrete scheme (2.10) is an upwind-like DG approximation to problem (2.1)-(2.2).

The following Lemma shows that the bilinear form $a(\mathbf{w},\mathbf{v})$ is positive definite.

**Lemma 2.1.** *Let bilinear form* $a(\mathbf{w},\mathbf{v})$ *be defined by* (2.11). *Then the following identity holds:*

$$\begin{aligned}a(\mathbf{w},\mathbf{w})=&\frac{1}{2}((B+B^T-div\boldsymbol{A})\mathbf{w},\mathbf{w})_h+\frac{1}{2}<M\mathbf{w},\mathbf{w}>_{\partial\Omega}\\&+\frac{1}{4}<D_n^*[\mathbf{w}],[\mathbf{w}]>_{\mathcal{E}_h^0},\quad\forall\,\mathbf{w}\in[H^1(\mathcal{T}_h)]^m,\end{aligned}\tag{2.13}$$

*where* $D_n^*=\sum_{i=1}^d(A_i^+-A_i^-)|n_i|\geq 0$.

*Proof.* By using formula (2.8) we have

$$(\mathcal{L}^*\mathbf{w},\mathbf{w})_K=\frac{1}{2}((B+B^T-div\boldsymbol{A})\mathbf{w},\mathbf{w})_K-\frac{1}{2}<D_n\mathbf{w},\mathbf{w}>_{\partial K},\ \ \forall\,K\in\mathcal{T}_h.\tag{2.14}$$

Hence, it implies from (2.11) that

$$\begin{aligned}a(\mathbf{w},\mathbf{w})=&\frac{1}{2}((B+B^T-div\boldsymbol{A})\mathbf{w},\mathbf{w})_h+\frac{1}{2}<M\mathbf{w},\mathbf{w}>_{\partial\Omega}\\&+<D_n(\widehat{\mathbf{w}}-\frac{1}{2}\mathbf{w}),\mathbf{w}>_{\mathcal{E}_h^0}\\=&\frac{1}{2}((B+B^T-div\boldsymbol{A})\mathbf{w},\mathbf{w})_h+\frac{1}{2}<M\mathbf{w},\mathbf{w}>_{\partial\Omega}\\&+\sum_{i=1}^d\Big(<A_i^+n_i\Big(\widehat{\mathbf{w}}-\frac{1}{2}\mathbf{w}\Big),\mathbf{w}>_{\mathcal{E}_h^0}+<A_i^-n_i\Big(\widehat{\mathbf{w}}-\frac{1}{2}\mathbf{w}\Big),\mathbf{w}>_{\mathcal{E}_h^0}\Big).\end{aligned}\tag{2.15}$$

Now, for each fixed $i$, we divide the element boundary $\partial K\in\mathcal{E}_h^0$ into two components

$$\partial K=\partial K_+\bigcup\partial K_-;\ \ \partial K_+=\{x\in\partial K:\,n_i\geq 0\},\ \ \partial K_-=\{x\in\partial K:\,n_i<0\}.\tag{2.16}$$

Then, by the definition of $A_i^+n_i\widehat{\mathbf{w}}$ in (2.12), we obtain

$$\begin{aligned}&<A_i^+n_i(\widehat{\mathbf{w}}-\frac{1}{2}\mathbf{w}),\mathbf{w}>_{\mathcal{E}_h^0}\\=&\frac{1}{2}\sum_{\partial K\in\mathcal{E}_h^0}\big\{<A_i^+n_i\mathbf{w},\mathbf{w}>_{\partial K_+}+2<A_i^+n_i\mathbf{w}^-,\mathbf{w}>_{\partial K_-}-<A_i^+n_i\mathbf{w},\mathbf{w}>_{\partial K_-}\big\}\\=&\frac{1}{2}\sum_{\partial K\in\mathcal{E}_h^0}\big\{-<A_i^+n_i\mathbf{w}^-,\mathbf{w}^->_{\partial K_-}+2<A_i^+n_i\mathbf{w}^-,\mathbf{w}>_{\partial K_-}-<A_i^+n_i\mathbf{w},\mathbf{w}>_{\partial K_-}\big\}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{\partial K\in\mathcal{E}_h^0} < A_i^+ n_i[\mathbf{w}],[\mathbf{w}] >_{\partial K_-}\\
&=\frac{1}{2}\sum_{\partial K\in\mathcal{E}_h^0} < A_i^+ |n_i|[\mathbf{w}],[\mathbf{w}] >_{\partial K_-}\\
&=\frac{1}{4}\sum_{\partial K\in\mathcal{E}_h^0} < A_i^+ |n_i|[\mathbf{w}],[\mathbf{w}] >_{\partial K} . \qquad (2.17)
\end{aligned}
$$

Here we have used the fact that

$$
\sum_{\partial K\in\mathcal{E}_h^0} < A_i^+ n_i\mathbf{w},\mathbf{w} >_{\partial K_+} = -\sum_{\partial K\in\mathcal{E}_h^0} < A_i^+ n_i\mathbf{w}^-,\mathbf{w}^- >_{\partial K_-} . \qquad (2.18)
$$

Similarly, we can derive

$$
\begin{aligned}
< A_i^- n_i\Big(\widehat{\mathbf{w}}-\frac{1}{2}\mathbf{w}\Big),\mathbf{w} >_{\mathcal{E}_h^0} &= -\frac{1}{2}\sum_{\partial K\in\mathcal{E}_h^0} < A_i^- n_i[\mathbf{w}],[\mathbf{w}] >_{\partial K_+}\\
&= -\frac{1}{2}\sum_{\partial K\in\mathcal{E}_h^0} < A_i^- |n_i|[\mathbf{w}],[\mathbf{w}] >_{\partial K_+}\\
&= -\frac{1}{4}\sum_{\partial K\in\mathcal{E}_h^0} < A_i^- |n_i|[\mathbf{w}],[\mathbf{w}] >_{\partial K} . \qquad (2.19)
\end{aligned}
$$

Combining (2.15), (2.17) and (2.19), we arrive at the conclusion of Lemma 2.1. □

According to Lemma 2.1, we can define the energy-norm

$$
\|\mathbf{u}\|_a^2 = a(\mathbf{u},\mathbf{u}),\quad \forall\, \mathbf{u}\in[H^1(\mathcal{T}_h)]^m. \qquad (2.20)
$$

Obviously, $\|\mathbf{u}\|_a \geq \sqrt{\sigma_0}\,\|\mathbf{u}\|$. Thus we immediately obtain the following conclusion: the DG solution of problem (2.10) uniquely exists and satisfies the stability estimate,

$$
\|\mathbf{u}_h\|_a \leq \frac{1}{\sqrt{\sigma_0}}\|\mathbf{f}\| . \qquad (2.21)
$$

In order to do the error analysis in the negative norm, we need to introduce a stronger norm. Define the mesh dependent norm:

$$
|||\mathbf{u}_h|||_h^2 = \|\mathbf{u}_h\|_a^2 + ||h_K^{\frac{1}{2}}\boldsymbol{A}\cdot\nabla\mathbf{u}_h\|_h^2. \qquad (2.22)
$$

Additionally, in what follows, we assume that the boundary matrix $M$ satisfies

$$
\begin{aligned}
&| <(M-D_n)\mathbf{w},\mathbf{v}>_{\partial\Omega} |\\
\leq & C_M <M\mathbf{w},\mathbf{w}>_{\partial\Omega}^{\frac{1}{2}} \|\mathbf{v}\|_{L_2(\partial\Omega)},\quad \forall\, \mathbf{w},\ \mathbf{v}\in[L_2(\partial\Omega)]^m, \qquad (2.23)
\end{aligned}
$$

where $C_M$ is a constant independent of $\mathbf{w}$ and $\mathbf{v}$. In fact, for many physics problems, we may choose the boundary matrix $M$ properly such that both the boundary value condition of the problem and assumption (2.23) can be satisfied meanwhile. For example, the boundary matrix $M$ in the Maxwell's equations mentioned above satisfies the assumption (2.23) with $C_M = 4$.

**Lemma 2.2.** *For all $\mathbf{v}_h \in S_h$, the following inf-sup stability holds*

$$|||\mathbf{v}_h|||_h \leq C \sup_{\mathbf{w}_h \in S_h} \frac{a(\mathbf{v}_h, \mathbf{w}_h)}{|||\mathbf{w}_h|||_h}. \tag{2.24}$$

*Proof.* For $\mathbf{v}_h \in S_h$, let $\mathbf{w}_h^* = h_K \boldsymbol{A}^c \cdot \nabla \mathbf{v}_h$, where $\boldsymbol{A}^c$ is the piecewise constant approximation of $\boldsymbol{A}$ on $\mathcal{T}_h$, which is defined by

$$\boldsymbol{A}^c|_K = \frac{1}{|K|}\int_K \boldsymbol{A} dx, \ \ \forall\, K \in \mathcal{T}_h.$$

By using the inverse inequality

$$h_K \|\nabla \mathbf{v}_h\|_{0,K} + h_K^{\frac{1}{2}} \|\mathbf{v}_h\|_{0,\partial K} \leq C \|\mathbf{v}_h\|_{0,K}$$

and noting that $|\boldsymbol{A} - \boldsymbol{A}^c|_{0,\infty,K} \leq C h_K \|\boldsymbol{A}\|_{1,\infty,K}$, we have

$$\|\mathbf{w}_h^*\|_{0,K}^2 = \|h_K \boldsymbol{A}^c \cdot \nabla \mathbf{v}_h\|_{0,K}^2 \leq C\|\mathbf{v}_h\|_{0,K}^2, \tag{2.25}$$

$$\begin{aligned} h_K \|\boldsymbol{A}\cdot\nabla \mathbf{w}_h^*\|_{0,K}^2 &\leq C h_K^{-1} \|\mathbf{w}_h^*\|_{0,K}^2 = C h_K \|\boldsymbol{A}^c \cdot \nabla \mathbf{v}_h\|_{0,K}^2 \\ &\leq C h_K \|\boldsymbol{A}\cdot \nabla \mathbf{v}_h\|_{0,K}^2 + C\|\mathbf{v}_h\|_{0,K}^2, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \sum_{\partial K \in \mathcal{E}_h^0} \int_{\partial K} D_n^*[\mathbf{w}_h^*]\cdot[\mathbf{w}_h^*] ds &= \sum_{\partial K \in \mathcal{E}_h^0} \int_{\partial K} h_K^2 D_n^*[\boldsymbol{A}^c \cdot \nabla \mathbf{v}_h]\cdot[\boldsymbol{A}^c\cdot\nabla \mathbf{v}_h] ds \\ \leq C \sum_{K\in\mathcal{T}_h} h_K \|\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h\|_{0,K}^2 &\leq C\sum_{K\in\mathcal{T}_h} h_K\|\boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_{0,K}^2 + C\|\mathbf{v}_h\|_{0,K}^2. \end{aligned} \tag{2.27}$$

Similarly, we have

$$< M\mathbf{w}_h^*, \mathbf{w}_h^* >_{\partial\Omega} \leq C \sum_{K\in\mathcal{T}_h} h_K \|\boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_{0,K}^2 + C\|\mathbf{v}_h\|_{0,K}^2. \tag{2.28}$$

Thus, from (2.25)-(2.28), we first obtain

$$|||\mathbf{w}_h^*|||_h \leq C|||\mathbf{v}_h|||_h. \tag{2.29}$$

Next, we need to bound $\|h_K^{\frac{1}{2}} \boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_h^2$ by means of $a(\mathbf{v}_h, \mathbf{w}_h^*)$. To this end, using formula (2.8), we rewrite bilinear form $a(\mathbf{u},\mathbf{v})$ (see (2.11)) in the following equivalent form

$$a(\mathbf{u},\mathbf{v}) = (\mathcal{L}\mathbf{u},\mathbf{v})_h + \frac{1}{2} < (M - D_n)\mathbf{u}, \mathbf{v} >_{\partial\Omega} + < D_n(\widehat{\mathbf{u}} - \mathbf{u}), \mathbf{v} >_{\mathcal{E}_h^0}. \tag{2.30}$$

Then, we have

$$\begin{aligned}
&(h_K \boldsymbol{A}\cdot\nabla\mathbf{v}_h, \boldsymbol{A}\cdot\nabla\mathbf{v}_h)_h \\
&= a(\mathbf{v}_h, h_K\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h)_h - (\boldsymbol{A}\cdot\nabla\mathbf{v}_h + B\mathbf{v}_h, h_K(\boldsymbol{A}^c-\boldsymbol{A})\cdot\nabla\mathbf{v}_h)_h - (B\mathbf{v}_h, h_K\boldsymbol{A}\cdot\nabla\mathbf{v}_h)_h \\
&\quad -\frac{1}{2} < (M-D_n)\mathbf{v}_h, h_K\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h >_{\partial\Omega} - < D_n(\widehat{\mathbf{v}}_h - \mathbf{v}_h), h_K\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h >_{\mathcal{E}_h^0} \\
&= a(\mathbf{v}_h, \mathbf{w}_h^*) + E_1 + E_2 + E_3 + E_4.
\end{aligned} \tag{2.31}$$

Below we estimate the terms $E_1 \sim E_4$. By using the inverse inequality and Cauchy inequality, we have

$$\begin{aligned}
E_1 + E_2 &= -(\boldsymbol{A}\cdot\nabla\mathbf{v}_h + B\mathbf{v}_h, h_K(\boldsymbol{A}^c-\boldsymbol{A})\cdot\nabla\mathbf{v}_h)_h - (B\mathbf{v}_h, h_K\boldsymbol{A}\cdot\nabla\mathbf{v}_h)_h \\
&\le \varepsilon\|h_K^{\frac{1}{2}}\boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_h^2 + C\|\mathbf{v}_h\|^2,
\end{aligned}$$

where $\varepsilon > 0$ can be chosen as small as needed. For $E_3$, using condition (2.23) and inverse inequality we obtain

$$\begin{aligned}
E_3 &= -\frac{1}{2} < (M-D_n)\mathbf{v}_h, h_K\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h >_{\partial\Omega} \\
&\le C_M < M\mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega}^{\frac{1}{2}} \|h_K\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h\|_{L_2(\partial\Omega)} \\
&\le C < M\mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega}^{\frac{1}{2}} \|h_K^{\frac{1}{2}}\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h\|_h \\
&\le \varepsilon\|h_K^{\frac{1}{2}}\boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_h^2 + C < M\mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega} + C\|\mathbf{v}_h\|^2.
\end{aligned}$$

For the last term, by the definition of $D_n\,\widehat{\mathbf{v}}_h$ and (2.16), we obtain

$$\begin{aligned}
E_4 &= -\sum_{i=1}^{d}\sum_{\partial K\in\mathcal{E}_h^0} < (A_i^+ n_i + A_i^- n_i)\,(\widehat{\mathbf{v}}_h - \mathbf{v}_h), h_K\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h >_{\partial K} \\
&= -\sum_{i=1}^{d}\sum_{\partial K\in\mathcal{E}_h^0}\Big( < -A_i^- n_i[\mathbf{v}_h], \mathbf{w}_h >_{\partial K_+} + < -A_i^+ n_i[\mathbf{v}_h], \mathbf{w}_h >_{\partial K_-}\Big) \\
&= -\sum_{i=1}^{d}\sum_{\partial K\in\mathcal{E}_h^0}\Big( < -A_i^- |n_i|[\mathbf{v}_h], \mathbf{w}_h >_{\partial K_+} + < A_i^+ |n_i|[\mathbf{v}_h], \mathbf{w}_h >_{\partial K_-}\Big) \\
&\le |D_n^*|_\infty < D_n^*[\mathbf{v}_h], [\mathbf{v}_h] >_{\mathcal{E}_h^0} \|\mathbf{w}_h\|_{\mathcal{E}_h^0} \\
&\le C < D_n^*[\mathbf{v}_h], [\mathbf{v}_h] >_{\mathcal{E}_h^0} \| h_K^{\frac{1}{2}}\boldsymbol{A}^c\cdot\nabla\mathbf{v}_h\|_h \\
&\le \varepsilon\|h_K^{\frac{1}{2}}\boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_h^2 + C < D_n^*[\mathbf{v}_h], [\mathbf{v}_h] >_{\mathcal{E}_h^0} + C\|\mathbf{v}_h\|^2.
\end{aligned}$$

Taking $\varepsilon = 1/6$ in the above estimates and using Lemma 2.1, we arrive at from (2.31) that

$$\frac{1}{2}\|h_K^{\frac{1}{2}}\boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_h^2 \le a(\mathbf{v}_h, \mathbf{w}_h^*) + C_0 a(\mathbf{v}_h, \mathbf{v}_h) = a(\mathbf{v}_h, \mathbf{w}_h^* + C_0\mathbf{v}_h).$$

That is

$$|||\mathbf{v}_h|||_h^2 = \|\mathbf{v}_h\|_a^2 + \|h_K^{\frac{1}{2}}\boldsymbol{A}\cdot\nabla\mathbf{v}_h\|_h^2 \le a(\mathbf{v}_h, 2\mathbf{w}_h^* + C_1\mathbf{v}_h),$$

where $C_1 = 2C_0 + 1$. Then, setting $\mathbf{w}_h = 2\mathbf{w}_h^* + C_1\mathbf{v}_h$ and using (2.29), it yields

$$|||\mathbf{v}_h|||_h|||\mathbf{w}_h|||_h \le C|||\mathbf{v}_h|||_h^2 \le Ca(\mathbf{v}_h, \mathbf{w}_h).$$

This completes the proof. □

To get the $(k+1/2)$-order error estimate in energy-norm $\|\cdot\|_a$, the stability result (2.13) is enough. The inf-sup stability (2.24) will be needed for the error analysis in the mesh-dependent norm $|||\cdot|||_h$.

## 3. Error analysis in the negative-norm

In this section, we will give the superconvergent result of DG method (2.10) in the negative-norm. To this end, we first establish the error estimate in norm $|||\cdot|||_h$.

Let $\mathbf{u} \in [H^1(\Omega)]^m$ be the solution of problem (2.1)-(2.2). From (2.9) and the definition of $a(\mathbf{u}, \mathbf{v})$, we have

$$a(\mathbf{u}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \;\; \forall\, \mathbf{v}_h \in S_h.$$

This shows that DG scheme (2.10) is a consistent scheme, so that we have the error equation:

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0, \;\; \forall\, \mathbf{v}_h \in S_h. \tag{3.1}$$

For $u \in H^1(\mathcal{T}_h)$, we introduce the $L_2$-projection function $P_h u$, restricted to $K \in \mathcal{T}_h$, $P_h u \in S_k(K)$ such that

$$(u - P_h u, v_h)_K = 0, \;\; \forall\, v_h \in S_k(K), \;\; K \in \mathcal{T}_h. \tag{3.2}$$

For this projection, the following approximation property holds.

$$\|u - P_h u\|_{0,K} + h_K^{\frac{1}{2}}\|u - P_h u\|_{0,\partial K} \le Ch_K^{k+1}|u|_{k+1,K}, \;\; k \ge 0, \;\; K \in \mathcal{T}_h. \tag{3.3}$$

In what follows, for vector function $\mathbf{u}$, we set $P_h\mathbf{u} = (P_h u_1, \cdots, P_h u_m)^T$ and denote by $w^c$ the piecewise constant approximation of function $w$ on $\mathcal{T}_h$. Since

$$(M - D_n)\mathbf{v}\cdot\mathbf{w} = -(M + D_n)\mathbf{w}\cdot\mathbf{v} + (M + M^T)\mathbf{v}\cdot\mathbf{w}$$

and $M + M^T \ge 0$, then assumption condition (2.23) implies

$$|< (M + D_n)\mathbf{w}, \mathbf{v} >_{\partial\Omega}| \le C'_M\|\mathbf{w}\|_{L_2(\partial\Omega)} < M\mathbf{v}, \mathbf{v} >_{\partial\Omega}^{\frac{1}{2}}, \;\; \forall\, \mathbf{w},\, \mathbf{v} \in [L_2(\partial\Omega)]^m. \tag{3.4}$$

**Theorem 3.1.** *Assume that $\mathcal{T}_h$ is a shape-regular triangulation and condition* (2.23) *holds, and let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of problems* (2.1)-(2.2) *and* (2.10)*, respectively. Then we have*

$$|||\mathbf{u} - \mathbf{u}_h|||_h \le Ch^{k+\frac{1}{2}}|\mathbf{u}|_{H^{k+1}(\mathcal{T}_h)}, \;\; k \ge 0. \tag{3.5}$$

*Proof.* From error equation (3.1), we obtain for $\mathbf{v}_h \in S_h$ that

$$
\begin{aligned}
&a(\mathbf{u}_h - P_h\mathbf{u}, \mathbf{v}_h) = a(\mathbf{u} - P_h\mathbf{u}, \mathbf{v}_h)\\
&= (\mathbf{u} - P_h\mathbf{u}, \mathcal{L}^*\mathbf{v}_h)_h + \frac{1}{2} < (M + D_n)(\mathbf{u} - P_h\mathbf{u}), \mathbf{v}_h >_{\partial\Omega} + < D_n(\mathbf{u} - \widehat{P_h\mathbf{u}}), \mathbf{v}_h >_{\mathcal{E}_h^0}\\
&= T_1 + T_2 + T_3. \qquad (3.6)
\end{aligned}
$$

Below we estimate the terms $T_i$, $i = 1, 2, 3$. By the definition of $P_h\mathbf{u}$ and noting that $\boldsymbol{A}^c \cdot \nabla\mathbf{v}_h \in S_h$, we first have

$$
\begin{aligned}
T_1 &= (\mathbf{u} - P_h\mathbf{u}, \mathcal{L}^*\mathbf{v}_h)_h\\
&= -(\mathbf{u} - P_h\mathbf{u}, (\boldsymbol{A} - \boldsymbol{A}^c) \cdot \nabla\mathbf{v}_h)_h + (\mathbf{u} - P_h\mathbf{u}, (B^T - div\boldsymbol{A})\mathbf{v}_h)_h\\
&\le C \sum_{K\in\mathcal{T}_h} \Big( h_K|\boldsymbol{A}|_{1,\infty}\|\mathbf{u} - P_h\mathbf{u}\|_{0,K}\|\nabla\mathbf{v}_h\|_{0,K} + \|\mathbf{u} - P_h\mathbf{u}\|_{0,K}\|\mathbf{v}_h\|_{0,K}\Big)\\
&\le C\|\mathbf{u} - P_h\mathbf{u}\|\,\|\mathbf{v}_h\|, \qquad (3.7)
\end{aligned}
$$

where we have used the inverse inequality. For $T_2$, from condition (3.4) we have

$$
T_2 = \frac{1}{2} < (M + D_n)(\mathbf{u} - P_h\mathbf{u}), \mathbf{v}_h >_{\partial\Omega} \le \frac{1}{2}C_M'\|\mathbf{u} - P_h\mathbf{u}\|_{L_2(\partial\Omega)} < M\mathbf{v}_h, \mathbf{v}_h >_{\partial\Omega}^{\frac{1}{2}}.
$$

We now need to estimate

$$
T_3 =< D_n(\mathbf{u} - \widehat{P_h\mathbf{u}}), \mathbf{v}_h >_{\mathcal{E}_h^0} = \sum_{i=1}^{d} < (A_i^+ n_i + A_i^- n_i)(\mathbf{u} - \widehat{P_h\mathbf{u}}), \mathbf{v}_h >_{\mathcal{E}_h^0}. \qquad (3.8)
$$

To this end, for each fixed $i$, we set $\partial K = \partial K_+ \bigcup \partial K_-$, see (2.16). By the definition of $A_i^+ n_i \widehat{w}$, we have

$$
\begin{aligned}
&< A_i^+ n_i(\mathbf{u} - \widehat{P_h\mathbf{u}}), \mathbf{v}_h >_{\mathcal{E}_h^0} = \sum_{\partial K\in\mathcal{E}_h^0} < A_i^+ n_i(\mathbf{u} - \widehat{P_h\mathbf{u}}), \mathbf{v}_h >_{\partial K}\\
&= \sum_{\partial K\in\mathcal{E}_h^0} \{< A_i^+ n_i(\mathbf{u} - P_h\mathbf{u}), \mathbf{v}_h >_{\partial K_+} + < A_i^+ n_i(\mathbf{u} - (P_h\mathbf{u})^-), \mathbf{v}_h >_{\partial K_-}\}\\
&= \sum_{\partial K\in\mathcal{E}_h^0} \{< A_i^+ n_i(\mathbf{u} - P_h\mathbf{u}), \mathbf{v}_h >_{\partial K_+} - < A_i^+ n_i(\mathbf{u} - P_h\mathbf{u}), \mathbf{v}_h^- >_{\partial K_+}\}\\
&= \sum_{\partial K\in\mathcal{E}_h^0} < A_i^+|n_i|(\mathbf{u} - P_h\mathbf{u}), [\mathbf{v}_h] >_{\partial K_+}, \qquad (3.9)
\end{aligned}
$$

where we have used identify (2.18). Since $A_i^+|n_i| \ge 0$, we have from (3.9) that

$$
\begin{aligned}
&< A_i^+ n_i(\mathbf{u} - \widehat{P_h\mathbf{u}}), \mathbf{v}_h >_{\mathcal{E}_h^0} = \sum_{\partial K\in\mathcal{E}_h^0} < A_i^+|n_i|(\mathbf{u} - P_h\mathbf{u}), [\mathbf{v}_h] >_{\partial K_+}\\
&\le \sum_{\partial K\in\mathcal{E}_h^0} \big( < A_i^+|n_i|(\mathbf{u} - P_h\mathbf{u}), \mathbf{u} - P_h\mathbf{u} >_{\partial K_+} \big)^{\frac{1}{2}} \big( < A_i^+|n_i|[\mathbf{v}_h], [\mathbf{v}_h] >_{\partial K_+} \big)^{\frac{1}{2}}\\
&\le \big( < A_i^+|n_i|(\mathbf{u} - P_h\mathbf{u}), \mathbf{u} - P_h\mathbf{u} >_{\mathcal{E}_h^0} \big)^{\frac{1}{2}} \big( < A_i^+|n_i|[\mathbf{v}_h], [\mathbf{v}_h] >_{\mathcal{E}_h^0} \big)^{\frac{1}{2}}. \qquad (3.10)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
&< A_i^- n_i(\mathbf{u} - \widehat{P_h\mathbf{u}}), \mathbf{v}_h >_{\mathcal{E}_h^0} = \sum_{\partial K \in \mathcal{E}_h^0} < -A_i^- |n_i| (\mathbf{u} - P_h\mathbf{u}), [\mathbf{v}_h] >_{\partial K_-} \\
\le& \big( < -A_i^- |n_i| (\mathbf{u} - P_h\mathbf{u}), \mathbf{u} - P_h\mathbf{u} >_{\mathcal{E}_h^0} \big)^{\frac{1}{2}} \big( < -A_i^- |n_i| [\mathbf{v}_h], [\mathbf{v}_h] >_{\mathcal{E}_h^0} \big)^{\frac{1}{2}}. \qquad (3.11)
\end{aligned}
$$

Combining (3.8), (3.10) and (3.11) and using Cauchy inequality, we obtain

$$
T_3 \le \big( < D_n^* (\mathbf{u} - P_h\mathbf{u}), \mathbf{u} - P_h\mathbf{u} >_{\mathcal{E}_h^0} \big)^{\frac{1}{2}} \big( < D_n^* [\mathbf{v}_h], [\mathbf{v}_h] >_{\mathcal{E}_h^0} \big)^{\frac{1}{2}}. \qquad (3.12)
$$

Substituting the estimates of $T_1$, $T_2$ and $T_3$ into (3.6), we arrive at

$$
\begin{aligned}
&a(\mathbf{u}_h - P_h\mathbf{u}, \mathbf{v}_h) \\
\le& C\Big( \|\mathbf{u} - P_h\mathbf{u}\| + \|\mathbf{u} - P_h\mathbf{u}\|_{L_2(\partial\Omega)} + \|\mathbf{u} - P_h\mathbf{u}\|_{L_2(\mathcal{E}_h^0)} \Big) |||\mathbf{v}_h|||_h, \quad \mathbf{v}_h \in S_h. \qquad (3.13)
\end{aligned}
$$

It implies from Lemma 2.2 and approximation property (3.2) that

$$
|||\mathbf{u}_h - P_h\mathbf{u}|||_h \le Ch^{k+\frac{1}{2}} |\mathbf{u}|_{H^{k+1}(\mathcal{T}_h)}, \quad k \ge 0.
$$

The proof is completed by using the triangle inequality. □

In Theorem 3.1, we have derived the error estimate in the mesh-dependent norm, this result is useful for us to do the error analysis in the negative norm. It is well known that the $(k+1/2)$-order error estimate in the $L_2$-norm is sharp for DG methods with general meshes [15], but for rectangular meshes [12] and some structured triangle meshes [4, 16], the optimal error estimate will be of order $(k+1)$ in solving single hyperbolic equation.

Now we are in the position to estimate error in the negative norm:

$$
\|u\|_{-l,\Omega} = \sup_{\varphi \in C_0^\infty(\Omega)} \frac{(u, \varphi)_\Omega}{\|\varphi\|_{l,\Omega}}, \quad l \ge 0.
$$

Obviously, for any $\Omega_1 \subset \Omega$, $\|u\|_{-l,\Omega_1} \le \|u\|_{-l,\Omega}$ holds.

Introduce the dual problem: for any fixed $\boldsymbol{\varphi} \in [C_0^\infty(\Omega)]^m$, find $\boldsymbol{\psi}$ such that

$$
\mathcal{L}^* \boldsymbol{\psi} = \boldsymbol{\varphi}, \quad x \in \Omega; \quad (M + D_n)^T \boldsymbol{\psi} = 0, \quad x \in \partial\Omega. \qquad (3.14)
$$

We know that if the problem datum are smooth enough, problem (3.14) has a unique solution $\boldsymbol{\psi} \in [H^l(\Omega)]^m$ and satisfies the regularity estimate [7, 17]

$$
\|\boldsymbol{\psi}\|_l \le C \|\boldsymbol{\varphi}\|_l, \quad l \ge 1. \qquad (3.15)
$$

**Theorem 3.2.** *Assume that $\mathcal{T}_h$ is a shape-regular triangulation and condition* (2.23) *holds, and let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of problems* (2.1)-(2.2) *and* (2.10)*, respectively. Then we have*

$$
\|\mathbf{u} - \mathbf{u}_h\|_{-l,\Omega} \le Ch^{k+l} |\mathbf{u}|_{H^{k+1}(\mathcal{T}_h)}, \quad 1 \le l \le k+1. \qquad (3.16)
$$

*Proof.* From (3.14) we have

$$
\begin{aligned}
(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\varphi}) = (\mathbf{u}-\mathbf{u}_h,\mathcal{L}^*\boldsymbol{\psi}) &= (\mathbf{u}-\mathbf{u}_h,\mathcal{L}^*\boldsymbol{\psi}) \\
&\quad + \frac{1}{2} < (M+D_n)(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\psi} >_{\partial\Omega} + < D_n(\mathbf{u}-\widehat{\mathbf{u}}_h),\boldsymbol{\psi} >_{\mathcal{E}_h^0} \\
&= a(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\psi}).
\end{aligned} \tag{3.17}
$$

Here we have used the boundary condition $(M+D_n)^T\boldsymbol{\psi} = 0$ and the fact that from (2.16) and (2.18) we have for $\boldsymbol{\psi} \in [H^1(\Omega)]^m$,

$$
\begin{aligned}
& < D_n(\mathbf{u}-\widehat{\mathbf{u}}_h),\boldsymbol{\psi} >_{\mathcal{E}_h^0} \\
= & \sum_{\partial K\in\mathcal{E}_h^0}\sum_{i=1}^{d}\Big( < A_i^+ n_i(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\psi} >_{\partial K_+} + < A_i^- n_i(\mathbf{u}-\mathbf{u}_h^-),\boldsymbol{\psi} >_{\partial K_+} \Big) \\
& + \sum_{\partial K\in\mathcal{E}_h^0}\sum_{i=1}^{d}\Big( < A_i^+ n_i(\mathbf{u}-\mathbf{u}_h^-),\boldsymbol{\psi} >_{\partial K_-} + < A_i^- n_i(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\psi} >_{\partial K_-} \Big) \\
= & \sum_{\partial K\in\mathcal{E}_h^0}\sum_{i=1}^{d}\Big( < A_i^+ n_i(\mathbf{u}-\mathbf{u}_h),[\boldsymbol{\psi}] >_{\partial K_+} + < A_i^- n_i(\mathbf{u}-\mathbf{u}_h),[\boldsymbol{\psi}] >_{\partial K_-} \Big) = 0,
\end{aligned}
$$

noting that $[\boldsymbol{\psi}] = 0$. Then, from (3.17), (3.1) and (2.30) we obtain

$$
\begin{aligned}
&(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\varphi}) = a(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\psi}) = a(\mathbf{u}-\mathbf{u}_h,\boldsymbol{\psi}-P_h\boldsymbol{\psi}) \\
=&(\mathcal{L}(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\psi}-P_h\boldsymbol{\psi})_h + \frac{1}{2} < (M-D_n)(\mathbf{u}-\mathbf{u}_h),\boldsymbol{\psi}-P_h\boldsymbol{\psi} >_{\partial\Omega} \\
&+ < D_n(\mathbf{u}-\widehat{\mathbf{u}}_h-(\mathbf{u}-\mathbf{u}_h)),\boldsymbol{\psi}-P_h\boldsymbol{\psi} >_{\mathcal{E}_h^0} \\
=&R_1+R_2+R_3.
\end{aligned} \tag{3.18}
$$

Below we estimate $R_1 \sim R_3$. First, by using Theorem 3.1 we have

$$
\begin{aligned}
R_1 &\le Ch^l\|\mathbf{u}-\mathbf{u}_h\|\|\boldsymbol{\psi}\|_l + C\sum_{K\in\mathcal{T}_h} h_K^l\|\boldsymbol{A}\cdot\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,K}\|\|\boldsymbol{\psi}\|_{l,K} \\
&\le Ch^{l-\frac{1}{2}}|||\mathbf{u}-\mathbf{u}_h|||_h\|\boldsymbol{\psi}\|_l \le Ch^{k+l}\|\mathbf{u}\|_{H^{k+1}(\mathcal{T}_h)}\|\boldsymbol{\varphi}\|_l.
\end{aligned}
$$

Next, using condition (2.23) and Theorem 3.1 we obtain

$$
R_2 \le Ch^{l-\frac{1}{2}} < M(\mathbf{u}-\mathbf{u}_h),\mathbf{u}-\mathbf{u}_h >_{\partial\Omega}^{\frac{1}{2}} \|\boldsymbol{\psi}\|_l \le Ch^{k+l}\|\mathbf{u}\|_{H^{k+1}(\mathcal{T}_h)}\|\boldsymbol{\varphi}\|_l.
$$

For $R_3$, let $\mathbf{v_h} = \mathbf{u} - \mathbf{u}_h$, $\mathbf{w}_h = \boldsymbol{\psi} - P_h\boldsymbol{\psi}$. Since

$$
\begin{aligned}
&< D_n(\widehat{\mathbf{v}}_h - \mathbf{v}_h), \mathbf{w}_h >_{\mathcal{E}_h^0} \\
=& \sum_{\partial K \in \mathcal{E}_h^0} \sum_{i=1}^{d} \Big( < A_i^- n_i(\mathbf{v}_h^- - \mathbf{v}_h), \mathbf{w}_h >_{\partial K_+} + < A_i^+ n_i(\mathbf{v}_h^- - \mathbf{v}_h), \mathbf{w}_h >_{\partial K_-} \Big) \\
=& \sum_{\partial K \in \mathcal{E}_h^0} \sum_{i=1}^{d} \Big( < -A_i^- |n_i| [\mathbf{v}_h], \mathbf{w}_h >_{\partial K_+} + < A_i^+ |n_i| [\mathbf{v}_h], \mathbf{w}_h >_{\partial K_-} \Big) \\
\leq& \sum_{\partial K \in \mathcal{E}_h^0} < D_n^*[\mathbf{v}_h], [\mathbf{v}_h] >_{\partial K}^{\frac{1}{2}} < D_n^* \mathbf{w}_h, \mathbf{w}_h >_{\partial K}^{\frac{1}{2}},
\end{aligned}
$$

then we have from Theorem 3.1 that

$$
R_3 \leq Ch^{l-\frac{1}{2}} < D_n^*[\mathbf{u} - \mathbf{u}_h], [\mathbf{u} - \mathbf{u}_h] >_{\mathcal{E}_h^0}^{\frac{1}{2}} \|\boldsymbol{\psi}\|_l \leq Ch^{k+l} \|\mathbf{u}\|_{H^{k+1}(\mathcal{T}_h)} \|\boldsymbol{\varphi}\|_l.
$$

Substituting $R_1 \sim R_3$ into (3.18), we complete the proof. $\square$

Taking $l = k+1$ in (3.16), we immediately obtain the superconvergence result in the negative norm

$$
\|\mathbf{u} - \mathbf{u}_h\|_{-(k+1),\Omega} \leq Ch^{2k+1} \|\mathbf{u}\|_{H^{k+1}(\mathcal{T}_h)}.
$$

## 4. Accuracy enhancement by convolution post-precessing technique

As indicated in Section 1, as long as we obtain high order error estimates in the negative norm, we can use the convolution post-processing technique to enhance the accuracy of DG solution in the $L_2$-norm, up to the order of error estimates in the negative norm. In this section, we will discuss the post-processing method proposed in [1,2] and provide some numerical experiments to illustrate the accuracy enhancement.

### 4.1. The post-processing technique

This post-processing technique was originally introduced by Bramble and Schatz [1] for elliptic equations and further was developed by Cockburn, Luskin, Shu and S$\ddot{u}$li [2] for time-dependent hyperbolic equations. The post-processing technique is carried out by a convolution operation applied to the numerical solution. This convolution operation can filter out the oscillations in the error so that the accuracy of the numerical solution can be enhanced. In what follows, we will apply the post-processing technique to the DG solution of problem (2.1)-(2.2) by using the standard framework proposed in [2].

We define the post-processed solution by the following convolution operation

$$
u_h^\star = K_h^{2k+1,k+1} \star u_h, \tag{4.1}
$$

where $u_h$ is the DG solution, $K_h^{2k+1,k+1}(x) = K^{2k+1,k+1}(x/h)/h^d$ is the convolution kernel, and $K^{2k+1,k+1}(x)$ is a linear combination of $B$-splines of order $k+1$. Specifically, the one dimensional convolution kernel is of form

$$K^{2k+1,k+1}(x) = \sum_{\gamma=-k}^{k} C_\gamma^{2k+1,k+1}\psi^{(k+1)}(x-\gamma),$$

where $\psi^{(1)} = \chi$, $\psi^{(k+1)} = \psi^{(k)} \star \chi$ for $k \geq 1$, and $\chi$ is the characteristic function

$$\chi = \begin{cases} 1, & x \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \text{otherwise}. \end{cases}$$

Further, for multi-integer $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_d)$ and any $x = (x_1, \cdots, x_d) \in R^d$, we set the multi-dimensional convolution kernel

$$K^{2k+1,k+1}(x) = \sum_{|\boldsymbol{\gamma}|\leq k} C_{\boldsymbol{\gamma}}^{2k+1,k+1}\psi^{(k+1)}(x-\boldsymbol{\gamma}), \tag{4.2}$$

where $\psi^{(k+1)}(x) = \psi^{(k+1)}(x_1)\cdots\psi^{(k+1)}(x_d)$ is the $d$-dimensional spline and the coefficients $C_{\boldsymbol{\gamma}}^{2k+1,k+1}$ are chosen such that $K^{2k+1,k+1}(x) \star p = p$ for any polynomial $p$ of order $2k$, see [2, 18].

It is easy to see that $u_h^\star$ is a piecewise polynomial of order $2k+1$ when $u_h$ is the DG solution using polynomial of order $k$. Hence, it is expectable that $u_h^\star$ will yield a higher order accuracy than that of $u_h$. As pointed out in [2], the convolution kernel has some favorable properties: Firstly, it has compact support. In fact, the $B$-splines kernel $K_h^{2k+1,k+1}(x)$ is locally supported in at most $2k+2$ elements. This property can reduce the computation cost of the post-processing. Secondly, the convolution is $2k$-polynomial invariable so that the accuracy is not destroyed by the post-processing.

In the above negative norm error estimates of the DG solution, the choice of meshes (triangle or rectangle) and function spaces ($P_k$ or $Q_k$) are flexible. But the post-processing kernel is applied in a tensor product fashion and therefore, in what follows, for the superconvergence extraction by the kernel we require that $Q_k$-polynomials are used. We further assume that the triangulation is uniform so that it is translation invariant, this condition is required in the superconvergence analysis, see [1, 2].

We define the difference quotient natation

$$\partial_{h,j}v(x) = \Big(v\Big(x+\frac{1}{2}he_j\Big) - v\Big(x-\frac{1}{2}he_j\Big)\Big)/h,$$

where $e_j$ is the multi-index whose $j$-component is 1 and all others 0. For any multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ we set $\alpha$-order difference quotient $\partial_h^\alpha v(x) = \partial_{h,1}^{\alpha_1}\cdots\partial_{h,d}^{\alpha_d}v(x)$. Now, we can state an important approximation result which allows that the post-processed solution has superconvergence by means of the negative norm estimates.

**Theorem 4.1** (Bramble [1], Cockburn [2]). *Let* $\mathbf{u}$ *and be the solution of problem* (2.1)-(2.2), $\Omega_0+2supp(K_h^{2k+1,k+1}(x)) \subset\subset \Omega_1 \subset\subset \Omega$*, and let* $\mathbf{u}_h \in L_2(\Omega)$ *be any approximation to* $\mathbf{u}$*. Then we have for* $s \geq 0$*,*

$$\|\mathbf{u} - K_h^{2k+1,k+1} \star \mathbf{u}_h\|_{0,\Omega_0} \leq C_1 h^{2k+1} |\mathbf{u}|_{2k+1,\Omega_1} + C_2 \sum_{|\alpha| \leq s} \|\partial_h^\alpha (\mathbf{u} - \mathbf{u}_h)\|_{-s,\Omega_1},$$

*where* $C_1$ *and* $C_2$ *depends solely on* $\Omega_0,\ \Omega_1,\ d,\ k,\ C_\gamma^{2k+1,k+1}$*, independent of* $h$*.*

Theorem 4.1 shows that the superconvergence can be obtained for the post-processed solution if we have a high order negative norm estimate for $\partial_h^\alpha(\mathbf{u} - \mathbf{u}_h)$. Now let $\mathbf{u}_h$ be the DG solution of problem (2.10). Because we are dealing with a linear equation and the triangulation is translation invariant, as noted in [2, 9], the negative norm error estimate for $\mathbf{u} - \mathbf{u}_h$ in Theorem 3.2 also holds for the difference quotient $\partial_h^\alpha(\mathbf{u} - \mathbf{u}_h)$ when $\{A_i\}$ are constant matrices. Thus, we immediately obtain from Theorem 4.1 and Theorem 3.2 that, if $\{A_i\}$ are constant matrices,

$$\|\mathbf{u} - K_h^{2k+1,k+1} \star \mathbf{u}_h\|_{0,\Omega_0} \leq C h^{2k+1} \big( \|\mathbf{u}\|_{2k+1,\Omega_1} + \|\mathbf{u}\|_{k+1,\Omega} \big). \tag{4.3}$$

This is the desired superconvergence result.

From Theorem 4.1, we see that the accuracy enhancement is dependent on the superconvergent estimate for the error of difference quotient $\partial_h^\alpha(\mathbf{u} - \mathbf{u}_h)$ in the negative norm. For constant coefficient equations, such superconvergence estimate can be derived by means of the superconvergence estimate of $\|\mathbf{u} - \mathbf{u}_h\|_{-s,\Omega}$, see [2, 9]. But, for variable coefficient equations, it is difficult to derive the superconvergence estimate of $\|\partial_h^\alpha(\mathbf{u} - \mathbf{u}_h)\|_{-s,\Omega_1}$, even if we have obtained the superconvergence estimate of $\|\mathbf{u} - \mathbf{u}_h\|_{-s,\Omega}$. Recently, Mirzaee et al. in [14] give the superconvergence estimate of $\|\partial_h^\alpha(\mathbf{u} - \mathbf{u}_h)\|_{-s,\Omega_1}$ for time-dependent problem with variable coefficients. But their method crucially relies on the time-dependent structure of the problem and is not available to time-independent problem (1.1). We need to do further research for variable coefficient case.

## 4.2. Numerical experiments

The effectiveness of the convolution post-processing technique has been verified by a large number of numerical experiments in [1, 2, 9, 18]. Here we only give two examples to show the superconvergence for our problems under consideration.

**Example 4.1.** Consider the hyperbolic problem: find $\mathbf{u} = (u, v)$ such that

$$\begin{aligned}
&2u_x - v_x + u_y + u = f_1, && (x,y) \in \Omega,\\
&v_x - u_x + v = f_2, && (x,y) \in \Omega,\\
&u = v = 0, && (x,y) \in \partial\Omega,
\end{aligned}$$

Table 1: Convergence rates before post-processing.

| | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|
| mesh $h$ | error | rate | error | rate | error | rate |
| 1/8 | 3.64E-2 | - | 1.12E-2 | - | 4.12E-3 | - |
| 1/16 | 9.22E-3 | 1.981 | 1.51E-3 | 2.892 | 2.72E-4 | 3.921 |
| 1/32 | 2.28E-3 | 2.015 | 1.88E-4 | 3.002 | 1.70E-5 | 4.002 |
| 1/64 | 5.70E-4 | 2.000 | 2.34E-5 | 3.003 | 1.00E-6 | 4.003 |
| 1/128 | 1.41E-4 | 2.013 | 2.83E-6 | 3.049 | 5.80E-8 | 4.109 |

Table 2: Convergence rates after post-processing.

| | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|
| mesh $h$ | error | rate | error | rate | error | rate |
| 1/8 | 2.64E-2 | - | 1.12E-3 | - | 7.12E-4 | - |
| 1/16 | 3.34E-3 | 2.981 | 3.06E-5 | 5.192 | 5.87E-6 | 6.921 |
| 1/32 | 4.14E-4 | 3.015 | 6.31E-7 | 5.602 | 4.58E-8 | 7.002 |
| 1/64 | 5.17E-5 | 3.000 | 9.84E-9 | 6.003 | 3.57E-10 | 7.003 |
| 1/128 | 6.41E-6 | 3.013 | 1.49E-10 | 6.049 | 2.59E-12 | 7.109 |

where $\Omega = (0,1)\times(0,1)$. The problem can be written as the Friedrichs' system by setting

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$D_n = \begin{pmatrix} 2n_1+n_2 & -n_1 \\ -n_1 & n_1 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The condition (2.23) holds with $C_M = 5$. We take the exact solution

$$\mathbf{u} = (x(1-x)\sin(\pi y),\ y(1-y)\sin(\pi x))^T.$$

Note that $A_1 \geq 0$, $A_2 \geq 0$, so we may take $A_i^+ = A_i$, $A_i^- = 0$ $(i=1,2)$ in the DG scheme, see Section 2. In the numerical computations, we partition $\Omega$ into uniform rectangular meshes and use the $Q_k$ polynomials. The post-processing domain is $\Omega_0 = (3/8,5/8)\times(3/8,5/8)$. The numerical results are given in Tables 1 and 2, in which the $L_2(\Omega_0)$-errors are presented for successively halving mesh size $h = 1/2^m$ and different polynomials of order $k$. The numerical convergence rate $\alpha$ is computed by using the formula $\alpha = \ln\left(e_h/e_{h/2}\right)/\ln 2$ where $e_h$ represents the $L_2$-error between the exact solution and the DG solution with mesh size $h$. We see that the $L_2$-error is of $(k+1)$-order before post-processing (see Table 1) and of at least $(2k+1)$-order after post-processing as the theory predicts (see Table 2).

**Example 4.2.** In order to examine the effectiveness of our method for hyperbolic systems with variable matrices $\{A_i\}$, in this example, we consider the following problem. Find $\mathbf{u} = (u, v)$ such that

$$\begin{aligned} &x^2 u_x - v_x + (y+1)u_y + 6u = f_1, && (x,y) \in \Omega, \\ &e^y v_x - u_x + x v_y + 4v = f_2, && (x,y) \in \Omega, \\ &u = v = 0, && (x,y) \in \partial\Omega, \end{aligned}$$

where $\Omega = (0,1) \times (0,1)$. The problem can be written as the Friedrichs' system by setting

$$\begin{aligned} &A_1 = \begin{pmatrix} x^2 & -1 \\ -1 & e^y \end{pmatrix}, \quad A_2 = \begin{pmatrix} y+1 & 0 \\ 0 & x \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}, \\ &D_n = \begin{pmatrix} x^2 n_1 + (y+1) n_2 & -n_1 \\ -n_1 & e^y n_1 + x n_2 \end{pmatrix}, \quad M = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}. \end{aligned}$$

The condition (2.23) holds with $C_M = 3$. We take the exact solution

$$\mathbf{u} = (\, 2\sin(\pi y)\sin(\pi x),\ 2y(1-y)\sin(\pi x)\,)^T.$$

In this numerical example, we take the same computation setting as that in Example 4.1. The numerical results obtained by using the post-processing method are given in Table 3. We see that the convergence rate is just about of order $2k+1$.

Table 3: Convergence rates after post-processing for variable $\{A_i\}$.

| | $k=1$ | | $k=2$ | | $k=3$ | |
|---|---|---|---|---|---|---|
| mesh $h$ | error | rate | error | rate | error | rate |
| 1/8 | 2.34E-2 | - | 3.12E-3 | - | 1.32E-3 | - |
| 1/16 | 2.96E-3 | 2.982 | 1.03E-4 | 4.921 | 1.17E-5 | 6.821 |
| 1/32 | 3.79E-4 | 2.965 | 3.30E-6 | 4.960 | 1.12E-7 | 6.702 |
| 1/64 | 4.80E-5 | 2.983 | 1.03E-7 | 5.003 | 9.49E-10 | 6.883 |
| 1/128 | 6.05E-6 | 2.988 | 3.11E-9 | 5.049 | 7.99E-12 | 6.892 |

## 5. Conclusions

In this paper, we investigate how to enhance the accuracy of the discontinuous Galerkin approximation to the hyperbolic systems written as Friedrichs' systems. We propose an upwind-like DG scheme that is different from those in existing literatures. The key element of our analysis is to establish a superconvergence order error estimate in the negative norm so that we can enhance the accuracy of the DG solution by using the convolution post-processing technique. Such negative norm estimate has been obtained by Cockburn, Mirzaee and Ji et al. in [2, 9, 14] for the DG approximations to some time-dependent hyperbolic problems, but their argument methods crucially rely on the time-dependent structure of the problems and are not available for the time-independent problems here considered. Based on the negative norm error estimate and the post-processing technique, we can enhance the accuracy in the $L_2$-norm up to the accuracy of $O(h^{2k+1})$-order in the negative norm. Numerical examples verify the effectiveness of this post-processing technique. Applying this post-processing technique to the variable coefficients Friedrichs' systems and the nonlinear problems is the subject of ongoing research.

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