

## Dynamic Evasion-Interrogation Games with Uncertainty in the Context of Electromagnetics

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**Abstract.** We consider two player electromagnetic evasion-pursuit games where each player must incorporate significant uncertainty into their design strategies to disguise their intention and confuse their opponent. In this paper, the evader is allowed to make dynamic changes to his strategies in response to the dynamic input with uncertainty from the interrogator. The problem is formulated in two different ways; one is based on the evolution of the probability density function of the intensity of reflected signal and leads to a controlled forward Kolmogorov or Fokker-Planck equation. The other formulation is based on the evolution of expected value of the intensity of reflected signal and leads to controlled backward Kolmogorov equations. In addition, a number of numerical results are presented to illustrate the usefulness of the proposed approach in exploring problems of control in a general dynamic game setting.

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### 1. Introduction

In an electromagnetic evasion-interrogation game, the evader wishes to minimize the intensity of the reflected signal to remain undetected in carrying out his mission while the interrogator wishes to maximize the intensity of reflected signal to detect the attacker. It was demonstrated in [9] that it is possible to design ferroelectric materials with appropriate dielectric permittivity and magnetic permeability to significantly attenuate reflections of electromagnetic interrogation signals from highly conductive targets such as airfoils and missiles. These results were further sharpened and illustrated in [10] where a series of different material designs were considered to minimize over a given set of input design frequencies the maximum reflected field from input signals. In addition, it was shown

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that if the evader employed a simple counter interrogation design based on a fixed set (assumed known) of interrogating frequencies, then by a rather simple counter-counter interrogation strategy (use of an interrogating frequency little more than 10% different from the assumed design frequencies), the interrogator can easily defeat the evader's material coatings counter interrogation strategy to obtain strong reflected signals. From the combined results of [9, 10] it is thus rather easily concluded that the evader and the interrogator must each try to confuse the other by introducing significant *uncertainty* in their design and interrogating strategies, respectively.

Static two-player non-cooperative games with uncertainty were considered in [6]. In these problems, the evader and the interrogator are each subject to uncertainties as to the actions of the other. The evader wants to choose a best coating design (i.e., best dielectric permittivities and magnetic permeabilities) while the interrogator wants to choose best angles of interrogation and interrogating frequencies for input signals. Each player must act in the presence of incomplete information about the other's action. Partial information regarding capabilities and tendencies of the adversary can be embodied in probability distributions for the choices to be made. That is, one may formalize this by assuming the evader may choose (with an as yet to be determined set of probabilities) dielectric permittivity and magnetic permeability parameters from given admissible sets while the interrogator chooses angles of interrogation and interrogating frequencies from appropriate admissible sets. The formulation in [6] is based on the *mixed strategies* proposals of von Neumann [2, 30, 31] and the ideas can be summarized as follows. The evader does not choose a single coating, but rather has a set of possibilities available for choice. He only chooses the probabilities with which he will employ the materials on a target. This, in effect, disguises his intentions from his adversary. By choosing his coatings randomly (according to a best strategy to be determined in, for example, a minmax game), he prevents adversaries from discovering which coating he will use—indeed, even he does not know which coating will be chosen for a given target. The interrogator, in a similar approach, determines best probabilities for choices of frequency and angle in the interrogating signals. Note that such a formulation tacitly assumes that the adversarial relationship persists with multiple attempts at evasion and detection.

The problems are mathematically formulated in [6] as two sided optimization problems over spaces of probability measures, i.e., minmax games over sets of probability measures. That work demonstrates the feasibility and the potential usefulness of developing theories for problems with uncertainty. In this paper, we move toward a more realistic dynamic modeling by introducing time dynamics into the problem for single evasion attempts. Specifically, we allow a single evader to make dynamic changes to his dielectric permittivity strategies in response to feedback entailing measures of the reflection signals based on dynamic information with uncertainty about the interrogator's choices. Thus, this new formulation is more in the spirit of the deterministic dynamical differential games as formulated, for example, in [20] except here uncertainties of the two players' actions are a major feature as in the static games of [6, 31]. The remainder of this paper is organized as follows. We begin in Section 2 by presenting a description of our problem formulation. We then outline a theoretical and computational framework in Section 3 that provides a

foundation for our solution methods. A number of computational results are presented in Section 4. We conclude the paper by some summary remarks and proposed future research efforts in Section 5.

## 2. Problem description

In this section, we will use a capital italic letter to denote a random variable unless otherwise indicated, and use the corresponding small letter to denote its realization.

We formulate a minimization problem with cost functional in terms of some reflection coefficient dependent on the evader's dielectric permittivity  $\epsilon$  as well as the interrogator frequency  $\omega$ . This reflection coefficient could be based on a simple planar geometry using Fresnel's formula for a perfectly conducting half plane which has a coating layer of thickness  $d$  with dielectric permittivity  $\epsilon$  and interrogating frequency  $\omega$  as detailed in [6]. This expression can be derived directly from Maxwell's equation by considering the ratio of reflected to incident wave for example in the case of parallel polarized ( $TE_x$ ) incident wave (see [9, 25]).

An alternative and much more computationally intensive approach, which may be necessitated by some target geometries (e.g., missiles), employs the far field pattern for reflected waves computed directly using Maxwell's equations. In two dimensions, for a reflecting body with a given coating layer with an interrogating plane wave  $E^{(i)}$ , the scattered field  $E^{(s)}$  satisfies the Helmholtz equation [18] as detailed in [6].

Throughout we assume for simplicity that the magnetic permeability for the evader is fixed as is the angle of incidence of the interrogating signal. We assume that the evader has the ability to choose the dielectric parameter  $\epsilon$  he uses in order to thwart detection, and the parameter  $\epsilon$  is changed adaptively depending on the frequency  $w$  that the interrogator is using (or rather depending on the reflections produced by the interrogator's frequency choices). In addition, we assume that the interrogator frequency process  $\{W_t : t \geq 0\}$  is an Itô diffusion process (Chapter 7 of [29]) satisfying the stochastic differential equation

$$dW_t = \mu(W_t)dt + \sigma(W_t)dB_t, \quad (2.1)$$

where both  $\mu$  and  $\sigma$  are non-random functions that are Lipschitz continuous, and  $B_t$  denotes the standard Brownian motion.

Below we will consider two different formulations, one is based on the evolution of the probability density function of the intensity of reflected signal, and the other is based on the evolution of expected value of the intensity of reflected signal.

### 2.1. Evolution of probability density function of intensity

Let  $Y_t = \varphi(W_t)$ , where  $\varphi$  is some chosen measure of intensity of the reflected signal for a given material dielectric parameter (for example,  $\varphi$  can be chosen as the magnitude of the reflection coefficient). In addition, we assume that  $\varphi$  is twice continuously

differentiable. Then by Itô's formula we find that

$$dY_t = \left[ \varphi'(W_t)\mu(W_t) + \frac{1}{2}\varphi''(W_t)\sigma^2(W_t) \right] dt + \varphi'(W_t)\sigma(W_t)dB_t.$$

If we further assume that  $\varphi^{-1}$  exists, then we can rewrite the right-hand side of the above equation in terms of  $Y_t$  given by

$$dY_t = \varphi'(\varphi^{-1}(Y_t))\sigma(\varphi^{-1}(Y_t))dB_t + \left[ \varphi'(\varphi^{-1}(Y_t))\mu(\varphi^{-1}(Y_t)) + \frac{1}{2}\varphi''(\varphi^{-1}(Y_t))\sigma^2(\varphi^{-1}(Y_t)) \right] dt.$$

If  $\varphi$  is chosen such that functions  $\varphi'(\varphi^{-1}(y))\mu(\varphi^{-1}(y)) + \frac{1}{2}\varphi''(\varphi^{-1}(y))\sigma^2(\varphi^{-1}(y))$  and  $\varphi'(\varphi^{-1}(y))\sigma(\varphi^{-1}(y))$  are both Lipschitz continuous, then  $\{Y_t : t \geq 0\}$  is also an Itô diffusion process. Let  $\rho(t, y)$  denote the probability density function of the random variable  $Y_t$ . Then it is well known that  $\rho$  satisfies Fokker-Planck equation (e.g., see [23, p. 118])

$$\begin{aligned} & \frac{\partial \rho(t, y)}{\partial t} + \frac{\partial}{\partial y} \left[ \left( \varphi'(\varphi^{-1}(y))\mu(\varphi^{-1}(y)) + \frac{1}{2}\varphi''(\varphi^{-1}(y))\sigma^2(\varphi^{-1}(y)) \right) \rho(t, y) \right] \\ &= \frac{1}{2} \frac{\partial^2}{\partial y^2} \left[ (\varphi'(\varphi^{-1}(y))\sigma(\varphi^{-1}(y)))^2 \rho(t, y) \right]. \end{aligned} \tag{2.2}$$

For our illustration here, we choose  $\varphi$  to be a first-order approximation of the chosen measure of intensity of the reflected signal, that is,  $\varphi(w) = c_1w + c_0$  (i.e.,  $y = c_1w + c_0$ ), where  $c_0$  and  $c_1$  are constants. Then (2.2) can be simplified as

$$\frac{\partial \rho(t, y)}{\partial t} + c_1 \frac{\partial}{\partial y} [\mu(\varphi^{-1}(y))\rho(t, y)] = \frac{1}{2}c_1^2 \frac{\partial^2}{\partial y^2} [\sigma^2(\varphi^{-1}(y))\rho(t, y)]. \tag{2.3}$$

Let  $\varrho(t, w) = \rho(t, y)$ . Then we have

$$\frac{\partial \varrho}{\partial t} = \frac{\partial \rho}{\partial t}, \quad \frac{\partial}{\partial y} = \frac{1}{c_1} \frac{\partial}{\partial w}, \quad \frac{\partial^2}{\partial y^2} = \frac{1}{c_1^2} \frac{\partial^2}{\partial w^2}.$$

Hence, we can rewrite (2.3) in terms of  $w$  as follows

$$\frac{\partial \varrho(t, w)}{\partial t} + \frac{\partial}{\partial w} [\mu(w)\varrho(t, w)] = \frac{1}{2} \frac{\partial^2}{\partial w^2} [\sigma^2(w)\varrho(t, w)]. \tag{2.4}$$

To allow for evader control of the system (2.4), we introduce some input of the form

$$-\lambda_r(\varrho(t, w) - u(t, \epsilon))$$

into the system (2.4), that is,

$$\frac{\partial \varrho(t, w)}{\partial t} + \frac{\partial}{\partial w} [\mu(w)\varrho(t, w)] = \frac{1}{2} \frac{\partial^2}{\partial w^2} [\sigma^2(w)\varrho(t, w)] - \lambda_r(\varrho(t, w) - u(t, \epsilon)), \tag{2.5}$$

where  $\lambda_r$  is the relaxation constant of the material. We note from (2.5) that once we introduce feedback controls into this system,  $\rho$  is no longer a probability density function (indeed we are trying to drive it to zero on most of its support).

Here we consider a generalized control  $u$ , which is defined by

$$u(t, w) = \int_{\mathcal{E}} r(w, \epsilon) d\mathcal{U}(t, \epsilon), \quad (2.6)$$

where  $r(w, \epsilon)$  is some given real-valued function of the reflection coefficient with given frequency  $w$  and dielectric parameter  $\epsilon$ , and  $\mathcal{U}$  is a time-dependent distribution of possible dielectric settings  $\epsilon$  in  $\mathcal{E}$ . The motivation for introducing distributional or generalized controls is two fold. First, this is natural when one is extending the static theory of [6] where the uncertainty in controls is embodied in probability measures on the static control parameters such as dielectric permittivities and interrogating frequencies. A second compelling motivation is prompted by a rich literature on closure theorems in the calculus of variations and optimal control associated with distinguished contributors such as Young [37, 38], McShane [26–28], Filippov [21], and Warga [33–35], among others. In some variational and control problems (and especially in two player differential games—see for example the discussions in [20] and the counter example of Berkovitz [16]), it has been known since the years of L. C. Young that one must often introduce generalized or relaxed controls (also called sliding regimes [21] or chattering controls) in order to obtain well posed optimization problems. In anticipation of treating these two player dynamical games where both the evader and interrogator have time dependent controllers, here we use generalized controls for the evader which thus introduces uncertainty in the evader controls as well as uncertainty in the interrogation frequencies via the stochastic dynamics (2.1).

We remark that the use of generalized controls has arisen naturally in a number of other modern applications including in smart materials with smoothed Preisach controls [12–14] where hysteretic control influence operators representing smart material actuators can be used to guarantee well posedness as well as to develop efficient computational algorithms.

One of the main benefits of relaxed controls is that the optimal relaxed controller can be approximated by “real controls” and we shall do that here. Indeed, for computational purposes, we will approximate the control  $u(t, w)$  with delta approximations (rigorous justification for such approximations in the context of the Prohorov [32] or weak\* (metric) topology on spaces of probability distributions [4, 17, 24] can be found in [3–5] as well as in the closure theories from [26–28, 33–35, 37, 38]. We restrict the set  $\mathcal{E}$  to a finite set  $\{\epsilon_j^*\}_{j=1}^M$ , thus obtaining the collection of choices of materials available to the evader

$$\begin{aligned} \mathcal{U}(t, \epsilon) &\approx \sum_{j=1}^M \epsilon_j(t) \Delta_{\epsilon_j^*}(\epsilon), \\ d\mathcal{U}(t, \epsilon) &\approx \sum_{j=1}^M \epsilon_j(t) \delta_{\epsilon_j^*}(\epsilon) d\epsilon, \end{aligned}$$

where  $\epsilon_j(t)$  denotes the time-dependent weighings for the material with dielectric permittivity  $\epsilon_j^*$  that the evader may choose, and  $\Delta$  and  $\delta$  are the Dirac distribution and density, respectively. Let  $b_j(w) = r(w, \epsilon_j^*)$ ,  $j = 1, 2, \dots, M$ . Then (2.6) can be rewritten as

$$u(t, w) \approx \int_{\mathcal{E}} r(w, \epsilon) \sum_{j=1}^M \epsilon_j(t) \delta_{\epsilon_j^*}(\epsilon) d\epsilon = \sum_{j=1}^M \epsilon_j(t) r(w, \epsilon_j^*) = b(w) \bar{\epsilon}(t), \tag{2.7}$$

where  $b(w) = (b_1(w), b_2(w), \dots, b_M(w))$  and  $\bar{\epsilon}(t) = (\epsilon_1(t), \epsilon_2(t), \dots, \epsilon_M(t))^T$ . Thus, using the control (2.7) we can write (2.5) as

$$\frac{\partial \varrho(t, w)}{\partial t} + \frac{\partial}{\partial w} [\mu(w) \varrho(t, w)] = \frac{1}{2} \frac{\partial^2}{\partial w^2} [\sigma^2(w) \varrho(t, w)] - \lambda_r (\varrho(t, w) - b(w) \bar{\epsilon}(t)), \tag{2.8}$$

which is a controlled forward Kolmogorov or Fokker-Planck equation [1, 22]. A reasonable linear quadratic regulator (LQR) control problem might involve minimizing the cost functional

$$J(\bar{\epsilon}) = \int_0^\infty \int_{\underline{w}}^{\bar{w}} |\varrho(t, w)|^2 dw dt + \int_0^\infty \beta |\bar{\epsilon}(t)|^2 dt, \tag{2.9}$$

subject to (2.8). Here  $[\underline{w}, \bar{w}]$  is the admissible range of interrogator frequencies.

### 2.2. Evolution of expected value of intensity

We next discuss an alternative formulation for our problem. Assuming that  $W_t$  satisfies (2.1) and for a given the material dielectric parameter value  $\epsilon_t$  at time  $t$ , we define

$$\tilde{v}(t, w) = \mathbb{E} \left[ \int_0^t \lambda e^{\lambda s} \tilde{r}(W_s, \epsilon_s) ds + v_0(W_t) \mid W_0 = w \right],$$

where  $\mathbb{E}[\cdot \mid \cdot]$  denotes the conditional expectation,  $\tilde{r}(w, \epsilon)$  again represents some scalar valued controlled intensity for the reflected signal (e.g., the magnitude of the reflection coefficient) depending on incoming frequency  $w$  and dielectric parameter  $\epsilon$ , and  $\lambda > 0$  is now a discount parameter. Following a standard technique [29, Section 10.3] for treating integrals, we next define

$$Z_t = \int_0^t \lambda e^{\lambda s} \tilde{r}(W_s, \epsilon_s) ds.$$

Then the process  $X_t = (W_t, Z_t)^T$  satisfies

$$d \begin{pmatrix} W_t \\ Z_t \end{pmatrix} = \begin{pmatrix} \mu(W_t) \\ \lambda e^{\lambda t} \tilde{r}(W_t, \epsilon_t) \end{pmatrix} dt + \begin{pmatrix} \sigma(W_t) \\ 0 \end{pmatrix} dB_t$$

and

$$\tilde{v}(t, w) = g(t, (w, 0)), \quad \text{for } g(t, (w, z)) \equiv \mathbb{E}[Z_t + v_0(W_t) \mid X_0 = (w, z)^T].$$

Here the generator of the Itô diffusion process  $\{X_t : t \geq 0\}$  is

$$\mathcal{L}\phi(w, z) = \mu(w) \frac{\partial}{\partial w} \phi(w, z) + \frac{1}{2} \sigma^2(w) \frac{\partial^2}{\partial w^2} \phi(w, z) + \lambda e^{\lambda t} \tilde{r}(w, \epsilon_t) \frac{\partial}{\partial z} \phi(w, z).$$

It then follows from Section 8.1 in [29] that  $g$  satisfies the *backward Kolmogorov* equation

$$\frac{\partial}{\partial t} g = \mathcal{L}g, \quad g(0, (w, z)) = z + v_0(w). \tag{2.10}$$

A discussion of the relationship between this state and the semigroup generated by  $\mathcal{L}$  can be found in [19]. Since  $g = \tilde{v} + z$  is the solution to (2.10), it follows that  $\tilde{v}$  satisfies

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{v}(t, w) &= \mu(w) \frac{\partial}{\partial w} \tilde{v}(t, w) + \frac{1}{2} \sigma^2(w) \frac{\partial^2}{\partial w^2} \tilde{v}(t, w) + \lambda e^{\lambda t} \tilde{r}(w, \epsilon_t), \\ \tilde{v}(0, w) &= v_0(w). \end{aligned}$$

Now let  $v(t, w) = e^{-\lambda t} \tilde{v}(t, w)$ . It is easy to show that  $v$  satisfies

$$\begin{aligned} \frac{\partial}{\partial t} v(t, w) &= \mu(w) \frac{\partial}{\partial w} v(t, w) + \frac{1}{2} \sigma^2(w) \frac{\partial^2}{\partial w^2} v(t, w) + \lambda (\tilde{r}(w, \epsilon_t) - v(t, w)), \\ v(0, w) &= v_0(w). \end{aligned} \tag{2.11}$$

We note that the state  $v$  in this formulation is

$$v(t, w) = \mathbb{E} \left[ \int_0^t \lambda e^{-\lambda(t-s)} \tilde{r}(W_s, \epsilon_s) ds + e^{-\lambda t} v_0(W_t) \mid W_0 = w \right],$$

the *expected value* of a measure of the reflected intensity.

In this formulation, the controlled reflection index  $\tilde{r}(w, \epsilon)$  can be extended to generalized controls as in (2.6), (2.7) where  $r = \tilde{r}$ . Thus we can rewrite Eq. (2.11) using the generalized control from Eq. (2.7) as follows

$$\frac{\partial}{\partial t} v(t, w) = \mu(w) \frac{\partial}{\partial w} v(t, w) + \frac{1}{2} \sigma^2(w) \frac{\partial^2}{\partial w^2} v(t, w) - \lambda (v(t, w) - b(w) \bar{\epsilon}(t)). \tag{2.12}$$

Note that the control for this formulation is the same as the control for the formulation in Section 2.1 although this is a controlled backward Kolmogorov equation [1, 22]. The primary difference is that for this latter formulation the control was naturally a part of the dynamics equation and we did not have to artificially introduce a control into the system as we did in Section 2.1. Thus, this second formulation is somewhat more direct and hence perhaps more desirable from an intuitive perspective. Moreover, the backward Kolmogorov equation of this formulation usually presents less formidable computational challenges.

### 2.3. Special case

For our presentation of theoretical, approximation and computational results for the above two control problems, we consider without loss of generality a special case where the above problems are the same. Specifically, from (2.8) and (2.12), we see that if we choose  $\mu \equiv 0$  and  $\sigma$  to be a positive constant function, then (2.8) and (2.12) are reduced to the same controlled system, a controlled diffusion equation. The choice of  $\mu = \text{constant} > 0$  is a more physically relevant case (resulting in a convection-diffusion equation with either positive or negative convective flow depending on which dynamics are chosen from above), but all that we present below applies to these cases albeit with more technical detail in the theoretical and approximation frameworks. In particular, the inclusion of the convective term can greatly complicate the computational problem in the forward Kolmogorov formulation. Thus, for our demonstration purposes here and for the sake of simplicity, we will consider the reduced controlled system given by

$$\frac{\partial v}{\partial t} = \eta \frac{\partial^2 v}{\partial w^2} - \lambda(v - b(w)\bar{\epsilon}(t)), \quad (2.13)$$

where  $\eta = \sigma^2/2 > 0$ . We further suppose that the frequencies that the interrogator is capable of transmitting are in the range of  $[\underline{w}, \bar{w}]$ , i.e., the support of the interrogator probability density is finite. Then the boundary and initial conditions are given respectively by

$$\begin{aligned} v(t, \underline{w}) &= v(t, \bar{w}) = 0, \\ v(0, w) &= v_0(w). \end{aligned} \quad (2.14)$$

In the context of the LQR control problem we thus must minimize the cost function

$$J(\bar{\epsilon}) = \int_0^\infty \int_{\underline{w}}^{\bar{w}} |v(t, w)|^2 dw dt + \int_0^\infty \beta |\bar{\epsilon}(t)|^2 dt, \quad (2.15)$$

subject to (2.13) and (2.14). We remark here that  $\beta$  is a design parameter which is chosen to balance the relative merits of reduction of reflection intensity versus control costs in the control objectives.

### 3. Sesquilinear forms: theory and numerical approximations

A fundamental framework for theory, approximation and computation for (2.15) subject to (2.13)–(2.14) is available in the context of an abstract control problem as developed in [7, 8] with a accessible summary given in [15]. For convenience and following standard conventions, we use an over dot ( $\dot{\cdot}$ ) to denote the derivative with respect to the time variable  $t$ . While the results presented below can readily be given for both general Kolmogorov formulations, for brevity we only consider the canonical case described in the previous section where  $\mu \equiv 0$ .

### 3.1. Sesquilinear forms

We first present theoretical underpinnings for our control calculations in a real Hilbert space setting. All of the results given here are summarized in more detail in [15]. Let the Hilbert spaces  $H$  and  $V$  be defined by  $H = L^2(\underline{w}, \overline{w})$  and  $V = H_0^1(\underline{w}, \overline{w})$ . We denote the topological dual space of  $V$  by  $V^* = H^{-1}(\underline{w}, \overline{w})$ . If we identify  $H$  with its dual  $H^*$  then  $V \hookrightarrow H = H^* \hookrightarrow V^*$  is a Gelfand triple [15, 36].

Define the linear operator  $\mathcal{A} : V \rightarrow V^*$  by

$$\mathcal{A}\phi = \eta \frac{\partial^2 \phi}{\partial w^2} - \lambda \phi, \quad \phi \in V,$$

where as usual derivatives are interpreted in the weak or distributional sense. We may now write (2.13) in the following abstract form

$$\dot{v}(t) = \mathcal{A}v(t) + \mathcal{B}\bar{\epsilon}(t), \quad v(0) = v_0, \tag{3.1}$$

where  $v(t)$  is used as the shorthand notation for the function  $v(t, \cdot)$  (this shorthand notation will be used throughout the remainder of this section), and the operator  $\mathcal{B} : \mathbb{R}^M \rightarrow V^*$  is defined by

$$\mathcal{B}\xi(w) = \lambda b(w)\xi, \quad \text{for all } \xi \in \mathbb{R}^M.$$

In the particular case studied here we actually have  $b \in L^2$  so that  $\mathcal{B} : \mathbb{R}^M \rightarrow L^2 \subset V^*$ . It is easy to argue that the adjoint  $\mathcal{B}^*$  of  $\mathcal{B}$  is given by

$$\mathcal{B}^*\phi = (\langle b_1, \phi \rangle, \langle b_2, \phi \rangle, \dots, \langle b_M, \phi \rangle)^T, \quad \text{for all } \phi \in V.$$

Using integration by parts, we obtain

$$\begin{aligned} \langle \mathcal{A}\phi, \psi \rangle_{V^*, V} &= \int_{\underline{w}}^{\overline{w}} \left( \eta \frac{\partial^2 \phi(w)}{\partial w^2} - \lambda \phi(w) \right) \psi(w) dw \\ &= - \int_{\underline{w}}^{\overline{w}} \eta \frac{\partial \phi(w)}{\partial w} \frac{\partial \psi(w)}{\partial w} dw - \int_{\underline{w}}^{\overline{w}} \lambda \phi(w) \psi(w) dw, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle_{V^*, V}$  denotes the usual duality product [15, 36]. We then define a sesquilinear form  $a$  on  $V \times V$  by  $a(\phi, \psi) = \langle -\mathcal{A}\phi, \psi \rangle_{V^*, V}$ , that is,

$$a(\phi, \psi) = \int_{\underline{w}}^{\overline{w}} \eta \frac{\partial \phi(w)}{\partial w} \frac{\partial \psi(w)}{\partial w} dw + \int_{\underline{w}}^{\overline{w}} \lambda \phi(w) \psi(w) dw. \tag{3.2}$$

We see immediately that  $a$  is symmetric, and hence the adjoint  $\mathcal{A}^*$  of  $\mathcal{A}$  defined by  $a(\phi, \psi) = \langle \phi, -\mathcal{A}^*\psi \rangle_{V, V^*}$  is equal to  $\mathcal{A}$ . Now we may rewrite (3.1) in weak form as:  $v(t) \in V$  for all  $t$  is the solution of

$$\langle \dot{v}(t), \psi \rangle_{V^*, V} + a(v(t), \psi) = \langle \mathcal{B}\bar{\epsilon}(t), \psi \rangle_{V^*, V}, \quad v(0) = v_0, \tag{3.3}$$

for all  $\psi \in V$ . By (3.2) and Poincaré’s inequality, we find that there exists some positive constant  $c$  such that

$$|a(\phi, \psi)| \leq (c\lambda + \eta)\|\phi\|_V\|\psi\|_V,$$

holds for any  $\phi, \psi \in V$ . Similarly for all  $\phi \in V$

$$a(\phi, \phi) = \eta\|\phi\|_V^2 + \lambda\|\phi\|_H^2 \geq \eta\|\phi\|_V^2.$$

One can then establish estimates and arguments as in [7, 8] to argue that  $\mathcal{A}$  generates an analytic semigroup on  $H, V$  and  $V^*$ . Furthermore, this semigroup is exponentially stable on  $H, V$  and  $V^*$ .

Turning next to the control problem for the abstract dynamics (3.1), we find that the above results along with Theorem 3.4 in [8] implies that the algebraic Riccati equation

$$(\mathcal{A}^*\Pi + \Pi\mathcal{A} - \Pi\mathcal{B}\beta^{-1}\mathcal{B}^*\Pi + \mathcal{I})\psi = 0, \quad \text{for all } \psi \in V \tag{3.4}$$

has a unique nonnegative solution  $\Pi \in \mathcal{L}(V^*, V)$  and

$$\mathcal{A} - \mathcal{B}\beta^{-1}\mathcal{B}^*\Pi$$

generates an exponentially stable semigroup on  $H, V$  and  $V^*$ . Moreover, the optimal feedback solution that minimizes cost functional (2.15) subject to (3.1) is given by

$$\bar{\epsilon}_{\text{opt}}(t) = -\frac{1}{\beta}\mathcal{B}^*\Pi v(t).$$

### 3.2. Numerical approximation

Our goal in this section is to present computational methods for solution of the feedback control systems under investigation here. We do this in the context of the abstract formulation developed in [7, 8, 15] and summarized above. We briefly outline a method based on a standard finite element approach. For the convenience, we use  $(\prime)$  for the derivative with respect to the space variable  $w$ .

We define the mesh points  $w_j^N, j = 0, \dots, N + 1$  as  $w_0^N = \underline{w}, w_j^N = w_{j-1}^N + h,$  for  $j = 1, \dots, N,$  and  $w_{N+1}^N = \bar{w}$  where  $h = (\bar{w} - \underline{w})/(N + 1)$ . Next we let  $V^N$  be a sequence of finite dimensional subspaces of  $V$ . In particular,  $V^N = \text{span}\{l_1^N(w), l_2^N(w), \dots, l_N^N(w)\} \subset V = H_0^1(\underline{w}, \bar{w})$  where the piecewise linear basis elements  $\{l_j^N(w)\}$  are defined as follows for  $j = 1, \dots, N$ :

$$l_j^N(w) = \begin{cases} \frac{w - w_{j-1}^N}{w_j^N - w_{j-1}^N}, & w_{j-1}^N \leq w < w_j^N, \\ \frac{w_{j+1}^N - w}{w_{j+1}^N - w_j^N}, & w_j^N \leq w < w_{j+1}^N, \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

We next define the operator  $\mathcal{A}^N : V^N \rightarrow V^N$  (which approximates  $\mathcal{A}$ ) by restriction of  $a$  to  $V^N \times V^N$ ; this yields

$$\langle -\mathcal{A}^N \phi, \psi \rangle = a(\phi, \psi), \quad \text{for all } \phi, \psi \in V^N.$$

For given  $\mathcal{B} : \mathbb{R}^M \rightarrow V^*$ , we define its approximation  $\mathcal{B}^N : \mathbb{R}^M \rightarrow V^N$  by

$$\langle \mathcal{B}^N \xi, \psi \rangle = \langle \xi, \mathcal{B}^* \psi \rangle, \quad \text{for all } \xi \in \mathbb{R}^M \text{ and } \psi \in V^N.$$

For this family of approximations, the corresponding  $N$ th approximate problem in  $V^N$  entails the minimization of the cost functional

$$J^N(\bar{\epsilon}) = \int_0^\infty \int_{\underline{w}}^{\bar{w}} |v^N(t, w)|^2 dw dt + \int_{-\infty}^0 \beta |\bar{\epsilon}(t)|^2 dt, \tag{3.6}$$

subject to

$$\frac{dv^N(t)}{dt} = \mathcal{A}^N v^N(t) + \mathcal{B}^N \bar{\epsilon}(t), \quad v^N(0) = \mathcal{P}^N v_0. \tag{3.7}$$

Here  $v^N(t)$  is the notation for  $v^N(t, \cdot)$ , and the operator  $\mathcal{P}^N$  denotes the usual orthogonal projection of  $H$  onto  $V^N$ . That is, for  $\phi \in H$ , we have  $\mathcal{P}^N \phi \in V^N$  is defined by

$$\langle \mathcal{P}^N \phi, \psi \rangle = \langle \phi, \psi \rangle, \quad \text{for all } \psi \in V^N. \tag{3.8}$$

The weak form of (3.7), i.e., the approximate problem corresponding to (3.3), can then be formulated as finding  $v^N(t) \in V^N$  which satisfies

$$\left\langle \frac{dv^N(t)}{dt}, \psi \right\rangle + a(v^N(t), \psi) = \langle \mathcal{B}^N \bar{\epsilon}(t), \psi \rangle, \quad \psi \in V^N, \\ v^N(0) = \mathcal{P}^N v_0. \tag{3.9}$$

It is well-known [11] that for any  $\phi \in V$ , there exist a sequence  $\phi^N \in V^N$  such that  $|\phi^N - \phi|_V \rightarrow 0$  as  $N \rightarrow \infty$ . Thus we can be assured that these approximations  $v^N(t)$  will approach  $v(t)$  for  $N$  sufficiently large. To obtain the matrix representations for the operators  $\mathcal{A}^N$  and  $\mathcal{B}^N$  in terms of the piecewise linear spline basis, we substitute

$$v(t) \approx v^N(t) = \sum_{j=1}^N v_j^N(t) l_j^N$$

into (3.9) and let  $\psi = l_i^N$  for  $i = 1, 2, \dots, N$ . We obtain the vector system

$$\sum_{j=1}^N \dot{v}_j^N(t) \langle l_j^N, l_i^N \rangle + \eta \sum_{j=1}^N v_j^N(t) \langle (l_j^N)', (l_i^N)' \rangle + \lambda \sum_{j=1}^N v_j^N(t) \langle l_j^N, l_i^N \rangle = \lambda \sum_{k=1}^M \epsilon_k(t) \langle b_k, l_i^N \rangle.$$

We note that the above equation can be written in the matrix form

$$F^N \dot{v}^N + \eta Q^N v^N + \lambda F^N v^N = \lambda G^N \bar{\epsilon}, \tag{3.10}$$

where  $v^N(t) = (v_1^N(t), v_2^N(t), \dots, v_N^N(t))^T$ ,  $F^N$  and  $Q^N$  are  $N \times N$  matrices with their  $(i, j)$ th elements defined by

$$\langle l_i^N, l_j^N \rangle = \begin{cases} \frac{2}{3}h, & \text{if } i = j, \\ \frac{1}{6}h, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \langle (l_i^N)', (l_j^N)' \rangle = \begin{cases} \frac{2}{h}, & \text{if } i = j, \\ -\frac{1}{h}, & \text{if } |i - j| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

respectively, and  $G^N$  is an  $N \times M$  matrix with its  $(i, j)$ th element being defined by  $\langle b_j, l_i^N \rangle$ . Note that (3.10) can be simplified as follows

$$\dot{v}^N(t) = A^N v^N(t) + B^N \bar{\epsilon}(t), \tag{3.11}$$

where

$$A^N = -(F^N)^{-1}(\eta Q^N + \lambda F^N), \quad B^N = \lambda (F^N)^{-1} G^N,$$

are the matrix representations for operators  $\mathcal{A}^N$  and  $\mathcal{B}^N$ , respectively. We consider the approximation  $v^N(0)$  to the initial condition  $v(0)$ . To do this, we substitute

$$v(0) \approx v^N(0) = \mathcal{P}^N v_0 = \sum_{j=1}^N v_{0,j}^N l_j^N$$

into (3.8) with  $\phi = v_0$ , and let  $\psi = l_i^N$  for  $i = 1, 2, \dots, N$ , and we find

$$\sum_{j=1}^N v_{0,j}^N \langle l_j^N, l_i^N \rangle = \langle v_0, l_i^N \rangle, \quad i = 1, 2, \dots, N.$$

Let  $v_0^N = (v_{0,1}^N, \dots, v_{0,N}^N)^T$ . Then from the above equation we have

$$v_0^N = (F^N)^{-1} \begin{pmatrix} \langle v_0, l_1^N \rangle \\ \langle v_0, l_2^N \rangle \\ \vdots \\ \langle v_0, l_N^N \rangle \end{pmatrix}. \tag{3.12}$$

We now solve (3.6) subject to (3.7) to obtain approximations to the optimal  $\bar{\epsilon}_{\text{opt}}$  denoted by  $\bar{\epsilon}_{\text{opt}}^N$ . Note that the injection from  $V$  to  $H$  is compact. Hence, by Theorem 4.8 in [8], for  $N$  sufficiently large, there exists a unique nonnegative self-adjoint solution  $\Pi^N$  to the algebraic Riccati equation in  $V^N$

$$(\mathcal{A}^N)^* \Pi^N + \Pi^N \mathcal{A}^N - \Pi^N \mathcal{B}^N \beta^{-1} (\mathcal{B}^N)^* \Pi^N + \mathcal{Q} = 0, \tag{3.13}$$

and the convergence of the Riccati and control operators are also obtained. In addition, the feedback system operator  $\mathcal{A} - \mathcal{B} \beta^{-1} (\mathcal{B}^N)^* \Pi^N$  (i.e., the approximate feedback controls

used in the original infinite dimensional system) generates an exponential stable analytic semigroup on  $H$  and for  $v_0 \in H$

$$|J(\bar{\epsilon}_{\text{opt}}^N) - J(\bar{\epsilon}_{\text{opt}})| \leq \gamma(N)\|v_0\|_H^2,$$

where  $\bar{\epsilon}_{\text{opt}}^N(t) = -\frac{1}{\beta}(\mathcal{B}^N)^* \Pi^N v^N(t)$ , and  $\gamma(N) \rightarrow 0$  as  $N \rightarrow \infty$ . In terms of matrix representation,  $\bar{\epsilon}_{\text{opt}}^N$  is given by

$$\bar{\epsilon}_{\text{opt}}^N(t) = -\frac{1}{\beta}(B^N)^T \pi^N v^N(t),$$

where  $\pi^N$  is an  $N \times N$  matrix representation of the operator  $\Pi^N$  given by the corresponding matrix representation for algebraic Riccati equation (3.13), i. e.,

$$(A^N)^T \pi^N + \pi^N A^N - \pi^N B^N \beta^{-1} (B^N)^T \pi^N + I = 0.$$

We can easily solve for  $\pi^N$  using the built in MATLAB function `are` or other available software.

We have thus gathered all of the information needed to solve for  $v^N$ . At this time recall Eq. (3.11) with the optimal  $\bar{\epsilon}_{\text{opt}}^N$  is given by

$$\dot{v}^N(t) = \left\{ A^N - \frac{1}{\beta} B^N (B^N)^T \pi^N \right\} v^N(t)$$

with initial conditions  $v^N(0) = v_0^N$  defined by (3.12). We use these approximations in the numerical results presented below.

### 4. Numerical results

In this section, a number of simulations are carried out to investigate the proposed approach and illustrate its usefulness in exploring questions of control in a dynamic player game study. All of the computational results presented in this section are obtained with  $\eta = 6$  and  $\lambda = 1$ . We use the piecewise linear approximations presented in the previous section. In the examples given here, we typically used  $N = 50$  basis elements. The number of basis elements used was arrived at after simulations studies to ascertain values of  $N$  required to insure convergence.

For the intensity function  $r(w, \epsilon)$  of (2.7) we used the magnitude of Fresnel reflection coefficient based on planar layer geometry [6] given by  $r(w, \epsilon) = |(a + b)/(1 + ab)|$ , where

$$a = \frac{\epsilon - \sqrt{\epsilon\mu}}{\epsilon + \sqrt{\epsilon\mu}} \quad \text{and} \quad b = e^{4i\pi\sqrt{\epsilon\mu}wd/c}, \tag{4.1}$$

with magnetic layer permeability  $\mu = 1$ , layer thickness  $d = 2.5$  mm and speed of light  $c = 3 \times 10^8$  m/sec. We chose  $\mathcal{E} = \{\epsilon_j^*\}_{j=1}^M$  by taking  $M = 50$  equal partition points  $\epsilon_j^*$  in the dielectric permittivity interval  $1 \leq \epsilon \leq 1000$ , and interrogator frequency range  $[\underline{w}, \bar{w}]$  with  $\underline{w} = 0.4$  GHz and  $\bar{w} = 1$  GHz.

#### 4.1. Single carrier frequency input

Our design parameter is chosen to be  $\beta = 0.25$  for all the results obtained in this part. Our first attempt for the initial condition  $v_0$  is a truncated normal (Gaussian) distribution. The plots for the numerical approximations of  $v^N(t, w)$  and the control  $u^N(t, w)$  are illustrated in Fig. 1. This reveals that when the most emphasis in interrogating frequencies is placed on frequencies around 0.7 GHz, that the best controls are ones corresponding to materials effective around the frequency 0.7 GHz.

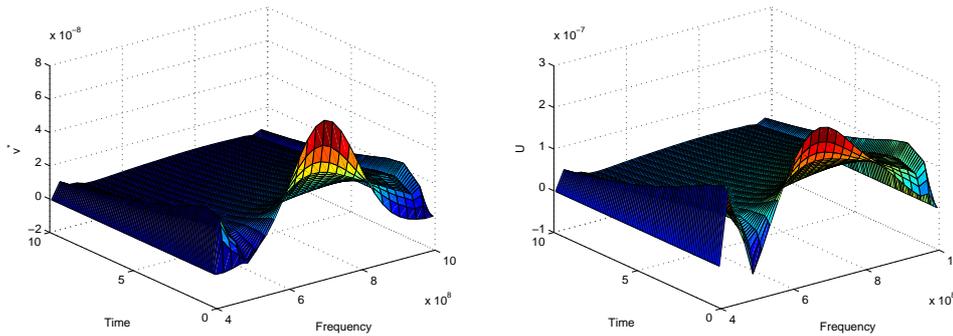


Figure 1: Numerical results obtained with initial condition  $v_0$  given by a truncated normal distribution. (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

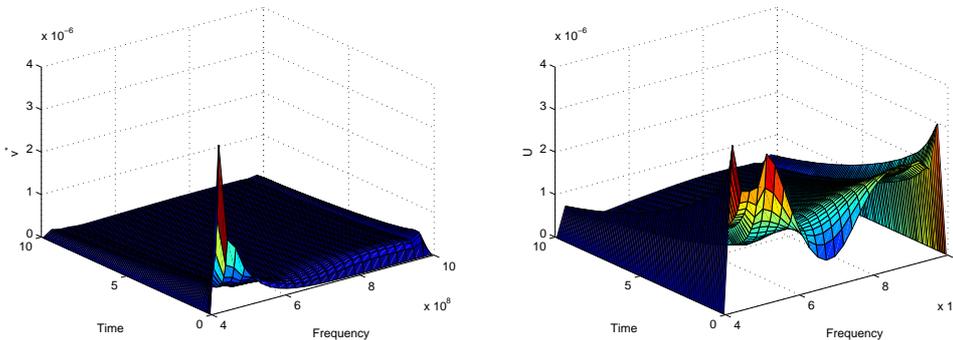


Figure 2: Numerical results obtained with initial condition  $v_0$  given by a truncated Gamma distribution. (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

In the next two trials, we choose distributions concentrated at the extreme frequencies for an initial choice for  $v_0$ . Specifically, the numerical results illustrated in Fig. 2 are obtained with a truncated *Gamma* distribution. We see from Fig. 2 that when the most emphasis is placed on frequencies around 0.4 GHz, that the best controls (materials) are ones effective around the frequency 0.4 GHz as well as some measure of control is exhibited at frequencies around 1 GHz. This agrees with what was observed in [6] in that a material that nulls well at 0.4 GHz also has some ability to null at 1 GHz.

Fig. 3 illustrates the numerical results obtained with a truncated Beta input. From this figure we see that when the most emphasis is placed on frequencies around 1 GHz, that the best materials to use are ones effective around the frequency 1 GHz. This illustrates what was seen in the static examples of [6] with a material that cloaks well at 1GHz.

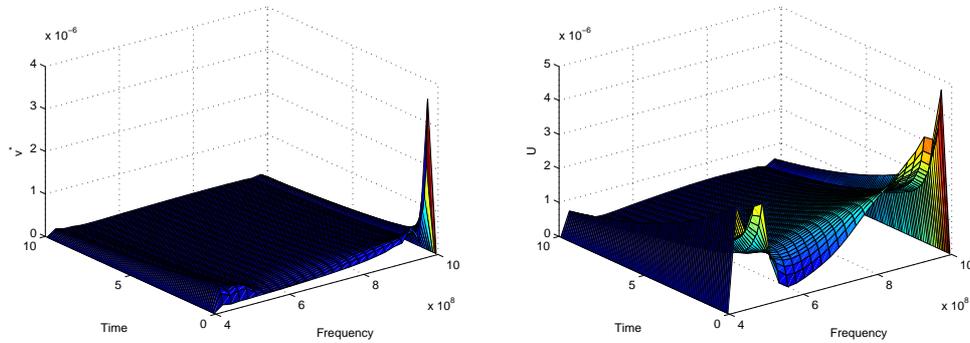


Figure 3: Numerical results obtained with initial condition  $v_0$  given by a truncated Beta distribution. (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

### 4.2. Multiple carrier frequency inputs

In this section, the numerical results are obtained with  $\beta = 0.25$ . The input is chosen from different distributions in a sequence of interrogating pulses. That is, the interrogator uses one distribution which is unknown to the evader to sample for interrogating pulse, then switches to choosing from a second distribution and then a third. This is done in an effort to confuse the evader in his choice of surface permittivities. These simulations

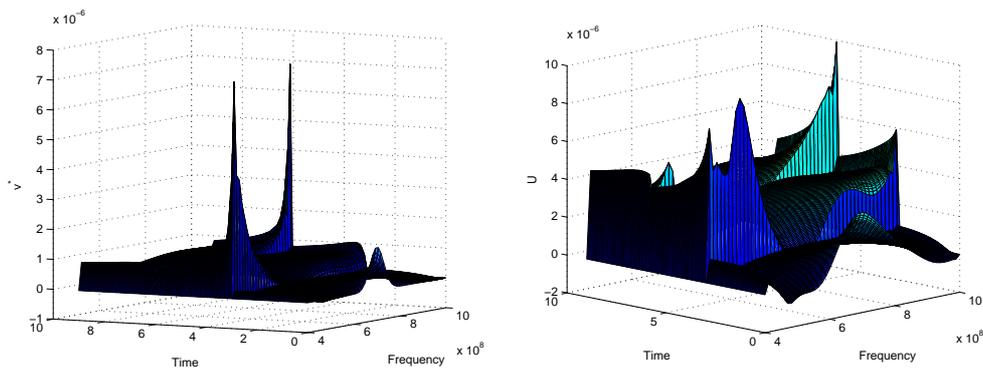


Figure 4: Numerical results obtained with a sequence of consecutive initial conditions  $v_0$  given by truncated normal–truncated Gamma–truncated Beta distributions. (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

are thus a rudimentary example of the situation where the interrogator also has a (non-feedback) time dependent control for the input frequency distributions. We simulate this by presenting graphs of responses to a sequence of consecutive initial condition inputs  $v_0$ . As we can see from a variety of different combinations in Figs. 4–6, the computational results demonstrate that the evader control quickly switches in time to accommodate the new choices in the interrogator frequency distributions. This suggests some level of robustness in the evader’s response to changing interrogator frequencies.

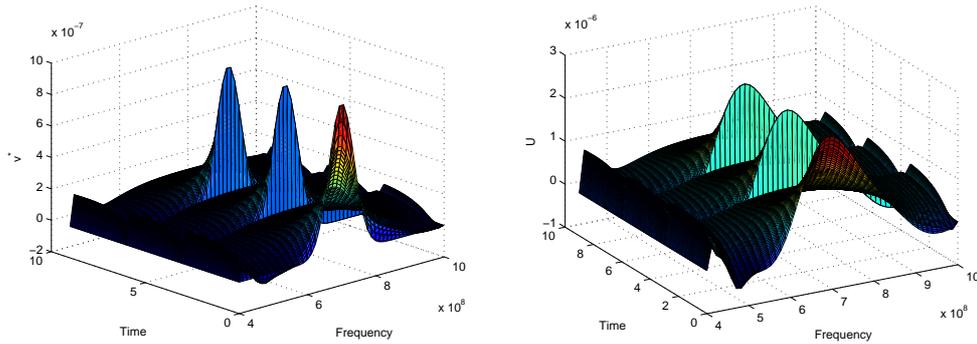


Figure 5: Numerical results obtained with a sequence of consecutive initial conditions  $v_0$  given by truncated normal–truncated normal–truncated normal distributions. (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

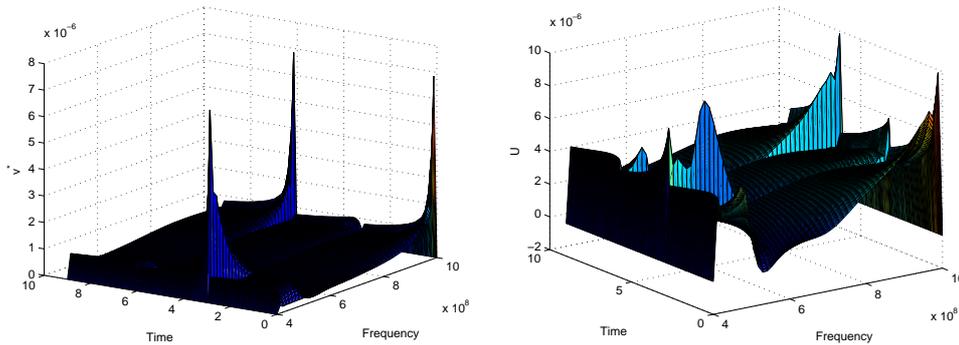


Figure 6: Numerical results obtained with a sequence of consecutive initial conditions  $v_0$  given by truncated Beta–truncated Gamma–truncated Beta distributions. (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

### 4.3. Effect of design parameter on the overall control

Finally, we consider how the choice of the design parameter  $\beta$  affects the overall control effectiveness. All the numerical results in this section are obtained with a truncated normal initial condition  $v_0$ . We note from Figs. 7–9 as  $\beta$  increases (i.e., the control gain decreases) the feedback control action is less effective and rapid.

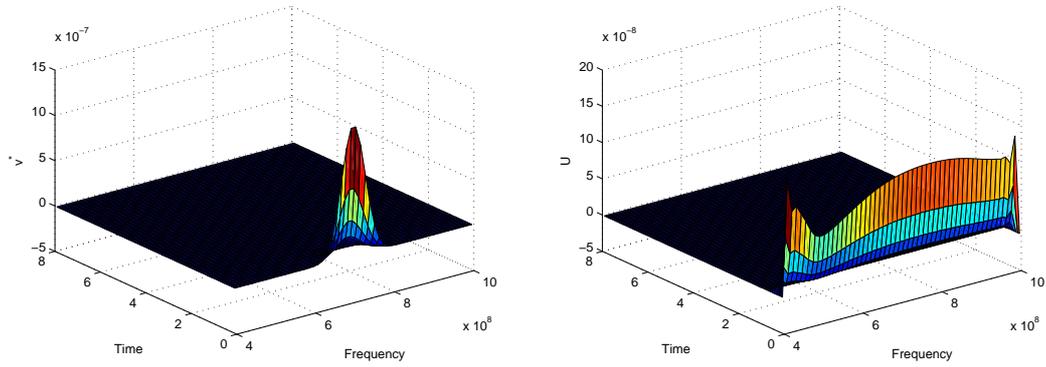


Figure 7: Numerical results obtained with  $\beta = 2.5 \times 10^{-4}$ . (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

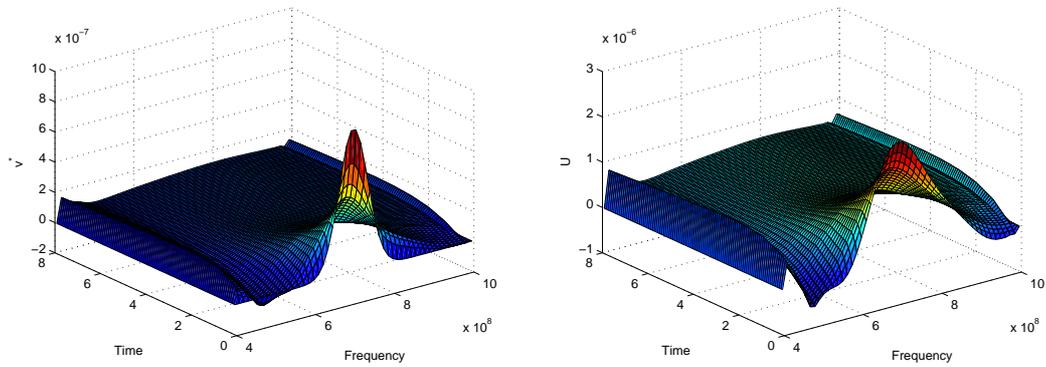


Figure 8: Numerical results obtained with  $\beta = 0.25$ . (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

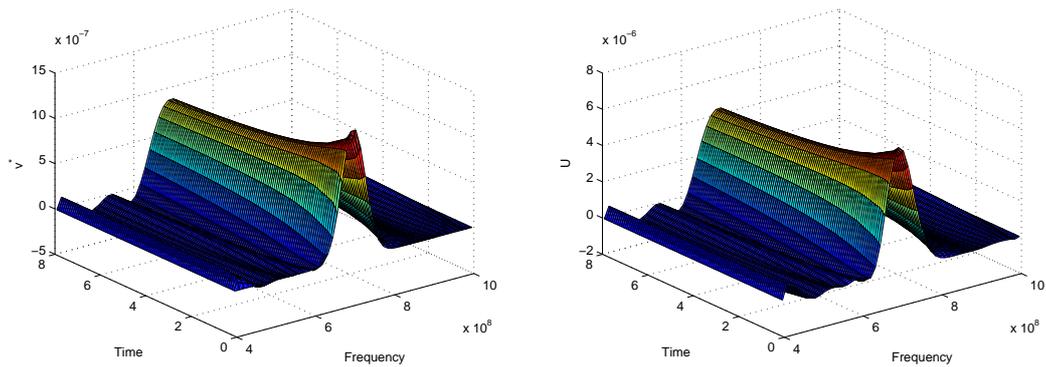


Figure 9: Numerical results obtained with  $\beta = 25000$ . (left):  $v^N(t, w)$ ; (right):  $u^N(t, w)$ .

## 5. Conclusion and future research efforts

In this paper, we consider a dynamic evasion-interrogation games with uncertainty in the context of electromagnetics. Two different formulations are considered: one is based on the probability density function of the intensity of the reflected signal, and the other is based on the expected value of the intensity of the reflected signal. We should note that we anticipate that the ideas presented here can be readily implemented in a number of other modern non-cooperative adversarial situations such as information warfare and network security.

There are several efforts we plan to pursue in the near future. One is to extend the ideas in this paper to include a stochastic process for the evader to obtain a true two player min-max dynamic differential game for the evader-interrogator problem. The other efforts include investigation of other means to introduce uncertainty in the dynamic two player games.

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