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Convergence Estimates for Some Regularization Methods to Solve a Cauchy Problem of the Laplace Equation

T. Wei*, H. H. Qin and H. W. Zhang

School of Mathematics and Statistics, Lanzhou University, P. R. China.

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Abstract. In this paper, we give a general proof on convergence estimates for some regularization methods to solve a Cauchy problem for the Laplace equation in a rectangular domain. The regularization methods we considered are: a non-local boundary value problem method, a boundary Tikhonov regularization method and a generalized method. Based on the conditional stability estimates, the convergence estimates for various regularization methods are easily obtained under the simple verifications of some conditions. Numerical results for one example show that the proposed numerical methods are effective and stable.

AMS subject classifications: 65N12, 65N15, 65N20, 65N21

Key words: Cauchy problem, Laplace equation, regularization methods, convergence estimates.

1. Introduction

In this paper, we consider a Cauchy problem for the Laplace equation in a rectangular domain as follows

$$\Delta u(x,y)=0, \qquad 0<x<\pi, \quad 0<y<a, \tag{1.1}$$

$$u(0,y)=u(\pi,y)=0, \quad 0\le y\le a, \tag{1.2}$$

$$u_y(x,0)=0, \qquad 0\le x\le \pi, \tag{1.3}$$

$$u(x,0)=\varphi(x), \qquad 0\le x\le \pi, \tag{1.4}$$

where a is a positive constant.

Define the family of rectangular regions with parameter $0<\sigma\le a$ by

$$D_\sigma=\Big\{(x,y)\ \Big|\ 0<x<\pi, 0<y<\sigma\Big\}. \tag{1.5}$$

*Corresponding author. *Email address:* `tingwei@lzu.edu.cn` (T. Wei)

Assume that the exact Dirichlet data $\varphi \in L^2(0,\pi)$ and the measured data $\varphi^\delta \in L^2(0,\pi)$ satisfy

$$\|\varphi^\delta - \varphi\| \le \delta, \tag{1.6}$$

where $\|\cdot\|$ denotes the L^2-norm and $\delta > 0$ is a noise level.

Further we assume that the following *a-priori* bound exists,

$$\|u(\cdot,a)\| \le E, \tag{1.7}$$

or a stronger *a-priori* bound assumption holds

$$\left\| \frac{\partial^p u}{\partial y^p}(\cdot,a) \right\| \le E_p, \tag{1.8}$$

where E and E_p are positive constants and $p \ge 1$ is an integer number.

The problem is to find an approximate solution for problem (1.1)–(1.4) with noisy data φ^δ instead of φ. This is a classical Cauchy problem for Laplace's equation in a special domain. It arises in many real applications, such as nondestructive testing [1, 7], geophysics [26] and cardiology [8]. It is well known that the problem is typically ill-posed. That is, any small changes of the Cauchy data may induce large changes of the solutions (e.g. [15,26]).

Under an additional *a-priori* bound assumption, a continuous dependence of the solution on the Cauchy data can be obtained. This is called conditional stability (e.g. [2,21]). We note that the conditional stability is closely related with the convergence of some regularization methods. For example, in [6] Cheng et al. provided a relationship between the convergence rate of the Tikhonov regularization method and conditional stability for an ill-posed operator equation. In [16] and [17], based on the conditional stability, the authors gave some convergence estimates for gradient-based methods and general linear regularization methods to treat with a linear ill-posed operator equation. In this paper, based on the conditional stabilities of a Cauchy problem, we give a general proof on the convergence estimates for three special methods for solving the Cauchy problem: the nonlocal boundary value problem method, the boundary Tikhonov regularization method and a generalized method. The first two methods have been investigated extensively in [9,23] where the authors presented convergence analysis based on the direct error estimates without using conditional stability. As we know, the convergence proof based on conditional stability for the Cauchy problem of Laplace equation is new issue and the generalized regularization method does not appear in references.

In [10], Eldén et al. gave an explicit and concrete stability result for problem (1.1), (1.3)–(1.4) with the homogenous Neumann condition at boundary $x = 0$ and $x = \pi$ in a square domain. By the method in [10] and a small modification, the stability estimate for a solution of problem (1.1)–(1.4) in a rectangular domain is also obtained which has a little difference from one in [10] and we show it in the following proposition.

Proposition 1.1. *Assume that the function u satisfies* (1.1)–(1.4) *and*

$$\int_0^\pi \varphi^2 dx \le \varepsilon, \qquad \|u\|^2_{L^2(D_a)} \le M,$$

then, for $0 < \sigma < a$, *the following stability estimate holds,*

$$\int_{D_\sigma} u^2 dxdy \le a^{1-\frac{\sigma}{a}} M^{\frac{\sigma}{a}} \varepsilon^{1-\frac{\sigma}{a}}. \tag{1.9}$$

From Proposition 1.1, we note that the stability estimate (1.9) is not useful at $\sigma = a$. To restore the continuous dependence of the solution at $y = a$, we use the stronger assumption (1.8) and obtain a conditional stability result on the boundary $y = a$ which is displayed in the following Proposition 1.2. The proof is given in Appendix based on the idea in [22].

Proposition 1.2. *Let the function u satisfy* (1.1)–(1.4) *and assume*

$$\|\varphi\| \le \epsilon, \qquad \left\| \frac{\partial^p u}{\partial y^p}(\cdot, a) \right\| \le E_p,$$

where constants $\epsilon > 0$, $E_p > 0$ *and integer* $p \ge 1$. *Then, we have the following stability estimate at* $y = a$,

$$\|u(\cdot, a)\| \le 2E_p \left(\ln \frac{2E_p}{\epsilon} \right)^{-p} + 2(1 - e^{-2a})^{-1} \max \left\{ \mu^{2p/3}, \frac{a^3}{2}\mu^{2p/3}, \frac{a^3}{2}\mu^2 \right\} E_p,$$

where

$$\mu = \left(\ln \left(\frac{2E_p}{\epsilon} (\ln \frac{2E_p}{\epsilon})^{-p} \right) \right)^{-1}.$$

Due to the ill-posedness of the Cauchy problem for the Laplace equation, numerical computations are very difficult if there is no *a-priori* assumption on the exact solution. A regularization technique is usually required to obtain a stable solution. In the past decades, several regularization methods have been proposed: Quasi-reversibility method [4, 19], Tikhonov regularization method [25], the iterative method [12, 20], the conjugate gradient method [13], moment method [5, 14], discretization method [3, 8, 24] for a Cauchy problem in a general domain and the modified method [22] in a special domain and so on. In particular, Takeuchi and Yamamoto in [25] used a Tikhonov regularization method in a reproducing kernel Hilbert space to solve the Cauchy problem for an elliptic equation in a general domain.

In this paper, we consider some regularization methods for solving problem (1.1)–(1.4). Based on Propositions 1.1 and 1.2, the convergence estimates for three special regularization methods can be easily obtained, see Sections 2–3. Numerical experiments are also presented in Section 4.

2. A general regularization method

By the separation of variables, it is easy to get the solution of problem (1.1)–(1.4) as follows

$$u(x, y) = \sum_{n=1}^{\infty} c_n X_n(x) \cosh(ny), \tag{2.1}$$

where

$$X_n(x) := \sqrt{\frac{2}{\pi}}\sin(nx), \quad n \ge 1, \tag{2.2}$$

and $c_n = \int_0^\pi \varphi(\xi)X_n(\xi)d\xi$.

Note that $\{X_n\}_{n=1}^\infty$ is an orthonormal basis of $L^2(0,\pi)$.

From the following statements we know that problem (1.1)–(1.4) is instable. Choose $u_n(x,y) = \sin(nx)\cosh(ny)/n^2$ ($n \ge 1$) as the exact solutions for problem (1.1)–(1.4) with initial data $\varphi_n(x) = \sin(nx)/n^2$. We find that $\sup_{x\in[0,\pi]}|\varphi_n(x)| \to 0$ as n tends to infinity, but $\sup_{x\in[0,\pi]}|u_n(x,y)| \to \infty$ as n tends to infinity for fixed $y > 0$. Hence, we can not use a classical numerical method to solve the Cauchy problem (1.1)–(1.4) and some regularization techniques are required [11, 15, 18].

For noisy data φ^δ, denote

$$c_n^\delta = \int_0^\pi \varphi^\delta(\xi)X_n(\xi)d\xi. \tag{2.3}$$

Then the condition (1.6) means

$$\sum_{n=1}^\infty (c_n^\delta - c_n)^2 \le \delta^2. \tag{2.4}$$

Let $q(\alpha,n) \ge 0$ be a filter function satisfying the following conditions for all $n \ge 1$:

(a) $|q(\alpha,n)\cosh(na)| \le \dfrac{1}{C_1(\alpha)}$;

(b) $|q(\alpha,n) - 1| \le K_1(\alpha)$;

(c) $|q(\alpha,n)| \le \dfrac{1}{C_2(\alpha)}$;

(d) $\left|\dfrac{q(\alpha,n)-1}{\cosh(na)}\right| \le K_2(\alpha)$.

The regularized solution is defined as follows,

$$u_\alpha^\delta(x,y) = \sum_{n=1}^\infty q(\alpha,n)c_n^\delta X_n(x)\cosh(ny). \tag{2.5}$$

It is easy to verify that the regularized solution satisfies the following equation and boundary conditions

$$\begin{aligned}
&\Delta u_\alpha^\delta(x,y) = 0, && 0 < x < \pi, \quad 0 < y < a,\\
&u_\alpha^\delta(0,y) = u_\alpha^\delta(\pi,y) = 0, && 0 \le y \le a,\\
&\partial_y u_\alpha^\delta u(x,0) = 0, && 0 \le x \le \pi.
\end{aligned}$$

According to Proposition 1.1, in order to get a convergence estimate, we only need to prove that $\|u_\alpha^\delta - u\|_{L^2(D_a)}$ is bounded and $\|\varphi - u_\alpha^\delta(\cdot,0)\|$ converges to zero as the noisy level tends to zero.

From (2.1) and (2.5), we have

$$\begin{aligned} u_\alpha^\delta(x,y) - u(x,y) &= \sum_{n=1}^{\infty}(q(\alpha,n)c_n^\delta - c_n)X_n(x)\cosh(ny), \\ &= \sum_{n=1}^{\infty}\Big[q(\alpha,n)(c_n^\delta - c_n) + (q(\alpha,n)-1)c_n\Big]X_n(x)\cosh(ny). \end{aligned} \tag{2.6}$$

Assumption (1.7) leads to $\sum_{n=1}^{\infty} c_n^2\cosh^2(na) \le E^2$. By the orthonormality of $\{X_n\}$, (2.4) and conditions (a)–(b), we obtain

$$\begin{aligned} &\|u_\alpha^\delta(\cdot,y) - u(\cdot,y)\|^2 \\ \le& 2\sum_{n=1}^{\infty}\Big[q^2(\alpha,n)(c_n^\delta - c_n)^2 + (q(\alpha,n)-1)^2c_n^2\Big]\cosh^2(ny), \\ \le& 2\left[\frac{\delta^2}{C_1^2(\alpha)} + E^2K_1^2(\alpha)\right]. \end{aligned} \tag{2.7}$$

Furthermore, we have

$$\|u_\alpha^\delta - u\|_{L^2(D_a)}^2 \le 2a\left[\frac{\delta^2}{C_1^2(\alpha)} + E^2K_1^2(\alpha)\right]. \tag{2.8}$$

As for the boundary error, by conditions (c)–(d), we have

$$\begin{aligned} &\|u_\alpha^\delta(\cdot,0) - u(\cdot,0)\|^2 \\ \le& 2\sum_{n=1}^{\infty}\Big[q^2(\alpha,n)(c_n^\delta - c_n)^2 + (q(\alpha,n)-1)^2c_n^2\Big], \\ \le& 2\left[\frac{\delta^2}{C_2^2(\alpha)} + E^2K_2^2(\alpha)\right]. \end{aligned} \tag{2.9}$$

Then we have the following convergence estimate between the regularized solution u_α^δ given by (2.5) and the exact solution u given by (2.1).

Theorem 2.1. *Let u be the solution of problem* (1.1)–(1.4) *with the exact data φ and u_α^δ be the regularized approximate solution given by* (2.5) *with the filter function $q(\alpha,n)$ satisfying the conditions* (a)–(d). *Let the measured data φ^δ fulfill* (1.6) *and the exact solution u at $y=a$ satisfy* (1.7). *If we choose a regularization parameter $\alpha=\alpha(\delta)$ such that*

(1) *$\delta/C_1(\alpha)$ is bounded;*

(2) *$K_1(\alpha)$ is bounded;*

(3) $\delta/C_2(\alpha) \to 0$, *as* $\delta \to 0$;

(4) $K_2(\alpha) \to 0$, *as* $\delta \to 0$,

then we have the following convergence estimate for $0 < \sigma < a$,

$$
\begin{aligned}
&\|u_\alpha^\delta - u)\|^2_{L^2(D_\sigma)} \\
\leq & 2a^{1-\frac{\sigma}{a}} \left(\frac{\delta^2}{C_1^2(\alpha)} + E^2 K_1^2(\alpha) \right)^{\frac{\sigma}{a}} \left(\frac{\delta^2}{C_2^2(\alpha)} + E^2 K_2^2(\alpha) \right)^{1-\frac{\sigma}{a}}.
\end{aligned} \tag{2.10}
$$

Note that the estimate (2.10) is not valid at $\sigma = a$. Thus in the following we try to give a convergence estimate at $y = a$ for $\|u_\alpha^\delta(\cdot, a) - u(\cdot, a)\|$.

According to Proposition 1.2, we only need to prove that $\left\|\frac{\partial^p (u_\alpha^\delta - u)}{\partial y^p}(\cdot, a)\right\|$ is bounded and $\|\varphi - u_\alpha^\delta(\cdot, 0)\|$ converges to zero as the noisy level δ tends to zero.

Hereafter we assume that the filter function $q(\alpha, n)$ satisfies conditions (b)–(d) as well as the following condition (aa) for integer $p \geq 1$:

(aa) $|q(\alpha, n) n^p \cosh(na)| \leq \frac{1}{C_3(\alpha)}$.

Suppose the *a-priori* bound condition (1.8) holds, then we know,

$$
\left\| \frac{\partial^p u}{\partial y^p}(\cdot, a) \right\|^2 = \begin{cases} \sum\limits_{n=1}^{\infty} c_n^2 n^{2p} \cosh^2(na) \leq E_p^2, \ p \text{ is even}, \\ \sum\limits_{n=1}^{\infty} c_n^2 n^{2p} \sinh^2(na) \leq E_p^2, \ p \text{ is odd}. \end{cases} \tag{2.11}
$$

From (2.6), we have

$$
\begin{aligned}
\frac{\partial^p (u_\alpha^\delta - u)}{\partial y^p}(x, y) &= \begin{cases} \sum\limits_{n=1}^{\infty} (q(\alpha, n) c_n^\delta - c_n) X_n(x) n^p \cosh(ny), & p \text{ is even}, \\ \sum\limits_{n=1}^{\infty} (q(\alpha, n) c_n^\delta - c_n) X_n(x) n^p \sinh(ny), & p \text{ is odd}, \end{cases} \\
&= \begin{cases} \sum\limits_{n=1}^{\infty} \left[q(\alpha, n)(c_n^\delta - c_n) + (q(\alpha, n) - 1) c_n \right] X_n(x) n^p \cosh(ny), & p \text{ is even}, \\ \sum\limits_{n=1}^{\infty} \left[q(\alpha, n)(c_n^\delta - c_n) + (q(\alpha, n) - 1) c_n \right] X_n(x) n^p \sinh(ny), & p \text{ is odd}. \end{cases}
\end{aligned}
$$

By the orthonormality of $\{X_n\}$ and (2.11), for any integer $p \geq 1$, we obtain

$$
\begin{aligned}
&\left\| \frac{\partial^p (u_\alpha^\delta - u)}{\partial y^p}(\cdot, a) \right\|^2 \\
\leq & \begin{cases} 2 \sum\limits_{n=1}^{\infty} \left[q^2(\alpha, n)(c_n^\delta - c_n)^2 + (q(\alpha, n) - 1)^2 c_n^2 \right] n^{2p} \cosh^2(na), & p \text{ is even}, \\ 2 \sum\limits_{n=1}^{\infty} \left[q^2(\alpha, n)(c_n^\delta - c_n)^2 + (q(\alpha, n) - 1)^2 c_n^2 \right] n^{2p} \sinh^2(na), & p \text{ is odd}, \end{cases} \\
\leq & 2 \left[\frac{\delta^2}{C_3^2(\alpha)} + E_p^2 K_1^2(\alpha) \right].
\end{aligned}
$$

For the boundary error, by conditions (c)–(d), note that

$$\frac{\cosh(na)}{\sinh(na)} \leq 2(1-e^{-2a})^{-1},$$

we have

$$\begin{aligned}
&\|u_\alpha^\delta(\cdot,0)-u(\cdot,0)\| \\
\leq&\left(\sum_{n=1}^{\infty} q^2(\alpha,n)(c_n^\delta - c_n)^2\right)^{\frac{1}{2}} + \left(\sum_{n=1}^{\infty}(q(\alpha,n)-1)^2 c_n^2\right)^{\frac{1}{2}} \\
\leq&\frac{\delta}{C_2(\alpha)} + \frac{2E_p}{1-e^{-2a}} K_2(\alpha).
\end{aligned}$$

Thus we have the following convergence estimate at boundary $y=a$.

Theorem 2.2. *Let u be the solution of problem* (1.1)–(1.4) *with the exact data φ and u_α^δ be the regularized approximate solution given by* (2.5) *with the filter function $q(\alpha,n)$ satisfying the conditions* (aa) *and* (b)–(d). *Let the measured data φ^δ fulfill* (1.6) *and the exact solution u at $y=a$ satisfy* (1.8). *We choose a regularization parameter $\alpha=\alpha(\delta)$ such that*

(i) $\delta/C_3(\alpha)$ *is bounded;*

(ii) $K_1(\alpha)$ *is bounded;*

(iii) $\delta/C_2(\alpha)\to 0$, *as* $\delta\to 0$;

(iv) $K_2(\alpha)\to 0$, *as* $\delta\to 0$.

Then the following convergence estimate holds at $y=a$,

$$\begin{aligned}
&\|u_\alpha^\delta(\cdot,a)-u(\cdot,a)\| \\
\leq&2\tilde{E}_p\left(\ln\frac{2\tilde{E}_p}{\tilde{\epsilon}}\right)^{-p} + 2(1-e^{-2a})^{-1}\max\left\{\tilde{\mu}^{2p/3}, \frac{a^3}{2}\tilde{\mu}^{2p/3}, \frac{a^3}{2}\tilde{\mu}^2\right\}\tilde{E}_p,
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{E}_p = \sqrt{2}\left[\frac{\delta}{C_3(\alpha)} + E_p K_1(\alpha)\right], \qquad \tilde{\epsilon} = \frac{\delta}{C_2(\alpha)} + \frac{2\tilde{E}_p}{1-e^{-2a}}K_2(\alpha), \\
&\tilde{\mu} = \left(\ln\left(\frac{2\tilde{E}_p}{\tilde{\epsilon}}(\ln\frac{2\tilde{E}_p}{\tilde{\epsilon}})^{-p}\right)\right)^{-1}.
\end{aligned}$$

3. Three special regularization methods

In this section, we consider three special regularization methods: the non-local boundary value problem method, the boundary Tikhonov regularization method and a generalized method. The convergence estimates for $0<y\leq\sigma<a$ and $y=a$ will be given in the following subsections, respectively.

3.1. The non-local boundary value problem method

Take the filter function as

$$q(\alpha,n)=\frac{1}{1+\alpha\cosh(na)}. \tag{3.1}$$

With this filter function, we can prove that u_α^δ given by (2.5) is a solution of the following well-posed mixed boundary value problem

$$\begin{aligned}
&\Delta u_\alpha^\delta(x,y)=0, && 0<x<\pi, \quad 0<y<a, && (3.2)\\
&u_\alpha^\delta(0,y)=u_\alpha^\delta(\pi,y)=0, && 0\le y\le a, && (3.3)\\
&\partial_y u_\alpha^\delta(x,0)=0, && 0\le x\le \pi, && (3.4)\\
&u_\alpha^\delta(x,0)+\alpha u_\alpha^\delta(x,a)=\varphi^\delta(x), && 0\le x\le \pi. && (3.5)
\end{aligned}$$

The convergence estimate for $0<y\le\sigma<a$ will be shown in the following theorem.

Theorem 3.1. *Let u be the solution of problem* (1.1)–(1.4) *and u_α^δ be the regularized solution given by* (3.2)–(3.5)*. Let the measured data φ^δ fulfill* (1.6) *and let the exact solution u at $y=a$ satisfy* (1.7)*. If we choose the regularization parameter $\alpha=c\delta$ for a constant $c>0$, then we have the convergence estimate for $0<\sigma<a$ as follows*

$$\|u_\alpha^\delta-u\|_{L^2(D_\sigma)}^2 \;\le\; 2a^{1-\frac{\sigma}{a}}c^{-\frac{2\sigma}{a}}\left(1+E^2c^2\right)\delta^{2(1-\frac{\sigma}{a})}. \tag{3.6}$$

Proof. Note that in this case the filter function q is given by (3.1), it is easy to check

(1) $|q(\alpha,n)\cosh(na)|=\frac{\cosh(na)}{1+\alpha\cosh(na)}\le\frac{1}{\alpha}$, thus $C_1(\alpha)=\alpha$, $\delta/C_1(\alpha)=1/c$ is bounded;

(2) $|q(\alpha,n)-1|\le 1$, thus $K_1(\alpha)=1$ is bounded;

(3) $|q(\alpha,n)|=\frac{1}{1+\alpha\cosh(na)}\le 1$, thus $C_2(\alpha)=1$ and $\delta/C_2(\alpha)\to 0$ as $\delta\to 0$;

(4) $|\frac{q(\alpha,n)-1}{\cosh(na)}|\le\alpha$, thus $K_2(\alpha)=\alpha=c\delta\to 0$ as $\delta\to 0$.

By Theorem 2.1, we know the estimate (3.6) is true. ☐

Remark 3.1. In this method, condition (aa) is not satisfied, so the convergence result at boundary $y=a$ can not be obtained by Proposition 2.2. However, the convergence at $y=a$ is also true, refer to [27] where taking $k=0$.

3.2. The boundary Tikhonov regularization method

Take the filter function as

$$q(\alpha,n)=\frac{1}{1+\alpha\cosh^2(na)}. \tag{3.7}$$

We can prove that for this case the regularized solution u_α^δ is a solution of a well-posed direct problem with a Tikhonov regularized boundary value at $y=a$.

Define an operator $K: f(x)\in L^2(0,\pi)\mapsto v(x,0)\in L^2(0,\pi)$, where $v(x,y)$ is a solution of the following direct problem,

$$\begin{aligned}
&\Delta v(x,y)=0, && 0<x<\pi,\quad 0<y<a, && (3.8)\\
&v(0,y)=v(\pi,y)=0, && 0\le y\le a, && (3.9)\\
&v_y(x,0)=0, && 0\le x\le \pi, && (3.10)\\
&v(x,a)=f(x), && 0\le x\le \pi. && (3.11)
\end{aligned}$$

It is easy to show that the solution

$$v(x,y)=\sum_{n=1}^{\infty}\int_0^\pi f(\xi)X_n(\xi)d\xi X_n(x)\cosh(ny)(\cosh(na))^{-1}. \tag{3.12}$$

Therefore

$$Kf:=\int_0^\pi k(x,\xi)f(\xi)d\xi=v(x,0), \tag{3.13}$$

where the kernel function $k(x,\xi)=\sum_{n=1}^{\infty}X_n(x)X_n(\xi)(\cosh(na))^{-1}$.

Due to the term $(\cosh(na))^{-1}$, we know that the kernel function $k(x,\xi)$ is in $C^\infty([0,\pi]\times[0,\pi])$, thus the operator K is linear, bounded, self-adjoint, compact from $L^2(0,\pi)$ to $L^2(0,\pi)$ (cf. [18]). It is not hard to check its singular system is $\{(\mu_n,X_n(\xi),X_n(x))\}_{n=1}^{\infty}$ where $\mu_n=(\cosh(na))^{-1}$.

Let f_α^δ be the solution for the following minimization problem,

$$\min_{f\in L^2(0,\pi)}\|Kf-\varphi^\delta\|^2+\alpha\|f\|^2. \tag{3.14}$$

By Theorem 2.11 of Chapter 2 in [18], we know that

$$f_\alpha^\delta(x)=\sum_{n=1}^{\infty}\frac{\cosh(na)}{1+\alpha\cosh^2(na)}c_n^\delta X_n(x), \tag{3.15}$$

where c_n^δ is given by (2.3).

Then we can prove that the regularized solution u_α^δ given by (2.5) with filter function (3.7) is the solution of the following direct problem

$$\begin{aligned}
&\Delta u_\alpha^\delta(x,y)=0, && 0<x<\pi,\quad 0<y<a, && (3.16)\\
&u_\alpha^\delta(0,y)=u_\alpha^\delta(\pi,y)=0, && 0\le y\le a, && (3.17)\\
&\partial_y u_\alpha^\delta(x,0)=0, && 0\le x\le \pi, && (3.18)\\
&u_\alpha^\delta(x,a)=f_\alpha^\delta(x), && 0\le x\le \pi. && (3.19)
\end{aligned}$$

In the following theorem, we will give the error estimates between the regularization solution u_α^δ given by (2.5) and the exact solution u given by (2.1).

Theorem 3.2. *Let u be the solution of problem* (1.1)–(1.4) *with the exact data φ and u_α^δ be the solution for* (3.16)–(3.19)*. Let the measured data φ^δ fulfill* (1.6)*. If the exact solution u at $y = a$ satisfy* (1.7) *and the regularization parameter α is chosen as $\alpha = c\delta^2$ where $c > 0$ is a constant, then for fixed $0 < \sigma < a$, we have the following convergence estimate,*

$$\|u_\alpha^\delta - u\|_{L^2(D_\sigma)}^2 \le 2a^{1-\frac{\sigma}{a}} \left(\frac{1}{4c} + E^2\right)^{\frac{\sigma}{a}} \left(1 + \frac{1}{4}cE^2\right)^{1-\frac{\sigma}{a}} \delta^{2(1-\frac{\sigma}{a})}. \tag{3.20}$$

Moreover, if the exact solution $u(\cdot, a)$ satisfy (1.8)*, and the regularization parameter is chosen as $\alpha = \tilde{c}\delta$ with a constant $\tilde{c} > 0$, then we can obtain the following convergence estimate,*

$$\begin{aligned} &\|u_\alpha^\delta(\cdot, a) - u(\cdot, a)\| \qquad (3.21)\\ \le& 2\tilde{E}_p \left(\ln \frac{2\tilde{E}_p}{\tilde{\epsilon}}\right)^{-p} + 2(1-e^{-2a})^{-1} \max\left\{\tilde{\mu}^{2p/3}, \frac{a^3}{2}\tilde{\mu}^{2p/3}, \frac{a^3}{2}\tilde{\mu}^2\right\} \tilde{E}_p, \end{aligned}$$

where

$$\begin{aligned} \tilde{E}_p &= \sqrt{2}\left[\frac{2p\,!}{\tilde{c}a^p} + E_p\right], \\ \tilde{\epsilon} &= \delta + \frac{\tilde{E}_p}{1-e^{-2a}} \tilde{c}^{1/2}\delta^{1/2}, \\ \tilde{\mu} &= \left(\ln(\frac{2\tilde{E}_p}{\tilde{\epsilon}}(\ln\frac{2\tilde{E}_p}{\tilde{\epsilon}})^{-p})\right)^{-1}. \end{aligned}$$

Proof. Note that $0 < y < \sigma$ and the filter function q is given by (3.7), it is easy to check

(1) $|q(\alpha, n)\cosh(na)| = \frac{\cosh(na)}{1+\alpha\cosh^2(na)} \le \frac{1}{2\sqrt{\alpha}}$, so $C_1(\alpha) = 2\sqrt{\alpha}$, $\delta/C_1(\alpha) = \frac{1}{2\sqrt{c}}$ is bounded;

(2) $|q(\alpha, n) - 1| \le 1$, so $K_1(\alpha) = 1$ is bounded;

(3) $|q(\alpha, n)| = \frac{1}{1+\alpha\cosh^2(na)} \le 1$, thus $C_2(\alpha) = 1$ and $\delta/C_2(\alpha) \to 0$ as $\delta \to 0$;

(4) $|\frac{q(\alpha,n)-1}{\cosh(na)}| \le \sqrt{\alpha}/2$, so $K_2(\alpha) = \sqrt{\alpha}/2 = \sqrt{c}\delta/2 \to 0$ as $\delta \to 0$.

By Theorem 2.1, we know the convergence estimate (3.20) is true.

It is easy to see the conditions (ii) and (iii) in Theorem 2.2 are satisfied. Furthermore, we have

(i) $|q(\alpha, n)n^p\cosh(na)| = \frac{n^p\cosh(na)}{1+\alpha\cosh^2(na)} \le a^{-p}\frac{(na)^p}{\alpha\cosh(na)} \le \frac{2p\,!}{a^p\alpha}$, thus $C_3(\alpha) = \frac{a^p\alpha}{2p\,!}$, by the choice of $\alpha = \tilde{c}\delta$, we know $\delta/C_3(\alpha) = \frac{2p\,!}{a^p\tilde{c}}$ is bounded;

(iv) $K_2(\alpha) = \sqrt{\alpha}/2 = \sqrt{\tilde{c}\delta}/2 \to 0$ as $\delta \to 0$.

Hence, by Theorem 2.2, the conclusion (3.21) is true. ∎

3.3. A generalized method

Take the filter function as

$$q(\alpha,n)=\frac{1}{1+\alpha\cosh^{\nu}(na)},\quad \nu>2. \tag{3.22}$$

Then we can obtain the following error estimates for u_α^δ.

Theorem 3.3. *Let u be the solution of problem (1.1)–(1.4) with the exact data φ and u_α^δ be the regularized approximation solution given by (2.5) with a filter function (3.22). Let the measured data φ^δ fulfill (1.6). If the exact solution u at $y=a$ satisfy (1.7) and the regularization parameter α is chosen as $\alpha=c\delta^\nu$ where $c>0$ is a constant, then for fixed $0<\sigma<a$, we have the following convergence estimate,*

$$\begin{aligned}&\|u_\alpha^\delta-u\|^2_{L^2(D_\sigma)}\\ \leq&2a^{1-\frac{\sigma}{a}}\left(\frac{(\nu-1)^2}{\nu^2(\nu-1)^{\frac{2}{\nu}}c^{\frac{2}{\nu}}}+E^2\right)^{\frac{\sigma}{a}}\left(1+\frac{(\nu-1)^2}{\nu^2(\nu-1)^{\frac{2}{\nu}}}c^{\frac{2}{\nu}}E^2\right)^{1-\frac{\sigma}{a}}\delta^{2(1-\frac{\sigma}{a})}.\end{aligned} \tag{3.23}$$

Moreover, if the exact solution $u(\cdot,a)$ satisfy (1.8), and the regularization parameter is chosen as $\alpha=\tilde{c}\delta^{\frac{\nu}{2}}$ with a constant $\tilde{c}>0$, then we can obtain the following convergence estimate,

$$\begin{aligned}&\|u_\alpha^\delta(\cdot,a)-u(\cdot,a)\|\\ \leq&2\tilde{E}_p\left(\ln\frac{2\tilde{E}_p}{\tilde{\epsilon}}\right)^{-p}+2(1-e^{-2a})^{-1}\max\left\{\tilde{\mu}^{2p/3},\frac{a^3}{2}\tilde{\mu}^{2p/3},\frac{a^3}{2}\tilde{\mu}^2\right\}\tilde{E}_p,\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}\tilde{E}_p&=\sqrt{2}\left[2a^{-p}p!\frac{(\nu-2)}{\nu}(2/(\nu-2))^{2/\nu}\tilde{c}^{-\frac{2}{\nu}}+E_p\right],\\ \tilde{\epsilon}&=\delta+\frac{2\tilde{E}_p}{1-e^{-2a}}\nu^{-1}(\nu-1)^{1-\frac{1}{\nu}}\tilde{c}^{\frac{1}{\nu}}\delta^{\frac{1}{2}},\\ \tilde{\mu}&=\left(\ln\left(\frac{2\tilde{E}_p}{\tilde{\epsilon}}(\ln\frac{2\tilde{E}_p}{\tilde{\epsilon}})^{-p}\right)\right)^{-1}.\end{aligned}$$

Proof. Note that $0<y<\sigma<a$ and the filter function q is given by (3.22), it is easy to check

(1) $|q(\alpha,n)\cosh(na)|=\frac{\cosh(na)}{1+\alpha\cosh^{\nu}(na)}$. Denote $f(\xi)=\frac{\cosh(\xi)}{1+\alpha\cosh^{\nu}(\xi)}$. Let $f'(\xi)=0$. Then we have $\cosh^{\nu}(\xi)=\frac{1}{\alpha(\nu-1)}$. Further we can prove that

$$f(\xi)\leq\frac{\nu-1}{\nu(\nu-1)^{\frac{1}{\nu}}}\alpha^{-\frac{1}{\nu}}\qquad\text{for } \xi\geq 0.$$

So in this case, $C_1(\alpha)=\frac{\nu(\nu-1)^{\frac{1}{\nu}}}{\nu-1}\alpha^{\frac{1}{\nu}}$ and $\delta/C_1(\alpha)=\frac{\nu-1}{\nu(\nu-1)^{\frac{1}{\nu}}c^{\frac{1}{\nu}}}$ is bounded;

(2) $|(q(\alpha,n)-1)\frac{\cosh(ny)}{\cosh(na)}| \le 1$, so $K_1(\alpha)=1$ is bounded;

(3) $|q(\alpha,n)| = \frac{1}{1+\alpha\cosh^\nu(na)} \le 1$, thus $C_2(\alpha)=1$ and $\delta/C_2(\alpha)\to 0$ as $\delta\to 0$;

(4) $|\frac{q(\alpha,n)-1}{\cosh(na)}| \le \frac{\alpha\cosh^{\nu-1}(na)}{1+\alpha\cosh^\nu(na)}$. Denote $g(\xi)=\frac{\cosh^{\nu-1}(\xi)}{1+\alpha\cosh^\nu(\xi)}$. Let $g'(\xi)=0$, we have $\cosh^\nu(\xi)=\frac{\nu-1}{\alpha}>0$. Further we can prove that

$$g(\xi)\le \nu^{-1}(\nu-1)^{1-\frac{1}{\nu}}\alpha^{\frac{1}{\nu}-1}$$

for all $\xi\ge 0$. Thus we have

$$\left|\frac{q(\alpha,n)-1)}{\cosh(na)}\right| \le \nu^{-1}(\nu-1)^{1-\frac{1}{\nu}}\alpha^{\frac{1}{\nu}}.$$

Consequently,

$$K_2(\alpha)=\nu^{-1}(\nu-1)^{1-\frac{1}{\nu}}\alpha^{\frac{1}{\nu}}=\nu^{-1}(\nu-1)^{1-\frac{1}{\nu}}c^{\frac{1}{\nu}}\delta\to 0 \quad \text{as } \delta\to 0.$$

By Theorem 2.1, the error estimate (3.23) is obtained.

For the filter function q given by (3.22), we can check that

(i) $|q(\alpha,n)n^p\cosh(na)| \le a^{-p}2p!\frac{\cosh^2(na)}{1+\alpha\cosh^\nu(na)}$. Denote $f(\xi)=\frac{\cosh^2(\xi)}{1+\alpha\cosh^\nu(\xi)}$. Let $f'(\xi)=0$. Then we have $\cos^\nu(\xi)=\frac{2}{\alpha(\nu-2)}$. Further we can prove that

$$Tf(\xi)\le \frac{\nu-2}{\nu}(2/(\nu-2))^{2/\nu}\alpha^{-\frac{2}{\nu}} \qquad \text{for } \xi\ge 0.$$

So in this case, $C_3(\alpha)=\frac{a^p}{2p!}\frac{\nu}{\nu-2}((\nu-2)/\nu)^{2/\nu}\alpha^{\frac{2}{\nu}}$ and $\delta/C_3(\alpha)=a^{-p}2p!\frac{\nu-2}{\nu}(2/(\nu-2))^{2/\nu}\tilde{c}^{-\frac{2}{\nu}}$ is bounded;

(iv) In this case, $K_2(\alpha)=\nu^{-1}(\nu-1)^{1-\frac{1}{\nu}}\alpha^{\frac{1}{\nu}}=\nu^{-1}(\nu-1)^{1-\frac{1}{\nu}}\tilde{c}^{\frac{1}{\nu}}\delta^{1/2}\to 0$ as $\delta\to 0$.

By Theorem 2.2, the estimate (3.24) is obtained. □

4. Numerical example

In this section, we test one numerical example to show the effectiveness of the three special regularization methods.

Consider the following direct problem for the Laplace equation:

$$\Delta u(x,y)=0, \qquad 0<x<\pi, \quad 0<y<a. \tag{4.1}$$

$$u(x,a)=x^2(\pi-x)^2+\sin x\cosh a, \qquad 0\le x\le\pi. \tag{4.2}$$

$$u_y(x,0)=0, \qquad 0\le x\le\pi. \tag{4.3}$$

$$u(0,y)=u(\pi,y)=0, \qquad 0\le y\le a. \tag{4.4}$$

Separation of variables leads to the solution of problem (4.1)–(4.4) as follows,

$$u(x,y)=\sum_{n=1}^{\infty}(u(x,a),X_n)X_n\cosh(ny)(\cosh(na))^{-1}, \tag{4.5}$$

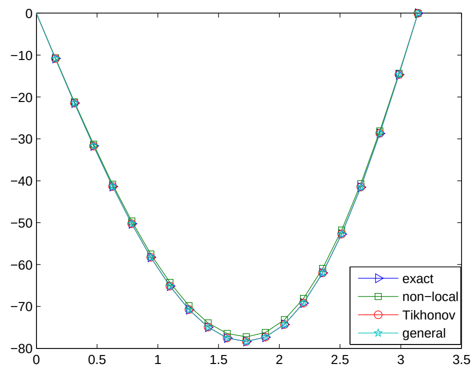
where X_n are given by (2.2).

Then, we choose

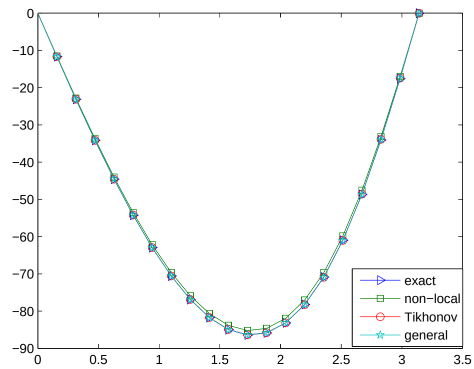
$$\varphi(x)=u(x,0)\approx\sum_{n=1}^{m}(u(x,a),X_n)X_n(\cosh(na))^{-1},$$

as the initial data for problem (1.1)–(1.4) with $m=21$. The measured data $\varphi^\delta(x)$ is given by $\varphi^\delta(x)=\varphi(x)+\varepsilon(3-x)(1-x)$, where ε denotes an error level.

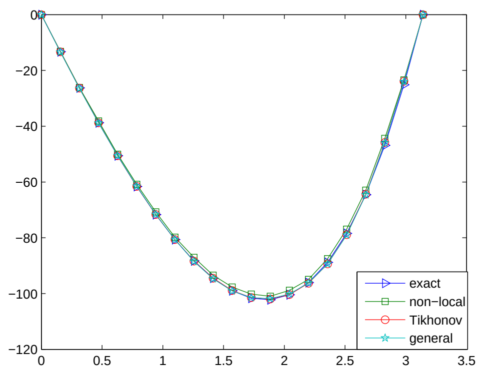
In the following numerical experiments, we always choose $a=1$.



(a) $y=0.2$.

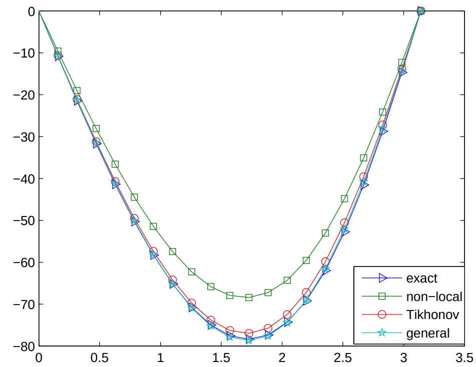


(b) $y=0.5$.

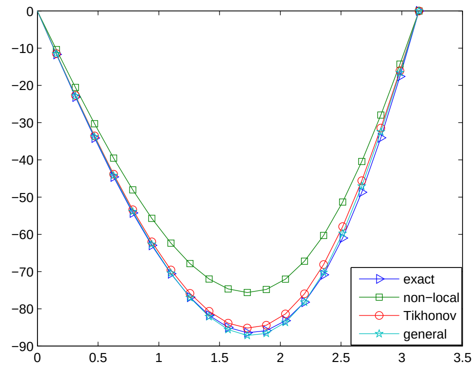


(c) $y=0.8$.

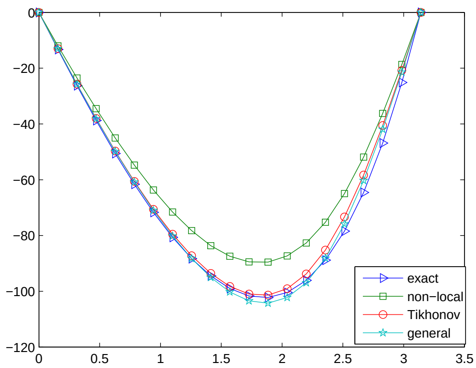
Figure 1: $u(\cdot,y)$ and $u_\alpha^\delta(\cdot,y)$ with $E=8.8$ and $\varepsilon=5\times10^{-3}$.



(a) $y = 0.2$.



(b) $y = 0.5$.



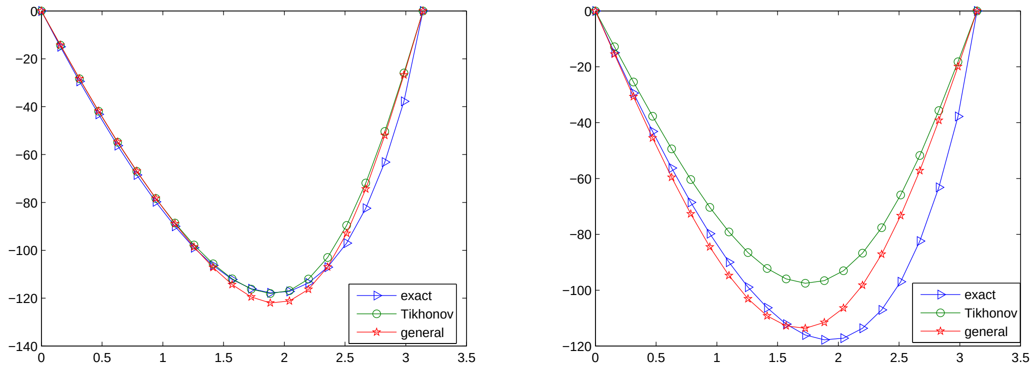
(c) $y = 0.8$.

Figure 2: $u(\cdot, y)$ and $u_\alpha^\delta(\cdot, y)$ with $E = 8.8$ and $\varepsilon = 5 \times 10^{-2}$.

The numerical results for $u(x, y)$ and $u_\alpha^\delta(x, y)$ with $\varepsilon = 5 \times 10^{-3}, 5 \times 10^{-2}$ and $\delta = 1.9\varepsilon$ at $y = 0.2, 0.5, 0.8$ are shown in Figs. 1–2 in which E is chosen by (1.7). In Figs. 1–2, "non-local" denotes numerical result with the filter function $q(\alpha, n)$ given by (3.1) and the regularization parameter $\alpha = \delta$; "Tikhonov" denotes numerical result with the filter function $q(\alpha, n)$ given by (3.7) and the regularization parameter is chosen as $\alpha = \delta^2$; "general" denotes numerical result with the filter function $q(\alpha, n)$ given by (3.22) and the regularization parameter is chosen as $\alpha = \delta^4$.

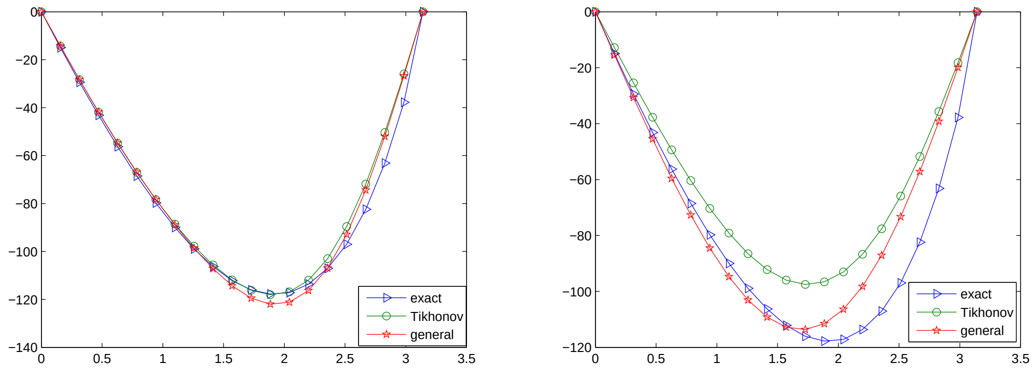
From Figs. 1–2, it can be observed that the proposed regularized methods work effectively. We also note that the numerical results become discouraging when the error level increases which indicates that the proposed methods are sensitive to the noise but have good improvements with the increase of parameter ν. Meanwhile, we find that the numerical results become worse when the value of y approaches to 1.

The numerical results for $u(\cdot, 1)$ and $u_\alpha^\delta(\cdot, 1)$ with $\varepsilon = 5 \times 10^{-3}$ and $\varepsilon = 5 \times 10^{-2}$ are shown in Figs. 3–4 in which E_p is chosen by (1.8).



(a) $\varepsilon = 5 \times 10^{-3}$. **(b)** $\varepsilon = 5 \times 10^{-2}$.

Figure 3: $u(\cdot, 1)$ and $u_\alpha^\delta(\cdot, 1)$ with $p = 1$ and $E_1 = 9.4$.



(a) $\varepsilon = 5 \times 10^{-3}$. **(b)** $\varepsilon = 5 \times 10^{-2}$.

Figure 4: $u(\cdot, 1)$ and $u_\alpha^\delta(\cdot, 1)$ with $p = 2$ and $E_2 = 17.2$.

In Figs. 3 and 4, "Tikhonov" denotes numerical result with the filter function $q(\alpha, n)$ given by (3.7) and the regularization parameter $\alpha = \delta$ and "general" denotes numerical results with the filter function $q(\alpha, n)$ given by (3.22) for $\nu = 4$ and the regularization parameter $\alpha = \delta^2$.

From Figs. 3–4, we observe that the proposed numerical methods are effective and give the good approximations at $y = a$. We also note that the numerical solutions at $y = a$ become discouraging with the increase of the noisy level.

5. Conclusions

In this paper, we consider a Cauchy problem for the Laplace equation in a rectangular domain. Three special regularization methods: the boundary Tikhonov regularization method, the non-local boundary value problem method and a generalized method are investigated. Convergence estimates based on the conditional stabilities are given under different conditions. Some numerical results show that the proposed methods are effective and stable. All proposed methods can be also used to solve the Cauchy problem with the Neumann boundary conditions at $x=0$ and $x=\pi$ without a big difference.

Although we consider the Cauchy problem only in a rectangular domain, since the stability results are also satisfied in a general domain, based on the similar idea, we can presume upon the convergence proof for some regularization methods while solving the Cauchy problem in a general domain. However, to deal with such a case, it will become more difficult because we do not have an explicit expression on solution. The detailed comparison with other available techniques, such as the proposed method in [19,25], will be considered in our future research. This paper indicates that the conditional stability is closely related with the convergence of some regularization methods.

Appendix

Proof of Theorem 2.1*:* The solution of problem (1.1)–(1.4) is given by (2.1), denote $\tau = n/\sqrt{1+\mu^2 n^2 a^2}$, then we have

$$u(x,a) = \sum_{n=1}^{\infty} c_n X_n \left(\cosh(na) - \cosh(\tau a)\right) + \sum_{n=1}^{\infty} c_n X_n \left(\cosh(\tau a)\right), \tag{A.1}$$

and

$$\|u(\cdot,a)\| \le \left[\sum_{n=1}^{\infty} c_n^2 \left(\cosh(na) - \cosh(\tau a)\right)^2\right]^{\frac{1}{2}} + \left[\sum_{n=1}^{\infty} c_n^2 \left(\cosh(\tau a)\right)^2\right]^{\frac{1}{2}}. \tag{A.2}$$

The condition $\|\varphi\| \le \epsilon$ leads to

$$\sum_{n=1}^{\infty} c_n^2 \le \epsilon^2. \tag{A.3}$$

The assumption $\left\|\frac{\partial^p u}{\partial y^p}(\cdot,a)\right\| \le E_p$ means

$$\sum_{n=1}^{\infty} c_n^2 n^{2p} \cosh^2(na) \le E_p^2, \qquad \text{for } p \text{ is even}, \tag{A.4}$$

or

$$\sum_{n=1}^{\infty} c_n^2 n^{2p} \sinh^2(na) \le E_p^2, \qquad \text{for } p \text{ is odd}. \tag{A.5}$$

Combining (A.3) and (A.4) or (A.5), we have estimate

$$\|u(\cdot,a)\| \le \sup_{n\ge 1} A(n) E_p + \sup_{n\ge 1} B(n)\epsilon, \tag{A.6}$$

where $B(n)=\cosh(\tau a)$ and

$$A(n)=\left|\frac{\cosh(na)-\cosh(\tau a)}{n^p\cosh(na)}\right|,\qquad \text{for } p \text{ is even,} \tag{A.7}$$

or

$$A(n)=\left|\frac{\cosh(na)-\cosh(\tau a)}{n^p\sinh(na)}\right|,\qquad \text{for } p \text{ is odd.} \tag{A.8}$$

Similar to the proof of Theorem 2.4 in [22] (P.484) and choosing

$$\mu=\left(\ln\left(\frac{2E_p}{\epsilon}\left(\ln\frac{2E_p}{\epsilon}\right)^{-p}\right)\right)^{-1},$$

we derive that

$$B(n)\epsilon\le 2E_p\left(\ln\frac{2E_p}{\epsilon}\right)^{-p},\qquad \text{for } n\ge 1,$$

which yields

$$\sup_{n\ge1}B(n)\epsilon\le 2E_p\left(\ln\frac{2E_p}{\epsilon}\right)^{-p}.$$

Moreover

$$A(n)E_p\le\max\left\{\mu^{2p/3},\frac{a^3}{2}\mu^{2p/3},\frac{a^3}{2}\mu^2\right\}E_p,\qquad \text{for } p \text{ is even.} \tag{A.9}$$

For odd p, we give the following estimate

$$\begin{aligned}A(n)E_p=&\left|\frac{\cosh(na)-\cosh(\tau a)}{n^p\cosh(na)}\right|\cdot\left|\frac{\cosh(na)}{\sinh(na)}\right|E_p\\ \le&2(1-e^{-2a})^{-1}\left|\frac{\cosh(na)-\cosh(\tau a)}{n^p\cosh(na)}\right|E_p\\ \le&2(1-e^{-2a})^{-1}\max\left\{\mu^{2p/3},\frac{a^3}{2}\mu^{2p/3},\frac{a^3}{2}\mu^2\right\}E_p.\end{aligned} \tag{A.10}$$

Finally, we have

$$\|u(\cdot,a)\|\le 2E_p\left(\ln\frac{2E_p}{\epsilon}\right)^{-p}+2(1-e^{-2a})^{-1}\max\left\{\mu^{2p/3},\frac{a^3}{2}\mu^{2p/3},\frac{a^3}{2}\mu^2\right\}E_p.$$

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