

A Collocation Method for Initial Value Problems of Second-Order ODEs by Using Laguerre Functions

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Abstract. We propose a collocation method for solving initial value problems of second-order ODEs by using modified Laguerre functions. This new process provides global numerical solutions. Numerical results demonstrate the efficiency of the proposed algorithm.

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1. Introduction

Many practical problems are governed by initial value problems of second-order ODEs. We may reformulate such problems to systems of first-order ODEs and then solve them numerically. Whereas, for saving work, it seems reasonable to solve them directly.

For notational convenience, we denote $\frac{d^r U}{dt^r}$ by $\partial_t^r U$. In many literatures, one focused on finite difference methods for the equation

$$\partial_t^2 U = f(U, t).$$

Generally, we divide those methods into two classes. In the first class of finite difference schemes, the coefficients depend on some known periods or frequencies of solutions, including exponential-fitted and trigonometrically-fitted methods, and linear multi-step method. In the second class of finite difference schemes, the coefficients are constants,

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such as Runge-Kutta-Nyström method, linear multi-step method, hybrid method, Störmer-Cowell method and prediction-correction method. Relatively, there have been less existing results on numerical methods for general equation

$$\partial_t^2 U = f(\partial_t U, U, t),$$

for which one often used Runge-Kutta-Nyström method, SDIRKN method and linear multi-step method.

Another efficient algorithm for solving initial value problems of ODEs is based on various collocations. The collocation method for first-order ODEs, could be regarded as Runge-Kutta method or linear multi-step method, see, e.g., [1, 5, 17–19, 22, 23, 27]. The collocation method has been also applied successfully to second-order ODEs, which is called as collocation-based Runge-Kutta-Nyström method in some literatures, see [16, 20, 21] and the references therein.

As is well known, spectral and pseudospectral methods employ orthogonal systems as the basis functions, and so usually provide accurate numerical results. Especially, we could use the Laguerre orthogonal approximation and interpolation to solve differential and integral equations on the half line, see [2, 4, 6, 7, 24–26, 28] and the references therein. Some authors designed Legendre and Laguerre collocation methods for initial value problems of first-order ODEs, cf. [9–12]. Meanwhile, the authors investigated Legendre and Laguerre collocation methods for second-order ODEs, see [13, 29]. One of advantages of Laguerre collocation method is the ability of producing global numerical solution for all time $t \geq 0$.

There are two kinds of Laguerre collocation method. In the first class of Laguerre collocation method, one took the modified Laguerre polynomials as the basis functions. They are convergent in certain Sobolev space with the weight $e^{-\beta t}$, $\beta > 0$, even if the exact solutions grow very rapidly as t increases. However, it does not ensure the high accuracy in the $C(0, \infty)$ -norm. In other words, even if the global weighted norm of error is small, the point-wise error might be big for large t . However, if the exact solutions are in the space $L^2(0, \infty)$, then we could use the second kind of Laguerre collocation method using the modified Laguerre functions. In this paper, we propose a new collocation method, in which we take the modified Laguerre functions as the basis functions and approximate the solutions of second-order ODEs directly.

The paper is organized as follows. The next section is for preliminary. In Section 3, we design the new numerical process, which has several merits:

- it provides the global numerical solution directly,
- it is simple to derive efficient algorithm,
- it is easy to be implemented for nonlinear problems,
- it is more natural, if the original problem is well posed in certain space without any weight.

In Section 4, we develop a multi-step version. More precisely, we first use the Laguerre-Gauss collocation method with moderate mode to obtain numerical results, and then refine them step by step. This technique simplifies computation and often leads to more accurate

numerical results. In Section 5, we present some numerical results demonstrating the efficiency of suggested methods.

2. Preliminary

Let $\omega_{\alpha,\beta}(t) = t^\alpha e^{-\beta t}$ with $\alpha > -1$ and $\beta > 0$. The modified generalized Laguerre polynomial of degree l is defined by

$$L_l^{(\alpha,\beta)}(t) = \frac{1}{l!} t^{-\alpha} e^{\beta t} \partial_t^l (t^{l+\alpha} e^{-\beta t}), \quad l = 0, 1, \dots$$

According to (2.2)-(2.4) of [14], we have

$$L_l^{(\alpha,\beta)}(0) = L_l^{(\alpha,1)}(0) = \frac{\Gamma(l + \alpha + 1)}{\Gamma(\alpha + 1)l!}, \quad l \geq 0, \quad (2.1)$$

$$\partial_t L_l^{(\alpha,\beta)}(t) = -\beta L_{l-1}^{(\alpha+1,\beta)}(t), \quad l \geq 1, \quad (2.2)$$

$$(l + 1)L_{l+1}^{(\alpha,\beta)}(t) = (2l + \alpha + 1 - \beta t)L_l^{(\alpha,\beta)}(t) - (l + \alpha)L_{l-1}^{(\alpha,\beta)}(t), \quad l \geq 1, \quad (2.3)$$

$$L_l^{(\alpha,\beta)}(t) = L_l^{(\alpha+1,\beta)}(t) - L_{l-1}^{(\alpha+1,\beta)}(t) = \beta^{-1}(\partial_t L_l^{(\alpha,\beta)}(t) - \partial_t L_{l+1}^{(\alpha,\beta)}(t)), \quad l \geq 1. \quad (2.4)$$

The set of modified generalized Laguerre polynomials is a complete $L^2_{\omega_{\alpha,\beta}}(0, \infty)$ -orthogonal system, namely,

$$\int_0^\infty L_l^{(\alpha,\beta)}(t)L_m^{(\alpha,\beta)}(t)\omega_{\alpha,\beta}(t)dt = \gamma_l^{(\alpha,\beta)}\delta_{l,m}, \quad \gamma_l^{(\alpha,\beta)} = \frac{\Gamma(l + \alpha + 1)}{\beta^{\alpha+1}l!}, \quad (2.5)$$

where $\delta_{l,m}$ is the Kronecker symbol.

We next recall the modified Laguerre-Gauss interpolation. For any positive integer N , $\mathcal{P}_N(0, \infty)$ stands for the set of all algebraic polynomials of degree at most N . Let $L_l^{(\beta)}(t) = L_l^{(0,\beta)}(t)$. We denote by $t_{\beta,j}^N$, $0 \leq j \leq N$, the modified Laguerre-Gauss interpolation nodes, which are the distinct zeros of $L_{N+1}^{(\beta)}(t)$, arranged in ascending order. The corresponding Christoffel numbers are as follows,

$$\omega_{\beta,j}^N = \frac{1}{t_{\beta,j}^N (\partial_t L_{N+1}^{(\beta)}(t_{\beta,j}^N))^2}, \quad 0 \leq j \leq N.$$

For any $\phi \in \mathcal{P}_{2N+1}(0, \infty)$ (cf. (2.8) of [8]),

$$\int_0^\infty \phi(t)\omega_\beta(t)dt = \sum_{j=0}^N \omega_{\beta,j}^N \phi(t_{\beta,j}^N). \quad (2.6)$$

We introduce the following discrete inner product and norm,

$$(u, v)_{\omega_{\beta,N}} = \sum_{j=0}^N u(t_{\beta,j}^N)v(t_{\beta,j}^N)\omega_{\beta,j}^N, \quad \|v\|_{\omega_{\beta,N}} = (v, v)_{\omega_{\beta,N}}^{\frac{1}{2}}.$$

Due to (2.6), we have

$$(\phi, \psi)_{\omega_\beta} = (\phi, \psi)_{\omega_{\beta,N}}, \quad \forall \phi, \psi \in \mathcal{P}_{2N+1}(0, \infty). \tag{2.7}$$

We now turn to the approximation using the modified Laguerre functions. The modified generalized Laguerre functions are defined by

$$\tilde{L}_l^{(\alpha, \beta)}(t) = e^{-\frac{1}{2}\beta t} L_l^{(\alpha, \beta)}(t) = \frac{1}{l!} t^{-\alpha} e^{\frac{1}{2}\beta t} \partial_t^l (t^{l+\alpha} e^{-\beta t}), \quad \alpha > -1, \beta > 0, l = 0, 1, \dots$$

By (2.1)-(2.3), we have (also see [15])

$$\tilde{L}_l^{(\alpha, \beta)}(0) = \tilde{L}_l^{(\alpha, 1)}(0) = \frac{\Gamma(l + \alpha + 1)}{\Gamma(\alpha + 1)l!}, \quad l \geq 0, \tag{2.8}$$

$$(l + 1)\tilde{L}_{l+1}^{(\alpha, \beta)}(t) = (2l + \alpha + 1 - \beta t)\tilde{L}_l^{(\alpha, \beta)}(t) - (l + \alpha)\tilde{L}_{l-1}^{(\alpha, \beta)}(t), \quad l \geq 1, \tag{2.9}$$

$$\partial_t \tilde{L}_0^{(\alpha, \beta)}(t) = -\frac{1}{2}\beta \tilde{L}_0^{(\alpha, \beta)}(t), \tag{2.10a}$$

$$\partial_t \tilde{L}_l^{(\alpha, \beta)}(t) = -\beta \tilde{L}_{l-1}^{(\alpha+1, \beta)}(t) - \frac{1}{2}\beta \tilde{L}_l^{(\alpha, \beta)}(t), \quad l \geq 1. \tag{2.10b}$$

Moreover, using (2.4), gives

$$\partial_t \tilde{L}_l^{(\alpha, \beta)}(t) = \partial_t \tilde{L}_{l-1}^{(\alpha, \beta)}(t) - \frac{1}{2}\beta \tilde{L}_l^{(\alpha, \beta)}(t) - \frac{1}{2}\beta \tilde{L}_{l-1}^{(\alpha, \beta)}(t), \quad l \geq 1, \tag{2.11}$$

whence

$$\partial_t \tilde{L}_l^{(\alpha, \beta)}(t) = -\frac{1}{2}\beta \tilde{L}_l^{(\alpha, \beta)}(t) - \beta \sum_{j=0}^{l-1} \tilde{L}_j^{(\alpha, \beta)}(t), \quad l \geq 1. \tag{2.12}$$

Let $(u, v)_{t^\alpha}$ and $\|v\|_{t^\alpha}$ be the inner product and the norm of the weighted space $L_{t^\alpha}^2(0, \infty)$, respectively. When $\alpha = 0$, they are denoted by (u, v) and $\|v\|$ for simplicity.

Thanks to (2.5), the set of modified generalized Laguerre functions is a complete $L_{t^\alpha}^2(0, \infty)$ -orthogonal system, i.e.,

$$(\tilde{L}_l^{(\alpha, \beta)}, \tilde{L}_m^{(\alpha, \beta)})_{t^\alpha} = \delta_{l,m} \gamma_l^{(\alpha, \beta)},$$

where $\gamma_l^{(\alpha, \beta)}$ is the same as in (2.5). Let $\tilde{L}_l^{(\beta)}(t) = \tilde{L}_l^{(0, \beta)}(t)$, $l \geq 0$. They form a complete $L^2(0, \infty)$ -orthogonal system. Thus, for any $v \in L^2(0, \infty)$,

$$v(t) = \sum_{l=0}^{\infty} \tilde{v}_{\beta, l} \tilde{L}_l^{(\beta)}(t), \quad \tilde{v}_{\beta, l} = \beta(v, \tilde{L}_l^{(\beta)}). \tag{2.13}$$

We next deal with the interpolation using the modified Laguerre functions. We set

$$\mathcal{Q}_N(0, \infty) = span\{\tilde{L}_0^{(\beta)}, \tilde{L}_1^{(\beta)}, \dots, \tilde{L}_N^{(\beta)}\}.$$

Let $t_{\beta,j}^N$ and $\omega_{\beta,j}^N$ be the same as in (2.6). The nodes and weights of the new interpolation are as follows,

$$\tilde{t}_{\beta,j}^N = t_{\beta,j}^N, \quad \tilde{\omega}_{\beta,j}^N = \frac{e^{\beta t_{\beta,j}^N}}{t_{\beta,j}^N (\partial_t (e^{\frac{1}{2}\beta t_{\beta,j}^N} \tilde{L}_{N+1}^{(\beta)}(t_{\beta,j}^N)))^2} = e^{\beta t_{\beta,j}^N} \omega_{\beta,j}^N, \quad 0 \leq j \leq N.$$

We introduce the discrete inner product and norm as

$$(u, v)_{\beta,N} = \sum_{j=0}^N u(t_{\beta,j}^N) v(t_{\beta,j}^N) \tilde{\omega}_{\beta,j}^N, \quad \|v\|_{\beta,N} = (v, v)_{\beta,N}^{\frac{1}{2}}.$$

For any $\phi_j \in \mathcal{Q}_{m_j}(0, \infty)$, we have $\phi_{m_j} = e^{\frac{1}{2}\beta t} \psi_{m_j}$ with $\psi_{m_j} \in \mathcal{P}_{m_j}(0, \infty)$. If $m_1 + m_2 \leq 2N + 1$, then by (2.7),

$$(\phi_1, \phi_2)_{\beta,N} = (\psi_1, \psi_2)_{\omega_{\beta,N}} = (\psi_1, \psi_2)_{\omega_{\beta}} = (\phi_1, \phi_2). \tag{2.14}$$

For $v \in C(0, \infty)$, the new Laguerre-Gauss interpolation $\tilde{\mathcal{I}}_{\beta,N} v \in \mathcal{Q}_N(0, \infty)$ is determined uniquely by

$$\tilde{\mathcal{I}}_{\beta,N} v(t_{\beta,j}^N) = v(t_{\beta,j}^N), \quad 0 \leq j \leq N.$$

By virtue of (2.14), for any $\phi \in \mathcal{Q}_{N+1}(0, \infty)$,

$$(\tilde{\mathcal{I}}_{\beta,N} v, \phi) = (\tilde{\mathcal{I}}_{\beta,N} v, \phi)_{\beta,N} = (v, \phi)_{\beta,N}. \tag{2.15}$$

We can expand $\tilde{\mathcal{I}}_{\beta,N} v(t)$ as

$$\tilde{\mathcal{I}}_{\beta,N} v(t) = \sum_{l=0}^N \tilde{d}_{\beta,l}^N \tilde{L}_l^{(\beta)}(t).$$

With the aid of (2.13) and (2.14), we obtain

$$\tilde{d}_{\beta,l}^N = \beta(\tilde{\mathcal{I}}_{\beta,N} v, \tilde{L}_l^{(\beta)}) = \beta(\tilde{\mathcal{I}}_{\beta,N} v, \tilde{L}_l^{(\beta)})_{\beta,N} = \beta(v, \tilde{L}_l^{(\beta)})_{\beta,N}, \quad 0 \leq l \leq N. \tag{2.16}$$

Remark 2.1. There is a close relation between $\mathcal{I}_{\beta,N}$ and $\tilde{\mathcal{I}}_{\beta,N}$. Indeed, by (3.7) of [9],

$$\tilde{\mathcal{I}}_{\beta,N} v(t) = e^{-\frac{1}{2}\beta t} \mathcal{I}_{\beta,N}(e^{\frac{1}{2}\beta t} v(t)).$$

3. Collocation method using modified Laguerre functions

In this section, we derive the new collocation method using the modified Laguerre functions. We consider the following model problem,

$$\begin{cases} \partial_t^2 U(t) = f(\partial_t U(t), U(t), t), & t > 0, \\ \partial_t U(0) = V_0, \quad U(0) = U_0, \end{cases} \tag{3.1}$$

where V_0 and U_0 describe the initial states of $\partial_t U(t)$ and $U(t)$ respectively, and $f(z_1, z_0, t)$ is a given function. We approximate the solution of (3.1) by $u^N(t) \in \mathcal{Q}_{N+2}(0, \infty)$ such that

$$\partial_t^2 u^N(t_{\beta,k}^N) = f(\partial_t u^N(t_{\beta,k}^N), u^N(t_{\beta,k}^N), t_{\beta,k}^N), \quad 0 \leq k \leq N, \tag{3.2a}$$

$$\partial_t u^N(0) = V_0, \quad u^N(0) = U_0. \tag{3.2b}$$

We now derive an efficient algorithm for solving (3.2). Let

$$u^N(t) = \sum_{l=0}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_l^{(\beta)}(t). \tag{3.3}$$

By (2.10a), we deduce that

$$\partial_t u^N(t) = \sum_{l=0}^{N+2} \tilde{u}_{\beta,l}^N \partial_t \tilde{L}_l^{(\beta)}(t) = -\beta \sum_{l=1}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_{l-1}^{(1,\beta)}(t) - \frac{\beta}{2} \sum_{l=0}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_l^{(\beta)}(t), \tag{3.4}$$

$$\partial_t^2 u^N(t) = \beta^2 \sum_{l=2}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_{l-2}^{(2,\beta)}(t) + \beta^2 \sum_{l=1}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_{l-1}^{(1,\beta)}(t) + \frac{\beta^2}{4} \sum_{l=0}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_l^{(\beta)}(t). \tag{3.5}$$

Clearly, $\tilde{L}_l^{(\beta)}(0) = 1$ and $\partial_t \tilde{L}_0^{(\beta)}(0) = -\beta/2$. Moreover, thanks to (2.8) and (2.10a), we have $\partial_t \tilde{L}_l^{(\beta)}(0) = -\frac{\beta}{2}(2l+1)$ for $l \geq 1$. Thus, with the aid of (3.3)-(3.5), we obtain from (3.2) that

$$\begin{aligned} & \beta^2 \sum_{l=2}^{N+2} \tilde{u}_{\beta,l}^N (\tilde{L}_{l-2}^{(2,\beta)}(t_{\beta,k}^N) + \tilde{L}_{l-1}^{(1,\beta)}(t_{\beta,k}^N) + \frac{1}{4} \tilde{L}_l^{(\beta)}(t_{\beta,k}^N)) \\ & \quad + \beta^2 \tilde{u}_{\beta,1}^N (\tilde{L}_0^{(1,\beta)}(t_{\beta,k}^N) + \frac{1}{4} \tilde{L}_1^{(\beta)}(t_{\beta,k}^N)) + \frac{\beta^2}{4} \tilde{u}_{\beta,0}^N \tilde{L}_0^{(\beta)}(t_{\beta,k}^N) \\ & = f(\partial_t u^N(t_{\beta,k}^N), u^N(t_{\beta,k}^N), t_{\beta,k}^N), \quad 0 \leq k \leq N, \end{aligned} \tag{3.6a}$$

$$-\frac{\beta}{2} \sum_{l=0}^{N+2} (2l+1) \tilde{u}_{\beta,l}^N = V_0, \quad \sum_{l=0}^{N+2} \tilde{u}_{\beta,l}^N = U_0, \tag{3.6b}$$

where

$$\begin{aligned} u^N(t_{\beta,k}^N) &= \sum_{l=0}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_l^{(\beta)}(t_{\beta,k}^N), \\ \partial_t u^N(t_{\beta,k}^N) &= -\beta \sum_{l=1}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_{l-1}^{(1,\beta)}(t_{\beta,k}^N) - \frac{\beta}{2} \sum_{l=0}^{N+2} \tilde{u}_{\beta,l}^N \tilde{L}_l^{(\beta)}(t_{\beta,k}^N). \end{aligned}$$

The scheme (3.6) can be rewritten as a compact matrix form. To do this, we introduce the $(N + 3) \times (N + 3)$ matrix \tilde{A}_β^N with the entries $\tilde{a}_{\beta,k,j}^N$ as follows,

$$\tilde{a}_{\beta,k,j}^N = \begin{cases} \beta^2(\tilde{L}_{j-2}^{(2,\beta)}(t_{\beta,k}^N) + \tilde{L}_{j-1}^{(1,\beta)}(t_{\beta,k}^N) + \frac{1}{4}\tilde{L}_j^{(\beta)}(t_{\beta,k}^N)), & \text{for } 0 \leq k \leq N, \quad 2 \leq j \leq N + 2, \\ \frac{\beta^2}{4}\tilde{L}_1^{(\beta)}(t_{\beta,k}^N) + \beta^2\tilde{L}_0^{(1,\beta)}(t_{\beta,k}^N), & \text{for } 0 \leq k \leq N, \quad j = 1, \\ \frac{\beta^2}{4}\tilde{L}_0^{(\beta)}(t_{\beta,k}^N), & \text{for } 0 \leq k \leq N, \quad j = 0, \\ -\frac{\beta}{2}(2j + 1), & \text{for } k = N + 1, \quad 0 \leq j \leq N + 2, \\ 1, & \text{for } k = N + 2, \quad 0 \leq j \leq N + 2, \\ 0, & \text{otherwise.} \end{cases}$$

Also, we define the $(N + 1) \times (N + 3)$ matrix \tilde{B}_β^N with the entries:

$$\tilde{b}_{\beta,k,0}^N = -\frac{\beta}{2}\tilde{L}_0^{(\beta)}(t_{\beta,k}^N), \quad \tilde{b}_{\beta,k,j}^N = -\beta(\tilde{L}_{j-1}^{(1,\beta)}(t_{\beta,k}^N) + \frac{1}{2}\tilde{L}_j^{(\beta)}(t_{\beta,k}^N)), \quad 0 \leq k \leq N, \quad 1 \leq j \leq N + 2,$$

and the $(N + 1) \times (N + 3)$ matrix \tilde{C}_β^N with the entries:

$$\tilde{c}_{\beta,k,l}^N = \tilde{L}_j^{(\beta)}(t_{\beta,k}^N), \quad 0 \leq k \leq N, \quad 0 \leq j \leq N + 2.$$

Further, let $\mathbf{u}_N = (\tilde{u}_{\beta,0}^N, \tilde{u}_{\beta,1}^N, \dots, \tilde{u}_{\beta,N+2}^N)^T$, and

$$\tilde{\mathbf{f}}_\beta^N(\tilde{\mathbf{u}}_N) = (f(\partial_t u^N(t_{\beta,0}^N), u^N(t_{\beta,0}^N), t_{\beta,0}^N), \dots, f(\partial_t u^N(t_{\beta,N}^N), u^N(t_{\beta,N}^N), t_{\beta,N}^N), V_0, U_0)^T,$$

where $\partial_t u^N(t_{\beta,k}^N)$ and $u^N(t_{\beta,k}^N)$ are the k 'th component of $\tilde{B}_\beta^N \mathbf{u}_N$ and $\tilde{C}_\beta^N \mathbf{u}_N$, respectively. Then we obtain from (3.6) that

$$\tilde{A}_\beta^N \tilde{\mathbf{u}}_N = \tilde{\mathbf{f}}_\beta^N(\tilde{\mathbf{u}}_N), \tag{3.7}$$

or equivalently,

$$\tilde{\mathbf{u}}_N = (\tilde{A}_\beta^N)^{-1} \tilde{\mathbf{f}}_\beta^N(\tilde{\mathbf{u}}_N). \tag{3.8}$$

We set $\mathbf{u}_N = (u^N(t_{\beta,0}^N), u^N(t_{\beta,1}^N), \dots, u^N(t_{\beta,N}^N))^T$. Then by virtue of (3.8),

$$\mathbf{u}_N = \tilde{C}_\beta^N (\tilde{A}_\beta^N)^{-1} \tilde{\mathbf{f}}_\beta^N(\tilde{\mathbf{u}}_N). \tag{3.9}$$

Remark 3.1. In actual computation, it requires to compute $(\tilde{A}_\beta^N)^{-1}$ only once. This allows us to save a significant amount of computational time.

Remark 3.2. When $f(z_1, z_0, t)$ is nonlinear, we first use certain iteration process to solve (3.8) and obtain $\tilde{u}_{\beta,l}^N, 0 \leq l \leq N + 2$. Then, we use (3.3) to obtain the global numerical solution $u^N(t), t \geq 0$. This fact shows another advantage of the method (3.2).

Remark 3.3. We could also use (3.8) and (3.9) to evaluate the values of $u^N(t_{\beta,k}^N)$. By (2.16) of [8], $t_{\beta,N}^N \cong 4\beta^{-1}N$. Thus, we obtain the values of numerical solution at large interpolation nodes, even with moderate mode N . In fact, this is also one of advantages of this new method.

The proposed method is also available for systems of ordinary differential equations. Let $\vec{U}(t) = (U^{(1)}(t), U^{(2)}(t), \dots, U^{(m)}(t))$ and

$$\begin{aligned} & \vec{f}(\partial_t \vec{U}(t), \vec{U}(t), t) \\ & = (f^{(1)}(\partial_t \vec{U}(t), \vec{U}(t), t), f^{(2)}(\partial_t \vec{U}(t), \vec{U}(t), t), \dots, f^{(m)}(\partial_t \vec{U}(t), \vec{U}(t), t)). \end{aligned}$$

We consider the model problem

$$\partial_t^2 \vec{U}(t) = \vec{f}(\partial_t \vec{U}(t), \vec{U}(t), t), \quad t > 0, \tag{3.10a}$$

$$\partial_t \vec{U}(0) = \vec{V}_0, \quad \vec{U}(0) = \vec{U}_0. \tag{3.10b}$$

The corresponding numerical method for solving system (3.10), is to seek $\vec{u}^N(t) \in (\mathcal{Q}_{N+2}(0, \infty))^m$, such that

$$\partial_t^2 \vec{u}^N(t_{\beta,j}^N) = \vec{f}(\partial_t \vec{u}^N(t_{\beta,j}^N), \vec{u}^N(t_{\beta,j}^N), t_{\beta,j}^N), \quad 0 \leq j \leq N, \tag{3.11a}$$

$$\partial_t \vec{u}^N(0) = \vec{V}_0, \quad \vec{u}^N(0) = \vec{U}_0. \tag{3.11b}$$

We can derive a compact matrix form of scheme (3.11), which is similar to (3.8).

4. Multi-step version of collocation method

In the last section, we proposed a numerical method for second-order ODEs. Since the distance between the adjacent nodes $t_{\beta,j}^N$ and $t_{\beta,j-1}^N$ increases rapidly as N and j increase. we could use moderate mode N to evaluate the unknown function at the nodes far from the origin $t = 0$. However, in actual computation, it is not convenient to use very large mode N . On the other hand, due to the big distances between large adjacent interpolation nodes, we may lose some information about the structure of solution which oscillates seriously between two big nodes. To remedy these deficiencies, we may use scheme (3.2) with moderate N repeatedly to refine numerical solutions.

Now, let N_0 be a moderate positive integer, $\beta_0 > 0$, and the set of nodes $\{t_{0,\beta_0,j}^{N_0}\}_{j=0}^{N_0} = \{t_{\beta_0,j}^{N_0}\}_{j=0}^{N_0}$. We use (3.2) to obtain the original numerical solution $u_0^N(t) = u^{N_0}(t)$, $0 \leq t < \infty$. In other words,

$$\begin{cases} \partial_t^2 u^{N_0}(t_{\beta_0,j}^{N_0}) = f(\partial_t u^{N_0}(t_{\beta_0,j}^{N_0}), u^{N_0}(t_{\beta_0,j}^{N_0}), t_{\beta_0,k}^{N_0}), & 0 \leq j \leq N_0, \\ \partial_t u^{N_0}(0) = V_0, \quad u^{N_0}(0) = U_0. \end{cases}$$

Then, we consider the auxiliary problem

$$\partial_t^2 U_1(t) = f(\partial_t U_1(t), U_1(t), t + t_{0,\beta_0,N_0}^{N_0}), \quad t > 0, \quad (4.1a)$$

$$\partial_t U_1(0) = \partial_t u_0^{N_0}(t_{0,\beta_0,N_0}^{N_0}), \quad U_1(0) = u_0^{N_0}(t_{0,\beta_0,N_0}^{N_0}). \quad (4.1b)$$

By using the numerical method (3.2) with the parameter β_1 and N_1 interpolation nodes $\{t_{1,\beta_1,j}^{N_1}\}_{j=0}^{N_1}$, coupled with a shifting transformation, we get the refined numerical solution $u_1^{N_1}(t)$ for $t_{0,\beta_0,N_0}^{N_0} < t < \infty$. Repeating the above procedure, we obtain the refined numerical solution $u_m^{N_m}(t)$ for $t_{m-1,\beta_{m-1},N_{m-1}}^{N_{m-1}} < t < \infty$. This algorithm saves work and provides more accurate numerical results oftentimes.

For understanding the above process more clearly, we let $t_0 = 0$, $t_m = t_{m-1,\beta_{m-1},N_{m-1}}^{N_{m-1}}$ for $m \geq 1$, and $U_m(t) = U(t + t_m)$. We consider the problem

$$\partial_t^2 U_m(t) = f(\partial_t U_m(t), U_m(t), t + t_m), \quad t > 0, \quad m \geq 0, \quad (4.2a)$$

$$\partial_t U_0(0) = V_0, \quad U_0(0) = U_0,$$

$$\partial_t U_m(0) = \partial_t U_{m-1}(t_m), \quad U_m(0) = U_{m-1}(t_m), \quad m \geq 1. \quad (4.2b)$$

Our method is to find a set of local solutions $u_m^{N_m}(t)$ such that

$$\partial_t^2 u_m^{N_m}(t_{m,\beta_m,k}^{N_m}) = f(\partial_t u_m^{N_m}(t_{m,\beta_m,k}^{N_m}), u_m^{N_m}(t_{m,\beta_m,k}^{N_m}), t_{m,\beta_m,k}^{N_m} + t_m), \quad 0 \leq k \leq N_m, \quad m \geq 0, \quad (4.3a)$$

$$\partial_t u_0^{N_0}(0) = V_0, \quad u_0^{N_0}(0) = U_0, \quad (4.3b)$$

$$\partial_t u_m^{N_m}(0) = \partial_t u_{m-1}^{N_{m-1}}(t_m), \quad u_m^{N_m}(0) = u_{m-1}^{N_{m-1}}(t_m), \quad m \geq 1. \quad (4.3c)$$

Obviously, the local numerical solution $u_m^{N_m}(t)$ is a proper approximation to the local exact solution $U_m(t)$ with the approximated initial values $\partial_t u_{m-1}^{N_{m-1}}(t_m)$ and $u_{m-1}^{N_{m-1}}(t_m)$. Finally, the numerical solution of original problem (3.1) is given by

$$u^N(t) = u_m^{N_m}(t - t_m), \quad t_m \leq t < t_{m+1}, \quad m = 0, 1, 2, \dots \quad (4.4)$$

Remark 4.1. Because of Gibbs phenomena, the numerical errors at the points t_m might be bigger than those at other interpolation nodes. To remedy this trouble, we may take $t_m = t_{m-1,\beta_{m-1},N_{m-1}-k_{m-1}}^{N_{m-1}}$, $k_{m-1} = 0, 1$ or 2 , in actual computation.

5. Numerical results

In this section, we present some numerical results. We will display the global absolute error $E_{\beta,G\alpha}^N = \|U - u^N\|_{\beta,N} \simeq \|U - u^N\|_{L^2(0,\infty)}$, and the point-wise absolute error $E_{\beta,p\alpha}^N(t) = |U(t) - u^N(t)|$.

Example 5.1. We use scheme (3.2) (labeled by LAGFC) to solve (3.1) with the nonlinear

term

$$\begin{aligned}
 f(\partial_t U(t), U(t), t) = & -45\partial_t U(t) - \frac{1}{2}\cos U(t) + e^{\frac{1}{5}\sin U(t)} \\
 & + \left(-\frac{2}{3}\cos t + \frac{5}{9} - \frac{8}{9}\sin t\right)e^{-\frac{t}{3}} + 45\left(\cos t - \frac{1}{3}(5 + \sin t)e^{-\frac{t}{3}}\right) \\
 & + \frac{1}{2}\cos(5 + \sin t)e^{-\frac{t}{3}} - e^{\frac{1}{5}\sin((5 + \sin t)e^{-\frac{t}{3}})}. \tag{5.1}
 \end{aligned}$$

The test function used is

$$U(t) = (5 + \sin t)e^{-\frac{t}{3}}, \tag{5.2}$$

which oscillates and decays exponentially as t increases.

In Fig. 1, we plot the values of $\log_{10} E_{\beta, Ga}^N$ vs. the mode N . The parameter β is chosen as 0.5, 1.5, and 2.5. Clearly, the numerical errors decay exponentially as N increases. In Fig. 2, we plot the point-wise absolute error $E_{\beta, Pa}^N(t)$ with $N = 100$ and $\beta = 1.5, 2$, respectively. It shows the small point-wise absolute errors.

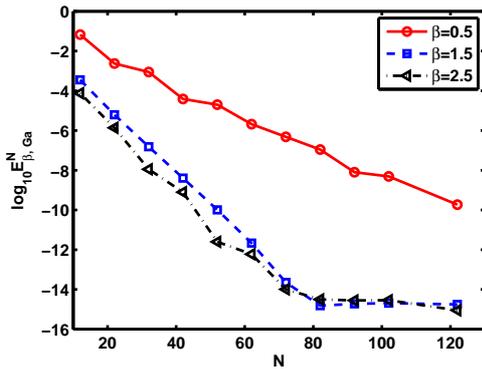


Figure 1: Global errors of LAGFC method against N , and 3 different values of β .

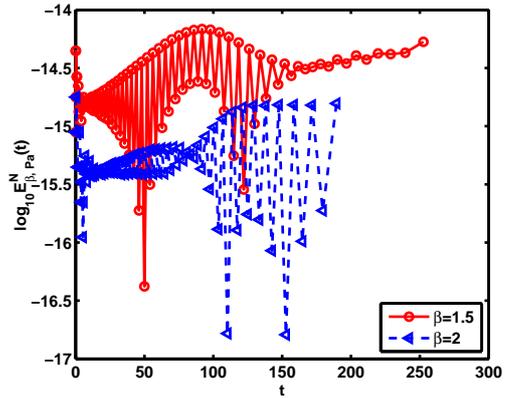


Figure 2: Point-wise errors of LAGFC method with $N = 100$, $\beta = 1.5$ and 2 .

Example 5.2. We next use scheme LAGFC method to solve (3.1) with the nonlinear term

$$\begin{aligned}
 & f(\partial_t U(t), U(t), t) \\
 = & -\partial_t U(t) - U^2(t) + \frac{\lambda(t+1)\cos \lambda t - m \sin \lambda t}{(t+1)^{m+1}} + \frac{\sin^2 \lambda t}{(t+1)^{2m}} \\
 & - \frac{\lambda^2(t+1)^2 \sin \lambda t + 2m\lambda(t+1)\cos \lambda t - m(m+1)\sin \lambda t}{(t+1)^{m+2}}, \quad m > \frac{1}{2}. \tag{5.3}
 \end{aligned}$$

The test function used is

$$U(t) = \frac{\sin \lambda t}{(t+1)^m}, \tag{5.4}$$

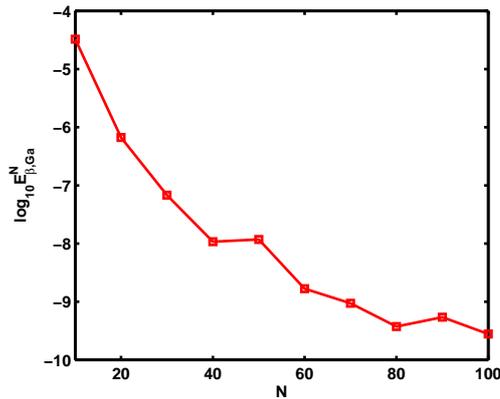


Figure 3: Global errors of LAGFC method against N , with $\beta = 2.5$.

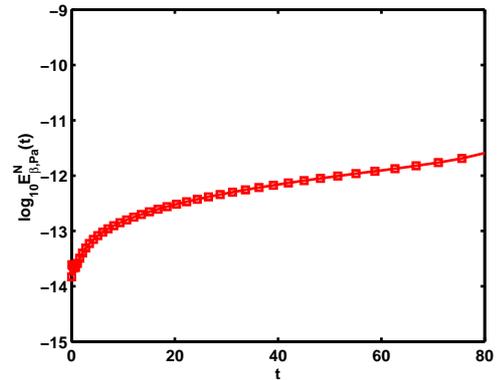


Figure 4: Point-wise errors of LAGFC method against t , with $N = 100$ and $\beta = 2.5$.

which oscillates and decays algebraically as t increases. In actual computation, we take $\lambda = 1$ and $m = 2$.

In Fig. 3, we plot the values of $\log_{10} E_{\beta, Ga}^N$ vs. the mode N . The parameter $\beta = 2.5$. As we expected, the numerical errors also decay fast as N increases. In Fig. 4, we plot the point-wise absolute errors $E_{\beta, Pa}^N(t)$ with $N = 100$ and $\beta = 2.5$. It shows the small point-wise absolute errors.

For comparison, we also use the method using the modified Laguerre polynomials given in [29] (labeled by LAGPC method), with the same test functions (4). In Fig. 5, we plot the point-wise absolute errors $E_{\beta, Pa}^N(t)$ of LAGFC and LAGPC methods, with $N = 20$ and $\beta = 2.5$. It indicates that for solution decaying as t increases, LAGFC method provides more accurate numerical approximations.

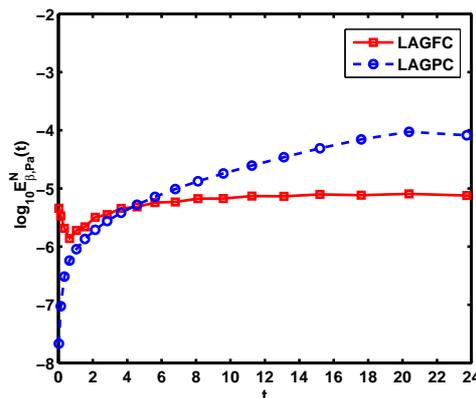


Figure 5: Comparison of the point-wise errors of LAGFC and LAGPC methods, with $N = 20$ and $\beta = 2.5$.

6. Concluding remarks

In this paper, we proposed a new collocation method and its multi-step version for solving initial value problems of second-order ordinary differential equations, by taking the modified Laguerre functions as the basis functions. This approach is applicable for solutions decaying to zero as t increases, and simplifies computation.

The numerical results demonstrated the high efficiency of suggested method. Indeed, unlike the Runge-Kutta-Nyström method, we used the modified Laguerre interpolation $\tilde{\mathcal{I}}_{\beta,N}$, which is stable even for large N . Moreover, the largest node of the interpolation $\tilde{\mathcal{I}}_{\beta,N}$ is about $\frac{4}{\beta}N$. Thus we could use larger time step size τ to save computational time.

Although we only considered a model problem in this paper, the main techniques developed in this work are also useful for other related problems, such as space-time spectral approximations to various nonlinear evolutionary partial differential equations.

How to estimate the errors of numerical solutions of the suggested method, is an interesting and still open problem so far.

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References

- [1] J. C. BUTCHER, *The Numerical Analysis of Ordinary Differential Equations, Runge-Kutta and General Linear Methods*, Wiley, New York, 1987.
- [2] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI, AND T. A. ZANG, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, Berlin, 1988.
- [3] J. M. FRANCO, *Runge-Kutta-Nyström method adapted to the numerical integration of perturbed oscillators*, *Commp. Phys. Comm.*, 147 (2002), pp. 770–787.
- [4] D. FUNARO, *Polynomial Approximations of Differential Equations*, Springer-Verlag, 1992.
- [5] A. GUILLOU AND J. L. SOULÉ, *La résolution numérique des problèmes différentiels aux conditions initiales par des méthodes de collocation*, *R.I.R.O.*, 3 (1969), pp. 17–44.
- [6] B.-Y. GUO, *Spectral Methods and Their Applications*, World Scietific, Singapore, 1998.
- [7] B.-Y. GUO AND J. SHEN, *Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval*, *Numer. Math.*, 86 (2000), pp. 635–654.
- [8] B.-Y. GUO, L.-L. WANG AND Z.-Q. WANG, *Generalized Laguerre interpolation and pseudospectral method for unbounded domains*, *SIAM J. Numer. Anal.*, 43 (2006), pp. 2567–2589.
- [9] B.-Y. GUO AND Z.-Q. WANG, *Numerical intergration based on Laguerre-Gauss interpolation*, *Comp. Meth. in Appl. and Engi.*, 196 (2007), pp. 3726–3741.
- [10] B.-Y. GUO AND Z.-Q. WANG, *Legendre-Gauss collocation methods for ordinary differential equations*, *Adv. in Comp. Math.*, 30 (2009), pp. 249–280.

- [11] B.-Y. GUO AND Z.-Q. WANG, *A spectral collocation method for solving initial value problems of first order ordinary differential equations*, Disc Cont. Dyna. Syst. B., 14 (2010), pp. 1029–1054.
- [12] B.-Y. GUO, Z.-Q. WANG, H.-J. TIAN AND L.-L. WANG, *Integration processes of ordinary differential equations based on Laguerre-Gauss interpolations*, Math. Comp., 77 (2008), pp. 181–199.
- [13] B.-Y. GUO AND J.-P. YAN, *Legendre-Gauss collocation methods for initial value problems of second order ordinary differential equations*, Appl. Numer. Math., 59 (2009), pp. 1386–1408.
- [14] B.-Y. GUO AND X.-Y. ZHANG, *A new generalized Laguerre approximation and its applications*, J. Comp. Appl. Math., 181 (2005), pp. 342–363.
- [15] B.-Y. GUO AND X.-Y. ZHANG, *Spectral method for differential equations of degenerate type on unbounded domains by using generalized Laguerre functions*, Appl. Numer. Math., 57 (2007), pp. 455–471.
- [16] E. HAIRER, *Unconditionally stable methods for second order differential equations*, Numer. Math., 32 (1979), pp. 373–379.
- [17] E. HAIRER, S. P. NORSETT AND G. WANNER, *Solving Ordinary Differential Equation I: Nonstiff Problems*, Springer-Verlag, Berlin, 1987.
- [18] E. HAIRER AND G. WANNER, *Solving Ordinary Differential Equation II: Stiff and Differential-Algebraic Problems*, Springer-Verlag, Berlin, 1991.
- [19] P. J. VAN DER HOUWEN AND B. P. SOMMEIJER, *Diagonally implicit Runge-Kutta(-Nyström) methods with reduced phase-errors for computing oscillating solutions*, SIAM J. Numer. Anal., 26 (1989), pp. 414–429.
- [20] P. J. VAN DEN HOUWEN, B. P. SOMMEIJER AND NGUYEN HUU CONG, *Stability of collocation-based Runge-Kutta-Nyström methods*, BIT, 31 (1991), pp. 469–481.
- [21] L. KRAMARZ, *Stability of collocation methods for the numerical solution of $y'' = f(x, y)$* , BIT, 20 (1980), pp. 215–222.
- [22] J. D. LAMBERT, *Numerical Methods for Ordinary Differential Systems, The Initial Value Problem*, John Wiley and Sons, Chichester, 1991.
- [23] I. LIE AND S. P. NORSETT, *Superconvergence for multistep collocation*, Math. Comp., 52 (1989), pp. 65–79.
- [24] Y. MADAY, B. PERNAUD-THOMAS AND H. VANDEVEN, *Reappraisal of Laguerre type spectral methods*, La Recherche Aérospatiale, 6 (1985), pp. 13–35.
- [25] G. MASTROIANNI AND G. MONEGATO, *Nyström interpolants based on zeros of Laguerre polynomials for some Weiner-Hopf equations*, IMA J. of Numer. Anal., 17 (1997), pp. 621–642.
- [26] J. SHEN, *Stable and efficient spectral methods in unbounded domains using Laguerre functions*, SIAM J. Numer. Anal., 38 (2000), pp. 1113–1133.
- [27] H. A. WATTS AND L. F. SHAMPINE, *A-stable block implicit one-step methods*, BIT, 12 (1972), pp. 252–266.
- [28] C.-L. XU AND B.-Y. GUO, *Laguerre pseudospectral method for nonlinear partial differential equations*, J. Comput. Math., 20 (2002), pp. 413–428.
- [29] J.-P. YAN AND B.-Y. GUO, *Laguerre-Gauss collocation method for initial value problems Of second order ODEs*, submitted.