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Stability Analysis of Runge-Kutta Methods for Nonlinear Neutral Volterra Delay-Integro-Differential Equations

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Abstract. This paper is concerned with the numerical stability of implicit Runge-Kutta methods for nonlinear neutral Volterra delay-integro-differential equations with constant delay. Using a Halanay inequality generalized by Liz and Trofimchuk, we give two sufficient conditions for the stability of the true solution to this class of equations. Runge-Kutta methods with compound quadrature rule are considered. Nonlinear stability conditions for the proposed methods are derived. As an illustration of the application of these investigations, the asymptotic stability of the presented methods for Volterra delay-integro-differential equations are proved under some weaker conditions than those in the literature. An extension of the stability results to such equations with weakly singular kernel is also discussed.

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Key words: Neutral differential equations, Volterra delay-integro-differential equations, Runge-Kutta methods, stability.

1. Introduction

Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote a given inner product and the corresponding induced norm in the complex N-dimensional space $\mathbf{C}^N$. In this paper we consider the stability of Runge-Kutta methods (RKMs) for nonlinear neutral Volterra delay-integro-differential equations (NVDIDEs) with constant delay $\tau > 0$,

$$y'(t) = f\left(t, y(t), y(t-\tau), \int_{t-\tau}^{t} K(t,\theta,y(\theta),y'(\theta))d\theta\right), \quad t \geq 0, \tag{1.1}$$

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subject to

$$y(t) = \phi(t), \quad t \in [-\tau, 0], \tag{1.2}$$

where $f : [0,\infty) \times \mathbf{C}^N \times \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{C}^N$ and $K : [0,\infty) \times [-\tau,\infty) \times \mathbf{C}^N \times \mathbf{C}^N \to \mathbf{C}^N$ are continuous functions, ϕ is a given C^1-function.

As special cases of Eq. (1.1), we have delay differential equations (DDEs)

$$y'(t) = f(t, y(t), y(t-\tau)), \quad t \geq 0, \tag{1.3}$$

and Volterra delay-integro-differential equations (VDIDEs)

$$y'(t) = f\left(t, y(t), y(t-\tau), \int_{t-\tau}^{t} K(t,\theta,y(\theta))d\theta\right), \quad t \geq 0. \tag{1.4}$$

Delay differential equations with constant delays have been investigated extensively in the past. For the literature concerned the stability of the true solution and the numerical solution to DDEs, we refer the reader to [19, 27, 34], and the references in [5, 24, 46]. Numerical methods for solving VDIDEs have been also studied by many authors (see [3, 10, 25] and references therein). Using the generalized Halanay inequality proved by Baker and Tang in [2], Zhang and Vandewalle [43] obtained the stability results for the true solution to

$$y'(t) = f\left(t, y(t), G\left(t, y(t-\tau), \int_{t-\tau}^{t} K(t,\theta,y(\theta))d\theta\right)\right), \quad t \geq 0. \tag{1.5}$$

They also investigated the stability of the numerical solution of a discretized form of (1.5). In [44, 45], they further considered the nonlinear stability of RKMs and general linear methods (GLMs) for VDIDEs (1.4), respectively.

There is a growing interest in developing numerical methods for solving NVDIDEs. This class of equations arises in many applications (see [10, 23, 39] and references therein) and often occurs in two forms: the general nonlinear delay IDEs of neutral type

$$y'(t) = f\left(t, y(t), y(t-\tau(t)), y'(t-\tau(t)), \int_{t-\tau(t)}^{t} K(t,\theta,y(\theta),y'(\theta))d\theta\right), \quad t \geq 0, \tag{1.6}$$

and the neutral equations of the "Hale's form"

$$\frac{d}{dt}\left[y(t) - \int_{t-\tau(t)}^{t} K(t,\theta,y(\theta))d\theta\right] = f\left(t, y(t), y(t-\tau(t)), y'(t-\tau(t))\right), \quad t \geq 0, \tag{1.7}$$

where $\tau(t) \leq t$ is a sufficiently smooth function. In [10] (see also [7, 13]), Brunner systematically discussed the existence and uniqueness of the solution to these two forms of equations and the convergence of collocation methods for them. Note that as far back as the 1980's, Jackiewicz gave the convergence results of Adams methods [20] and quasi-linear multi-step methods and variable step predictor-corrector methods [21] for solving more general neutral functional differential equations (NFDEs) which contains problem (1.6) as a particular case. Brunner [7] and Enright and Hu [16] considered the convergence of continuous RKMs for NVDIDEs

$$y'(t) = f(t, y(t)) + \int_{t-\tau}^{t} K(t, \theta, y(\theta), y'(\theta))d\theta, \quad t \geq 0. \tag{1.8}$$

Wang and Li studied the convergence of one-leg and Runge-Kutta methods for (1.1)–(1.2) in [37] and [35] (see also [36]), respectively. In recent years, much attention has been directed toward the stability analysis of numerical methods for NVDIDEs. Brunner and Vermiglio [9] considered the stability of continuous RKMs for NVDIDEs of the "Hale's form". Several researchers investigated the stability of numerical method for nonlinear NVDIDEs where the kernel $K(t, \theta, y)$ does not depend on y' and its linear version (see [40–42, 47–49]).

However, few studies have been done on the stability of numerical methods for nonlinear NVDIDEs (1.1)–(1.2) in which the kernel $K(t, \theta, y, y')$ also depends on y'. The purpose of this paper is to analyze the stability properties of the true solution and the numerical solution to (1.1)–(1.2). To obtain the stability results of the true solution to (1.1)–(1.2), we use the Halanay inequality proved by Liz and Trofimchuk [29] in Section 2. In Section 3, we consider to discretize Eqs. (1.1)–(1.2) by RKM with the compound quadrature rule (CQ). The nonlinear stability of RKM with this class of quadrature technique is studied in Section 4. In Section 5, some examples for the application of the theories obtained in the present paper are considered. An extension of the stability results to weakly singular NVDIDEs (1.1)–(1.2) is also discussed in this section. Finally, in Section 6 we provide some numerical examples to illustrate our results.

2. Stability properties of the true solution

Before stating the main results on the stability of the true solution, we mention the regularity of solution to (1.1)–(1.2). From the theoretical analysis in [11] (see also [10]), we find that when f, K and ϕ are sufficiently smooth, the smoothing properties of the solution to (1.1)–(1.2) is determined by the "DDE part", and is not influenced by the integral $\int_{t-\tau}^{t} K(t, \theta, y(\theta), \phi'(\theta))d\theta$. In fact, we can easily show that if f, K and ϕ in (1.1)–(1.2) are C^d-functions on their respective domains, then: (i) The (unique) solution of the initial-value problem for (1.1)–(1.2) is $(d+1)-$times continuously differentiable on each left-open macro-interval $((i-1)\tau, i\tau]$ $i = 1, 2, \cdots$, and has a bounded first derivative on $[0, \infty)$; (ii) At $t = i\tau$ $(i = 0, 1, \cdots, d)$ we have $\lim_{t\to i\tau^-} y^{(i)}(t) = \lim_{t\to i\tau^+} y^{(i)}(t)$, while the $(i+1)$st derivative of y is in general not continuous at $t = i\tau$. The solution possesses a continuous $(d+1)$st derivative on $[d\tau, \infty)$.

In this paper we restrict ourselves to discussion of problem (1.1) where f satisfies a one-sided Lipschitz condition and some Lipschitz conditions. For brevity, the class of problems (1.1)–(1.2) with f and K satisfying

$$\mathscr{R}e\langle f(t,y_1,u,v)-f(t,y_2,u,v),y_1-y_2\rangle \le \alpha\|y_1-y_2\|^2, \tag{2.1}$$
$$\left\|f(t,y,u_1,v_1)-f(t,y,u_2,v_2)\right\| \le \beta\|u_1-u_2\|+\gamma\|v_1-v_2\|, \tag{2.2}$$
$$\begin{aligned}&\left\|K(t,\theta,y_1,f(\theta,y_1,u,v))-K(t,\theta,y_2,f(\theta,y_2,u,v))\right\|\\ &\qquad \le L_K\|y_1-y_2\|, \qquad (t,\theta)\in\mathbf{D}, \quad \theta\ge 0,\end{aligned} \tag{2.3}$$
$$\left\|K(t,\theta,y,u_1)-K(t,\theta,y,u_2)\right\| \le \mu\|u_1-u_2\|, \qquad (t,\theta)\in\mathbf{D}, \tag{2.4}$$

is denoted by $\mathscr{D}(\alpha,\beta,\gamma,L_K,\mu)$, where $\alpha,\beta,\gamma,L_K,\mu$ are real constants, $t\in[0,+\infty)$; $\mathbf{D}=\{(t,\theta): t\in[0,+\infty),\theta\in[t-\tau,t]\}$; $y,y_1,y_2,u,u_1,u_2,v,v_1,v_2\in\mathbf{C}^N$.

The class of problems (1.1)–(1.2) with f and K satisfying (2.1)–(2.2), (2.4) and

$$\left\|f(t,y_1,u,v)-f(t,y_2,u,v)\right\| \le L_y\|y_1-y_2\|, \tag{2.5}$$
$$\left\|K(t,\theta,y_1,u)-K(t,\theta,y_2,u)\right\| \le L_\mu\|y_1-y_2\|, \qquad (t,\theta)\in\mathbf{D}, \tag{2.6}$$

is denoted by $\mathscr{L}(\alpha,\beta,\gamma,L_y,L_\mu,\mu)$, where $\alpha,\beta,\gamma,L_y,L_\mu,\mu$ are real constants, $t\in[0,+\infty)$; $y,y_1,y_2,u,v\in\mathbf{C}^N$.

Our assumptions that conditions (2.2) and (2.4)–(2.6) are satisfied ensure that system (1.1)–(1.2) possesses a unique solution (see [17,23]). Although conditions (2.1)–(2.4) do not ensure that system (1.1)–(1.2) possesses a unique solution, the existence of a unique solution of (1.1)–(1.2) will be assumed.

Remark 2.1. (i) The class $D_{p,q}$ for DDEs introduced by Huang *et al.* [19] can be viewed as the class $\mathscr{D}(p,q,0,0,0)$ for NVDIDEs.

(ii) The class $RI(\alpha,\beta,\sigma,\gamma)$ for VDIDEs introduced by Zhang and Vandewalle [44] can be viewed as the class $\mathscr{D}(\alpha,\beta,\sigma,\gamma,0)$ for NVDIDEs.

(iii) Obviously, $\mathscr{L}(\alpha,\beta,\gamma,L_y,L_\mu,\mu)$ is a sub-class of $\mathscr{D}(\alpha,\beta,\gamma,L_K,\mu)$. The following simple example illustrates that they are not identical. System

$$y'(t)=-(e^t+5)y(t)+y(t-1)+0.1\left[\int_{t-1}^{t}\sin\theta\left(e^\theta y(\theta)+y'(\theta)\right)d\theta\right], \quad t\ge 0, \tag{2.7}$$
$$y(t)=\phi(t), \qquad t\le 0 \tag{2.8}$$

belongs to the class $\mathscr{D}(\alpha,\beta,\gamma,L_K,\mu)$ with $\alpha=-6$, $\beta=1$, $\gamma=0.1$, $L_K=-5$, $\mu=1$ but not in the class $\mathscr{L}(\alpha,\beta,\gamma,L_y,L_\mu,\mu)$.

Before stating our main results in this section, we need the following Halanay inequality.

Lemma 2.1. ([29,38]) *Consider inequalities*

$$u'(t)\le -Au(t)+B\max_{\theta\in[t-\tau,t]}u(\theta)+C\max_{\theta\in[t-\tau,t]}w(\theta), \qquad t\ge t_0, \tag{2.9}$$
$$w(t)\le G\max_{\theta\in[t-\tau,t]}u(\theta)+H\max_{\theta\in[t-\tau,t]}w(\theta), \qquad t\ge t_0, \tag{2.10}$$

where t_0 is a constant. If $A,\ B,\ C,\ G,\ H \geq 0$ and $H < 1$, then for every $\epsilon > 0$, there exist $\delta(\epsilon) \to \delta+ < 0,\ \epsilon \to 0+$, such that

$$u(t) \leq (1+\epsilon) \max_{\theta \in [t_0-\tau, t_0]} u(\theta) e^{\delta(\epsilon)(t-t_0)}, \qquad t \geq t_0, \tag{2.11}$$

$$w(t) \leq (1+\epsilon) \max_{\theta \in [t_0-\tau, t_0]} w(\theta) e^{\delta(\epsilon)(t-t_0)}, \qquad t \geq t_0 \tag{2.12}$$

for every nonnegative solution $(u, w) : [t_0 - \tau, +\infty) \to \mathbf{R}_+^2$ of the inequality (2.9)–(2.10) if and only if

$$-A + B + \frac{CG}{1-H} < 0. \tag{2.13}$$

Theorem 2.1. *Suppose problem (1.1) belongs to the class $\mathscr{D}(\alpha, \beta, \gamma, L_K, \mu)$ with*

$$\gamma\tau\mu < 1, \qquad \alpha + \frac{\beta + \gamma\tau L_K}{1 - \gamma\tau\mu} < 0. \tag{2.14}$$

Then we have

$$\|y(t) - z(t)\| \leq \max_{\theta \in [0,\tau]} \|y(\theta) - z(\theta)\|, \qquad t \geq \tau \tag{2.15}$$

and

$$\lim_{t \to +\infty} \|y(t) - z(t)\| = 0, \tag{2.16}$$

where and in what follows, $z(t)$ denotes a solution of the perturbed problem

$$z'(t) = f\left(t, z(t), z(t-\tau), \int_{t-\tau}^{t} K(t, \theta, z(\theta), z'(\theta)) d\theta\right), \qquad t \geq 0, \tag{2.17}$$

$$z(t) = \psi(t), \qquad t \in [-\tau, 0]. \tag{2.18}$$

Here the initial function $\psi(t)$ is continuously differentiable.

Proof. For simplicity, let us introduce the notation $Y(t) = \|y(t) - z(t)\|$ and

$$\begin{aligned} \Phi(t) = \bigg\| & f\left(t, z(t), y(t-\tau), \int_{t-\tau}^{t} K(t, s, y(s), y'(s)) ds\right) \\ & - f\left(t, z(t), z(t-\tau), \int_{t-\tau}^{t} K(t, s, z(s), z'(s)) ds\right) \bigg\|. \end{aligned}$$

It follows that

$$Y'(t) \leq \alpha Y(t) + \Phi(t), \qquad t \geq 0. \tag{2.19}$$

It follows from (2.1)–(2.4) that

$$\begin{aligned}\Phi(t) \le&\beta Y(t-\tau)+\gamma\tau \max_{s\in[t-\tau,t]}\|K(t,s,y(s),y'(s))-K(t,s,z(s),z'(s))\|\\ \le&\beta Y(t-\tau)+\gamma\tau \max_{s\in[t-\tau,t]}\left\|K\left(t,s,y(s),f\left(s,y(s),y(s-\tau),\int_{s-\tau}^{s}K(s,r,y(r),y'(r))dr\right)\right)\right.\\ &\left.-K\left(t,s,z(s),f\left(s,z(s),z(s-\tau),\int_{s-\tau}^{s}K(s,r,z(r),z'(r))dr\right)\right)\right\|\\ \le&\beta Y(t-\tau)+\gamma\tau L_K \max_{s\in[t-\tau,t]}Y(s)+\gamma\tau\mu \max_{s\in[t-\tau,t]}\Phi(s)\\ \le&[\beta+\gamma\tau L_K]\max_{s\in[t-\tau,t]}Y(s)+\gamma\tau\mu\max_{s\in[t-\tau,t]}\Phi(s), \qquad t\ge\tau. \end{aligned} \tag{2.20}$$

Note that condition (2.14) implies $\alpha<0$. By virtue of Lemma 2.1, it is sufficient to prove the theorem from (2.19)–(2.20).

Theorem 2.2. *Suppose problem* (1.1) *belongs to the class* $\mathscr{L}(\alpha,\beta,\gamma,L_y,L_\mu,\mu)$ *with*

$$\gamma\tau\mu<1, \qquad \alpha+\frac{\beta+\gamma\tau(L_\mu+\mu L_y)}{1-\gamma\tau\mu}<0. \tag{2.21}$$

Then we have (2.16) *and*

$$\|y(t)-z(t)\|\le \max_{\theta\in[-\tau,0]}\|\phi(\theta)-\psi(\theta)\|, \ \ t\ge 0. \tag{2.22}$$

Proof. Define $\tilde{Y}(t)=\|y'(t)-z'(t)\|$. Note that

$$\begin{aligned} Y'(t)\le&\alpha Y(t)+\beta Y(t-\tau)+\gamma\tau L_\mu\max_{s\in[t-\tau,t]}Y(s)+\gamma\tau\mu\max_{s\in[t-\tau,t]}\tilde{Y}(s), \qquad t\ge 0,\\ \tilde{Y}(t)\le&L_y Y(t)+\beta Y(t-\tau)+\gamma\tau L_\mu\max_{s\in[t-\tau,t]}Y(s)+\gamma\tau\mu\max_{s\in[t-\tau,t]}\tilde{Y}(s), \qquad t\ge 0.\end{aligned}$$

The proof of the theorem follows directly from the above facts. □

Remark 2.2. (i) Observe that (2.15) differ from (2.22) in their right-hand side. If we note condition (2.3) and the proofs of Theorems 2.1 and 2.2, then we find that this is because (1.1) holds only for $t\ge 0$ and this fact is used to assume that condition (2.3) holds. If we assume that (1.1) holds for $t\ge-\tau$, then (2.20) holds for $t\ge 0$ and thus (2.22) holds under the conditions of Theorem 2.1.

(ii) It should be pointed out that if the kernel K does not depend on y', that is, Eq. (1.1) is non-neutral VDIDEs, then condition (2.14) is equivalent to condition (2.21) and our results is identical to the results obtained in [44].

(iii) If $K\equiv 0$ in (1.1), Theorems 2.1 and 2.2 reduce to a well-known stability result for DDEs: $\alpha+\beta<0$ implies that the solution to (1.2)–(1.3) is asymptotically stable [34].

We conclude this section with three examples.

Example 2.1. The above theorems can be applied to (1.8), the special but important class of equations, see, e.g., [7, 10, 16]). When the functions f and K in (1.8) satisfy

$$\mathscr{R}e\langle f(t,y_1)-f(t,y_2),y_1-y_2\rangle \le \alpha\|y_1-y_2\|^2, \tag{2.23}$$

$$\Big\|K(t,\theta,y_1,f(\theta,y_1)+u)-K(t,\theta,y_2,f(\theta,y_2)+u)\Big\| \le L_K\|y_1-y_2\|, \quad (t,\theta)\in\mathbf{D}, \quad \theta\ge 0, \tag{2.24}$$

$$\Big\|K(t,\theta,y,u_1)-K(t,\theta,y,u_2)\Big\| \le \mu\|u_1-u_2\|, \qquad (t,\theta)\in\mathbf{D}, \tag{2.25}$$

this class of equations belongs to class $\mathscr{D}(\alpha,0,1,L_K,\mu)$. Thus, they satisfy stability properties (2.15)–(2.16) whenever

$$\tau\mu<1, \qquad \alpha+\frac{\tau L_K}{1-\tau\mu}<0.$$

When the functions f and K in (1.8) satisfy (2.23), (2.25) and

$$\|f(t,y_1)-f(t,y_2)\| \le L_y\|y_1-y_2\|, \tag{2.26}$$

$$\|K(t,\theta,y_1,u)-K(t,\theta,y_2,u)\| \le L_\mu\|y_1-y_2\|, \qquad (t,\theta)\in\mathbf{D}, \tag{2.27}$$

this class of equations belongs to class $\mathscr{L}(\alpha,0,1,L_y,L_\mu,\mu)$. Consequently, they satisfy stability properties (2.16) and (2.22) whenever

$$\tau\mu<1, \ \ \alpha+\frac{\tau(L_\mu+\mu L_y)}{1-\tau\mu}<0.$$

Example 2.2. Consider the Hammerstein type of equation (cf. [10])

$$y'(t)=f(t,y(t))+\int_{t-\tau}^{t} K_\sigma(t-\theta)G(y'(\theta))d\theta, \qquad t\ge 0 \tag{2.28}$$

subject to (1.2), where

$$K_\sigma(t-\theta)=\begin{cases} K_0(t-\theta), & \text{if } \sigma=0, \text{ where } K_0 \text{ is smooth,}\\ (t-\theta)^\sigma, & \text{if } 0<\sigma<1. \end{cases} \tag{2.29}$$

On the basis of Theorem 2.2, we can assert that the solution to problem (2.28) with initial condition (1.2) is asymptotically stable if conditions (2.23), (2.26),

$$\|G(u_1)-G(u_2)\| \le \tilde{\mu}\|u_1-u_2\|, \tag{2.30}$$

and

$$\tau\mu<1, \qquad \alpha+\frac{\tau\mu L_y}{1-\tau\mu}<0 \tag{2.31}$$

are satisfied, where $\mu=\tilde{\mu}\sup_{(t,\theta)\in\mathbf{D}}K_\sigma(t-\theta)$.

Example 2.3. Condition (2.14) in Theorem 2.1 and condition (2.21) in Theorem 2.2 are similar. The difference between the two conditions is that L_K is replaced by $L_\mu + \mu L_y$. However, in general, condition (2.14) is much weaker than condition (2.21) since $L_K \le L_\mu + \mu L_y$. For example, consider the system

$$\begin{aligned} y'(t) = & L(t)y(t) + M(t)y(t-\tau) \\ & + N(t)\left[\int_{t-\tau}^{t} \big(A(\theta)y(\theta) + B(\theta)y'(\theta)\big)\,d\theta\right], \qquad t \ge 0, \end{aligned} \tag{2.32a}$$

$$y(t) = \phi(t), \qquad t \le 0, \tag{2.32b}$$

where $L(t), M(t), N(t), A(\theta), B(\theta)$ denote complex matrices functions. It is easy to verify that it belongs to the class $\mathscr{D}(\alpha, \beta, \gamma, L_K, \mu)$, where

$$\begin{aligned} &\alpha = \sup_{t\ge 0} \mu[L(t)], \qquad \beta = \sup_{t\ge 0} \|M(t)\|, \qquad \gamma = \sup_{t\ge 0} \|N(t)\|, \\ &L_K = \sup_{\theta \ge 0} \|A(\theta) + B(\theta)L(\theta)\|, \qquad \mu = \sup_{\theta \ge -\tau} \|B(\theta)\|, \end{aligned}$$

and $\mu[\cdot]$ stands for the logarithmic norm induced by $\langle \cdot, \cdot \rangle$. It also belongs to the class $\mathscr{L}(\alpha, \beta, \gamma, L_y, L_\mu, \mu)$, where

$$L_y = \sup_{t \ge 0} \|L(t)\|, \qquad L_\mu = \sup_{\theta \ge -\tau} \|A(\theta)\|.$$

Obviously, $L_K \le L_\mu + \mu L_y$, which means that condition (2.14) is much weaker than condition (2.21).

3. Runge-Kutta discretization

Now we approximate the solution of (1.1)–(1.2) numerically using a fixed time-stepping RKM. Let (A, b^T, c) denote a given s stage RKM with $s \times s$ matrix $A = (a_{ij})$ and vectors $b = [b_1, \cdots, b_s]^T$, $c = [c_1, \cdots, c_s]^T$. In this paper we will always assume that the method is consistent, which implies that $\sum_{i=1}^{s} b_i = 1$, and satisfies $c_i \in [0, 1]$, $i = 1, 2, \cdots, s$. Let $h = \tau/m > 0$ be the fixed step-size, where integer $m \ge 1$. Then the RKM for NVDIDEs (1.1)–(1.2) has the form

$$Y_i^{(n)} = y_n + h \sum_{j=1}^{s} a_{ij} f(t_{n,j}, Y_j^{(n)}, Y_j^{(n-m)}, K_j^{(n)}), \qquad i = 1, 2, \cdots, s, \tag{3.1}$$

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f(t_{n,j}, Y_j^{(n)}, Y_j^{(n-m)}, K_j^{(n)}), \qquad n \ge 0. \tag{3.2}$$

Here $t_{n,j} = t_n + c_j h$, and y_n approximates the true solution $y(t_n)$ at $t_n = nh$, in particular, $y_0 = \phi(0)$. The argument $Y_j^{(n)}$ denotes an approximation to $y(t_n + c_j h)$, and the argument $K_j^{(n)}$ denotes an approximation to $\int_{t_{n-m,j}}^{t_{n,j}} K(t, \theta, y(\theta), y'(\theta))d\theta$ that is obtained by the compound quadrature formula (CQ formula)

$$K_j^{(n)} = h\sum_{i=0}^{m} v_i K\big(t_{n,j}, t_{n-i,j}, Y_j^{(n-i)}, \tilde{Y}_j^{(n-i)}\big), \tag{3.3}$$

where $\tilde{Y}_j^{(n-i)}$ is an approximate value of $y'(t_{n-i}+c_jh)$ and is produced by

$$\tilde{Y}_j^{(n-i)} = f\big(t_{n-i,j}, Y_j^{(n-i)}, Y_j^{(n-m-i)}, K_j^{(n-i)}\big). \tag{3.4}$$

When $-m \le n \le -1$, $Y_j^{(n)}$ and $\tilde{Y}_j^{(n)}$ are given by

$$Y_j^{(n)} = \phi(t_n + c_jh), \qquad \tilde{Y}_j^{(n)} = \phi'(t_n + c_jh). \tag{3.5}$$

RKMs (3.1)–(3.2) with CQ formula (CQRKMs) has been applied to integro-differential equations (IDEs) and VDIDEs by many authors [1,6,43,44]. The convergence of this class of RKMs for NVDIDEs (1.1)–(1.2) has been reported by Wang and Li in [35]. So, in this paper we consider only the stability of this class of RKMs for NVDIDEs (1.1)–(1.2).

We will assume throughout the paper that for implicit equations (3.1) there always exists a unique solution $[Y_1^{(n)T}, Y_2^{(n)T}, \cdots, Y_s^{(n)T}]^T \in \mathbf{C}^{Ns}$. In Section 6, we will discuss the iterative scheme required to solve the nonlinear implicit equations (3.1).

The following definition can be found in [14].

Definition 3.1. *Let k,l be real constants. A Runge-Kutta method (A, b^T, c) is said to be (k,l)-algebraically stable if there exists a diagonal non-negative matrix $D = diag(d_1, d_2, \cdots, d_s)$ such that $\mathscr{M} = [\mathscr{M}_{ij}]$ is non-negative definite, where*

$$\mathscr{M} = \begin{bmatrix} k-1-2le^TDe & e^TD - b^T - 2le^TDA \\ De - b - 2lA^TDe & DA + A^TD - bb^T - 2lA^TDA \end{bmatrix}, \tag{3.6}$$

and $e = [1, 1, \cdots, 1]^T$. In particular, a $(1,0)$-algebraically stable method is called algebraically stable.

4. Stability of the numerical solution

In this section, we focus on the stability analysis of (k,l)- algebraically stable RKMs with respect to the nonlinear problem classes $\mathscr{D}(\alpha,\beta,\gamma,L_K,\mu)$ and $\mathscr{L}(\alpha,\beta,\gamma,L_y,L_\mu,\mu)$. Let $y_n \in \mathbf{C}^N$ and $z_n \in \mathbf{C}^N$ be the numerical solutions produced by the CQRKM (3.1)–(3.4) applied to (1.1)–(1.2) and (2.17)–(2.18), respectively. $Z_j^{(n)}$ and $\tilde{K}_j^{(n)}$ denote the approximations to $z(t_n + c_jh)$ and $\int_{t_{n-m,j}}^{t_{n,j}} K(t,\theta,z(\theta),z'(\theta))d\theta$, respectively. It is convenient to introduce the following notations:

$$\omega_n = y_n - z_n, \quad W_j^{(n)} = Y_j^{(n)} - Z_j^{(n)}, \quad \tilde{W}_j^{(n)} = \tilde{Y}_j^{(n)} - \tilde{Z}_j^{(n)},$$
$$Q_j^{(n)} = h\Big[f(t_{n,j}, Y_j^{(n)}, Y_j^{(n-m)}, K_j^{(n)}) - f(t_{n,j}, Z_j^{(n)}, Z_j^{(n-m)}, \tilde{K}_j^{(n)})\Big], \qquad j = 1, \cdots, s.$$

Then it follows from (3.1)–(3.2) that

$$W_i^{(n)} = \omega_n + \sum_{j=1}^{s} a_{ij} Q_j^{(n)}, \qquad i = 1, \cdots, s, \tag{4.1}$$

$$\omega_{n+1} = \omega_n + \sum_{j=1}^{s} b_j Q_j^{(n)}. \tag{4.2}$$

In what follows, the notation

$$\nu = \max\left\{ |\nu_0| + |\nu_m|, \max_{1 \le i \le m-1} |\nu_i| \right\}$$

is frequently used.

Lemma 4.1. *Assume that a (k,l)-algebraically stable RKM (A, b^T, c) is applied to the problem* (1.1)–(1.2) *and its perturbed problem* (2.17)–(2.18) *which satisfy condition* (2.1). *Then*

$$\|\omega_{n+1}\|^2 \le k\|\omega_n\|^2 + 2\sum_{j=1}^{s} d_j \left[(\alpha h - l)\|W_j^{(n)}\|^2 + h\|W_j^{(n)}\|\Phi_j^{(n)} \right], \tag{4.3}$$

where $\Phi_j^{(n)}$ is defined by

$$\Phi_j^{(n)} = \left\| f(t_{n,j}, Z_j^{(n)}, Y_j^{(n-m)}, K_j^{(n)}) - f(t_{n,j}, Z_j^{(n)}, Z_j^{(n-m)}, \tilde{K}_j^{(n)}) \right\|.$$

Proof. The (k,l)-algebraic stability of the method implies (see, for example, [14, 26])

$$\|\omega_{n+1}\|^2 \le k\|\omega_n\|^2 + 2\sum_{j=1}^{s} d_j \mathscr{R}e\langle W_j^{(n)}, Q_j^{(n)} - lW_j^{(n)} \rangle. \tag{4.4}$$

On the other hand, from (2.1), we have

$$\mathscr{R}e\langle W_j^{(n)}, Q_j^{(n)} \rangle \quad \le \quad h\alpha\|W_j^{(n)}\|^2 + h\|W_j^{(n)}\|\Phi_j^{(n)}. \tag{4.5}$$

Inserting (4.5) into (4.4), we have (4.3) and complete the proof of the lemma. □

4.1. Stability analysis for $\mathscr{D}(\alpha, \beta, \gamma, L_K, \mu)$

We first give a lemma which gives a upper bound for $\Phi_j^{(n)}$ defined in Lemma 4.1.

Lemma 4.2. *Assume that the problem* (1.1)–(1.2) *belongs to the class $\mathscr{D}(\alpha, \beta, \gamma, L_K, \mu)$. Then for any $n \ge m$, there exist integers $\varpi \ge 0$, $0 < r_i \le m$ $(i = 1, 2, \cdots, \varpi)$, such that one of the* *following inequalities holds:*

$$\Phi_j^{(n)} \le (\gamma\mu\nu\tau)^{\varpi}\Phi_j^{\left(n-\sum_{i=1}^{\varpi} r_i\right)} + \sum_{i=0}^{\varpi-1}(\gamma\mu\nu\tau)^i \left(\beta \left\| W_j^{\left(n-m-\sum_{q=0}^{i} r_q\right)} \right\| + \gamma h L_K \sum_{r=0}^{m} |\nu_r| \left\| W_j^{\left(n-r-\sum_{q=0}^{i} r_q\right)} \right\| \right), \qquad n - \sum_{i=1}^{\varpi} r_i < m; \tag{4.6}$$

$$\begin{aligned}\Phi_j^{(n)} \le & \frac{(\gamma\mu\nu\tau)^{\varpi}}{1-\gamma\mu\nu\tau}\left(\beta \left\| W_j^{\left(n-m-\sum_{q=0}^{\varpi} r_q\right)} \right\| + \gamma h L_K \sum_{r=0}^{m} |\nu_r| \left\| W_j^{\left(n-r-\sum_{q=0}^{\varpi} r_q\right)} \right\| \right) \\ & + \sum_{i=0}^{\varpi-1}(\gamma\mu\nu\tau)^i \left(\beta \left\| W_j^{\left(n-m-\sum_{q=0}^{i} r_q\right)} \right\| \right. \\ & \left. + \gamma h L_K \sum_{r=0}^{m} |\nu_r| \left\| W_j^{\left(n-r-\sum_{q=0}^{i} r_q\right)} \right\| \right), \qquad n - \sum_{i=1}^{\varpi} r_i \ge m,\end{aligned} \tag{4.7}$$

where $r_0 = 0$. Here and in what follows, we shall always assume $\sum_{r=j}^{i} = 0$ whenever $i < j$.

Proof. We consider two cases. First, for any $0 \le i \le n-m$, if $\max_{0\le r\le m}\Phi_j^{(n-r-i)} = \Phi_j^{(n-i)}$, we have

$$\Phi_j^{(n-i)} \le \beta\|W_j^{(n-m-i)}\| + \gamma h L_K \sum_{r=0}^{m}|\nu_r|\|W_j^{(n-r-i)}\| + \gamma\tau\nu\mu\Phi_j^{(n-i)},$$

and therefore

$$\Phi_j^{(n-i)} \le \frac{\beta}{1-\gamma\nu\mu\tau}\|W_j^{(n-m-i)}\| + \frac{\gamma h L_K}{1-\gamma\nu\mu\tau}\sum_{r=0}^{m}|\nu_r|\|W_j^{(n-r-i)}\|. \tag{4.8}$$

Second, if $\max_{0\le r\le m}\Phi_j^{(n-r-i)} = \Phi_j^{(n-r_q-i)}$ with $0 < r_q \le m$, we have

$$\Phi_j^{(n-i)} \le \beta\|W_j^{(n-m-i)}\| + \gamma h L_K \sum_{r=0}^{m}|\nu_r|\|W_j^{(n-r-i)}\| + \gamma\tau\nu\mu\Phi_j^{(n-r_q-i)}. \tag{4.9}$$

As a special case, for $\Phi_j^{(n)}$, we have (4.8) or (4.9) too. If (4.8) holds, we have (4.7) with $\varpi = 0$; if (4.9) holds, by induction we can obtain (4.6) or (4.7). ☐

Theorem 4.1. *Suppose the RKM (A, b^T, c) is (k,l)-algebraically stable for a non-negative diagonal matrix $D = diag(d_1, d_2, \cdots, d_s) \in \mathbf{R}^{s\times s}$, where $0 < k \le 1$. Then, the numerical solution produced by the CQRKM for the class $\mathscr{D}(\alpha,\beta,\gamma,L_K,\mu)$ satisfies*

$$\begin{aligned}\|\omega_n\|^2 \le & \|\omega_m\|^2 + \sum_{j=1}^{s} d_j \left[\frac{3\tau(\beta+\gamma\tau\nu L_K)}{1-\gamma\tau\mu\nu}\max_{-m\le i<m}\|W_j^{(i)}\|^2 \right. \\ & \left. + \frac{\tau}{(1-\gamma\tau\mu\nu)(\beta+\gamma\tau\nu L_K)}\max_{0\le i<m}\left(\Phi_j^{(i)}\right)^2\right], \qquad n \ge m,\end{aligned} \tag{4.10}$$

whenever

$$\gamma\tau\mu\nu < 1, \qquad \left[\alpha + \frac{\beta + \gamma\tau\nu L_K}{1-\gamma\tau\mu\nu}\right] h \le l. \tag{4.11}$$

Proof. We consider two cases.

Case 1. Inequality (4.6) holds. In this case, substituting (4.6) into (4.3) and using Cauchy inequality, we get

$$\begin{aligned}\|\omega_{n+1}\|^2 \le & k\|\omega_n\|^2 + \sum_{j=1}^{s} d_j \left[2(\alpha h - l)\|W_j^{(n)}\|^2 + \frac{h(\beta+\gamma\tau\nu L_K)}{1-\gamma\tau\mu\nu}\|W_j^{(n)}\|^2\right. \\ & + h\sum_{i=0}^{\varpi-1}(\gamma\mu\nu\tau)^i \left.\left(\beta\left\|W_j^{\left(n-m-\sum_{q=0}^{i} r_q\right)}\right\|^2 + \gamma h L_K \sum_{r=0}^{m}|\nu_r|\left\|W_j^{\left(n-r-\sum_{q=0}^{i} r_q\right)}\right\|^2\right)\right] \\ & + \frac{h(\gamma\mu\nu\tau)^{\varpi}}{\beta+\gamma\tau\nu L_K}\max_{0\le i<m}\left(\Phi_j^{(i)}\right)^2. \end{aligned} \tag{4.12}$$

Noting $0 < k \le 1$, by induction one further gives

$$\begin{aligned}\|\omega_{n+1}\|^2 \le & \|\omega_m\|^2 + \sum_{j=1}^{s} d_j \left[\left(2\alpha h + \frac{h(\beta+\gamma\tau\nu L_K)}{1-\gamma\tau\mu\nu} - 2l\right)\sum_{i=m}^{n}\|W_j^{(i)}\|^2\right. \\ & + \frac{\tau}{(\beta+\gamma\tau\nu L_K)(1-\gamma\tau\mu\nu)}\max_{0\le i<m}\left(\Phi_j^{(i)}\right)^2 \\ & \left. + \frac{h}{1-\gamma\mu\nu\tau}\left(\beta\sum_{i=0}^{n-m}\left\|W_j^{(i)}\right\|^2 + \gamma\tau\nu L_K\sum_{i=-m}^{n}\left\|W_j^{(i)}\right\|^2\right)\right]. \end{aligned} \tag{4.13}$$

An application of condition (4.11) yields (4.10).

Case 2. Inequality (4.7) holds. In this case, noting that

$$\frac{(\gamma\tau\mu\nu)^{\varpi}}{1-\gamma\tau\mu\nu} + \sum_{i=0}^{\varpi-1}(\gamma\tau\mu\nu)^i = \frac{1}{1-\gamma\tau\mu\nu}, \tag{4.14}$$

substitute (4.7) into (4.3) and use Cauchy inequality to give

$$\begin{aligned}\|\omega_{n+1}\|^2 \le & k\|\omega_n\|^2 + \sum_{j=1}^{s} d_j \left[2(\alpha h - l)\|W_j^{(n)}\|^2 + \frac{h(\beta+\gamma\tau\nu L_K)}{1-\gamma\tau\mu\nu}\|W_j^{(n)}\|^2\right. \\ & + h\frac{(\gamma\mu\nu\tau)^{\varpi}}{1-\gamma\mu\nu\tau}\left(\beta\left\|W_j^{\left(n-m-\sum_{q=0}^{\varpi} r_q\right)}\right\|^2 + \gamma h L_K\sum_{r=0}^{m}|\nu_r|\left\|W_j^{\left(n-r-\sum_{q=0}^{\varpi} r_q\right)}\right\|^2\right) \\ & + h\sum_{i=0}^{\varpi-1}(\gamma\mu\nu\tau)^i\left.\left(\beta\left\|W_j^{\left(n-m-\sum_{q=0}^{i} r_q\right)}\right\|^2 + \gamma h L_K\sum_{r=0}^{m}|\nu_r|\left\|W_j^{\left(n-r-\sum_{q=0}^{i} r_q\right)}\right\|^2\right)\right]. \end{aligned} \tag{4.15}$$

Then an induction to (4.15) generates the following result

$$\|\omega_{n+1}\|^2 \le \|\omega_m\|^2,$$

where we have used the conditions $0 < k \le 1$ and (4.11).

This means (4.10) holds for any $n \ge m$, which completes the proof of Theorem 4.1. □

Now we consider the asymptotic stability of CQRKMs and give the following theorem.

Theorem 4.2. *Suppose the RKM (A, b^T, c) is (k,l)-algebraically stable for a non-negative diagonal matrix $D = diag(d_1, d_2, \cdots, d_s) \in \mathbf{R}^{s\times s}$, where $0 < k < 1$. Then the numerical solution produced by the CQRKM for the class $\mathscr{D}(\alpha,\beta,\gamma,L_K,\mu)$ satisfies*

$$\lim_{n\to\infty} \|y_n - z_n\| = 0, \tag{4.16}$$

whenever

$$\gamma\tau\mu\nu < 1, \qquad \left[\alpha + \frac{\beta+\gamma\tau\nu L_K}{1-\gamma\tau\mu\nu}\right] h < l. \tag{4.17}$$

Note that the asymptotic stability equality (4.16) *can be regarded as numerical analogues of* (2.16) *for the true solution to the problem* (1.1)–(1.2).

Proof. Let us put

$$\delta = \left[2\alpha + \frac{\beta+\gamma\tau\nu L_K}{1-\gamma\tau\mu\nu}\right] h - 2l$$

and define the quantity ϑ as

$$\vartheta = \max\left\{ k, \left[\frac{(\beta+\gamma\tau\nu L_K)h}{-\delta(1-\gamma\tau\mu\nu)}\right]^{\frac{1}{m}} \right\}.$$

With $0 < k < 1$ and (4.17), it can be deduced that $0 < \vartheta < 1$. Then it follows from (4.12) that

$$\begin{aligned}\|\omega_{n+1}\|^2 \le& \vartheta^{n+1-m}\|\omega_m\|^2 + \sum_{j=1}^{s} d_j \left\{ \sum_{i=m}^{n} \vartheta^{n-m-i}\left[\delta\vartheta^m + \frac{(\beta+\gamma\tau\nu L_K)h}{1-\gamma\tau\mu\nu}\right] \|W_j^{(i)}\|^2 \right.\\ &+ \frac{\beta+\gamma\tau\nu L_K}{1-\gamma\tau\mu\nu}\sum_{q=m}^{2m-1}\vartheta^{n-q}\max_{0\le i\le m-1}\|W_j^{(i)}\|^2\\ &\left. + \frac{\tau}{\beta+\gamma\tau\nu L_K}\sum_{i=0}^{\varpi}(\gamma\tau\mu\nu)^i\vartheta^{\varpi-i}\max_{0\le q<m}\left(\Phi_j^{(q)}\right)^2 \right\}.\end{aligned}$$

Noting that in this case $\varpi \to +\infty$ as $n \to +\infty$ and

$$\delta\vartheta^m + \frac{(\beta+\gamma\tau\nu L_K)h}{1-\gamma\tau\mu\nu} \le 0,$$

from $d_j \ge 0$ and $0 < \vartheta < 1$, we have (4.16). For the case that (4.15) holds, noting that (4.14) holds for any $\varpi \ge 0$, similarly, we can obtain (4.16). This completes the proof. □

Remark 4.1. The main difference between Theorems 4.1 and 4.2 lies in the strict inequalities present in $k < 1$ and

$$\left[\alpha + \frac{\beta + \gamma\tau\nu L_K}{1 - \gamma\tau\mu\nu}\right] h < l.$$

For algebraically stable RKMs, we have the following result.

Theorem 4.3. *Suppose the RKM (A, b^T, c) with* $\det A \neq 0$ *is algebraically stable for a positive diagonal matrix $D > 0$ and satisfies $|1 - b^T A^{-1} e| < 1$. Then the numerical solution produced by the CQRKM for the class $\mathcal{D}(\alpha, \beta, \gamma, L_K, \mu)$ is asymptotically stable whenever*

$$\gamma\tau\mu\nu < 1, \qquad \alpha + \frac{\beta + \gamma\tau\nu L_K}{1 - \gamma\tau\mu\nu} < 0. \tag{4.18}$$

Proof. It follows from (4.13) or (4.15) that

$$\lim_{n\to\infty} \|W_j^{(n)}\| = 0, \quad j = 1, \cdots, s. \tag{4.19}$$

On the other hand, $\det A \neq 0$ implies A is non-singular. Set $G = [g_{ij}] = A^{-1}$. Then it follows from (4.1)–(4.2) that

$$\omega_{n+1} = (1 - b^T A^{-1} e)\omega_n + \sum_{i=1}^{s}\sum_{j=1}^{s} b_i g_{ij} W_j^{(n)}.$$

Hence (4.16) is easily obtained from $|1 - b^T A^{-1} e| < 1$ and (4.19). The proof of Theorem 4.3 is complete. □

4.2. Stability analysis for $\mathcal{L}(\alpha, \beta, \gamma, L_y, L_\mu, \mu)$

For the class $\mathcal{L}(\alpha, \beta, \gamma, L_y, L_\mu, \mu)$, we can obtain the same results that we have obtained for the class $\mathcal{D}(\alpha, \beta, \gamma, L_K, \mu)$, except for the fact that the constant L_K is to be replaced by $L_\mu + \mu L_y$. However, since $\mathcal{L}(\alpha, \beta, \gamma, L_y, L_\mu, \mu)$ is a sub-class of $\mathcal{D}(\alpha, \beta, \gamma, L_K, \mu)$, we can obtain some better results for $\mathcal{L}(\alpha, \beta, \gamma, L_y, L_\mu, \mu)$.

First, it should be noted that in Theorem 4.1 we give the bound of $\|y_n - z_n\|$ only for $n \geq m$. As mentioned in Section 2, conditions (2.5) and (2.6) allow us to give the bound of $\|y_n - z_n\|$ for any $n \geq 0$. The following theorem states the fact.

Theorem 4.4. *Suppose the RKM (A, b^T, c) is (k, l)-algebraically stable for a non-negative diagonal matrix $D = diag(d_1, d_2, \cdots, d_s) \in \mathbf{R}^{s\times s}$, where $0 < k \leq 1$. Then, the numerical solution produced by the CQRKM for the class $\mathcal{L}(\alpha, \beta, \gamma, L_y, L_\mu, \mu)$ satisfies*

$$\|\omega_n\|^2 \leq C \max\left\{ \max_{\theta\in[-\tau,0]} \|\phi(\theta) - \psi(\theta)\|, \max_{\theta\in[-\tau,0]} \|\phi'(\theta) - \psi'(\theta)\| \right\}, \tag{4.20}$$

whenever

$$\gamma\tau\mu\nu < 1, \qquad \left[\alpha + \frac{\beta + \gamma\tau\nu(L_\mu + \mu L_y)}{1 - \gamma\tau\mu\nu}\right] h \leq l, \tag{4.21}$$

where the constant C is defined by

$$C=\sqrt{1+\frac{[\beta+\gamma\tau\nu(L_\mu+\mu L_y)]^2+\tau(\beta+\gamma\tau\nu L_\mu+\gamma\tau\mu\nu)^2}{[\beta+\gamma\tau\nu(L_\mu+\mu L_y)](1-\gamma\tau\mu\nu)}\,d\tau},$$

and $d=\sum_{j=1}^{s}d_j$.

Proof. Note that for $n-m<i\le n$, we have

$$\Phi_j^{(n-i)}\quad\le\quad\beta\|W_j^{(n-i-m)}\|+\gamma h\sum_{r=0}^{m}|\nu_r|\left(L_\mu\|W_j^{(n-i-r)}\|+\mu\|\tilde{W}_j^{(n-i-r)}\|\right). \tag{4.22}$$

Then as in the proof of Theorem 4.1, we can obtain

$$\begin{aligned}\|\omega_{n+1}\|^2\le\|\omega_0\|^2&+\sum_{j=1}^{s}d_j\left\{2\alpha h+\frac{[\beta+\gamma\tau\nu(L_\mu+\mu L_y)]h}{1-\gamma\tau\mu\nu}-2l\right\}\sum_{i=0}^{n}\|W_j^{(i)}\|^2\\&+\frac{h}{1-\gamma\tau\mu\nu}\sum_{j=1}^{s}d_j\left[\beta\sum_{i=-m}^{n-m}\|W_j^{(i)}\|^2+\gamma\tau\nu(L_\mu+\mu L_y)\sum_{i=-m}^{n}\|W_j^{(i)}\|^2\right.\\&\left.+\frac{\tau(\beta+\gamma\tau\nu L_\mu+\gamma\tau\mu\nu)^2}{[\beta+\gamma\tau\nu(L_\mu+\mu L_y)](1-\gamma\tau\mu\nu)}\max\left\{\max_{-m\le i\le 0}\|W_j^{(i)}\|^2,\ \max_{-m\le i\le 0}\|\tilde{W}_j^{(i)}\|^2\right\}\right],\end{aligned} \tag{4.23}$$

and therefore the inequality (4.20) follows. This completes the proof of the theorem. □

Below, we will give another asymptotic stability results on CQRKM for the problem class $\mathscr{L}(\alpha,\beta,\gamma,L_y,L_\mu,\mu)$. The similar stability results on CQRKM for the problem class $\mathscr{D}(\alpha,\beta,\gamma,L_K,\mu)$ cannot be obtained at present.

Theorem 4.5. *Suppose the RKM (A,b^T,c) is (k,l)-algebraically stable for a positive diagonal matrix $D=diag(d_1,d_2,\cdots,d_s)\in\mathbf{R}^{s\times s}$, where $0<k\le 1$. Then, the numerical solution produced by the CQRKM for the class $\mathscr{L}(\alpha,\beta,\gamma,L_y,L_\mu,\mu)$ satisfies* (4.16) *whenever*

$$\gamma\tau\mu\nu<1,\qquad\left[\alpha+\frac{\beta+\gamma\tau\nu(L_\mu+\mu L_y)}{1-\gamma\tau\mu\nu}\right]h<l. \tag{4.24}$$

Proof. Noting (4.24) and $d_j>0$, from (4.23), we have

$$\lim_{n\to\infty}\|W_j^{(n)}\|=0,\quad j=1,\cdots,s. \tag{4.25}$$

On the other hand, using arguments similar to those in Theorem 4.3, we have two similar inequalities to (4.6) and (4.7), except for the fact that L_K is replaced by $L_\mu+\mu L_y$. Considering m being a fixed integer and (4.25), we have

$$\lim_{n\to\infty}\Phi_j^{(n)}=0,\quad j=1,\cdots,s. \tag{4.26}$$

Recalling that

$$\|Q_j^{(n)}\| \leq h\big[L_y\|W_j^{(n)}\| + \Phi_j^{(n)}\big] \tag{4.27}$$

and applying (4.25) and (4.27), we obtain

$$\lim_{n\to\infty}\|Q_j^{(n)}\| = 0, \quad j = 1,\cdots,s. \tag{4.28}$$

Therefore from (4.1) we can show that (4.16) holds. This completes the proof. □

5. Applications and extension

We illustrate the preceding theory by applying it to some concrete numerical methods and to some concrete problems.

5.1. Applications to some classical RKMs

We begin with some general results on various common RKMs when the stability and asymptotic stability results derived in above section are applied to them. Only need to note that the s stage RKMs of type Gauss, Radau IA, Radau IIA and Lobatto IIIC are algebraically stable and satisfy $b_j > 0$ ($j = 1,2,\cdots,s$), it is easy to obtain the following theorem.

Theorem 5.1. *Suppose the RKM (A, b^T, c) is of type Gauss, Radau IA, Radau IIA or Lobatto IIIC. Then, the corresponding CQRKM is stable for the class $\mathscr{D}(\alpha,\beta,\gamma,\ L_K,\mu)$ with $d_j = b_j$ whenever condition*

$$\gamma\tau\mu\nu < 1, \qquad \alpha + \frac{\beta+\gamma\tau\nu L_K}{1-\gamma\tau\mu\nu} \leq 0 \tag{5.1}$$

holds.

From [26], we can also see that all the s ($s \geq 1$) stage Radau IA, Radau IIA and s ($s \geq 2$) stage Lobatto IIIC Runge-Kutta methods satisfy the assumptions of Theorem 4.5 with $|1 - b^TA^{-1}e| = 0$. So, we have the following results.

Theorem 5.2. *Suppose the s stage RKM (A, b^T, c) is of type Radau IA ($s \geq 1$), Radau IIA ($s \geq 1$) or Lobatto IIIC ($s \geq 2$). Then, the corresponding CQRKM is asymptotically stable for the class $\mathscr{D}(\alpha,\beta,\gamma,\ L_K,\mu)$ with $d_j = b_j$ whenever condition* (4.18) *holds.*

Table 1: Value ν for different compound quadrature rules.

CT rule m is an integer	CS rule m is an even integer	CN rule m is a multiple of four	CG rule m is an integer
$\nu = 1$	$\nu = \dfrac{4}{3}$	$\nu = \dfrac{64}{45}$	$\nu = \dfrac{13}{12}$

Comparison of conditions (2.14) and (4.18) suggests that the value ν plays key role in preserving the stability of analytical solution for numerical methods. In Table 1, we give a value for ν (compare Zhang and Vandewalle [45]) for the compound trapezoidal (CT) rule, the compound Simpson (CS) rule, the compound Newton-Cotes (CN) rule and the compound Gregory (CG) rule (cf. [6]).

From Table 1, we find that ν approximate to 1 for some common quadrature formulas, especially, $\nu = 1$ for the CT rule. This means that the Radau IA ($s \geq 1$), Radau IIA ($s \geq 1$) and Lobatto IIIC ($s \geq 2$) methods with the CT rule can completely preserve the asymptotic stability of the underlying system.

Replacing L_K by $L_\mu + \mu L_y$ in the conditions of the above theorems, we can obtain the analogues of Theorems 5.1–5.2 for the class $\mathscr{L}(\alpha,\beta,\gamma,\ L_y,L_\mu,\mu)$. In addition, from Theorem 4.5, we can know that the CQRKM extended by Gauss type RKM is asymptotically stable for the class $\mathscr{L}(\alpha,\beta,\gamma,\ L_y,L_\mu,\mu)$ whenever condition (4.18) where L_K is replaced by $L_\mu + \mu L_y$ holds.

5.2. Application to VDIDEs

Applying our results obtained in Section 4 to VDIDEs (1.4) which can be viewed as a particular case of NVDIDEs (1.1), we can obtain the corresponding results.

Slightly modifying the proof of Theorem 4.1 and noting $\mu = 0$, we easily obtain the following corollary.

Corollary 5.1. *The CQRKM* (3.1)–(3.2) *is stable, i.e.,*

$$\|y_n - z_n\| \leq \mathscr{C} \max_{t\in[-\tau,0]} \|\phi(t) - \psi(t)\|, \qquad \forall\ n \geq 0$$

holds for class $\mathscr{D}(\alpha,\beta,\gamma,L_K,0)$ *with stability constant*

$$\mathscr{C} = \sqrt{1 + d\tau(1+\tau)(\beta + \gamma\tau L_K \nu)} \tag{5.2}$$

under the assumptions:

$$\text{the RKM } (A, b^T, c) \text{ is } (k,l)\text{-algebraically stable with } 0 < k \leq 1; \tag{5.3a}$$

$$h(\alpha + \beta + \gamma L_K \tau\nu) \leq l. \tag{5.3b}$$

Theorem 4.2 leads directly to

Corollary 5.2. *The CQRKM* (3.1)–(3.2) *is asymptotically stable, i.e.,* (4.16) *holds, for class* $\mathscr{D}(\alpha,\beta,\gamma,L_K,0)$ *under the assumptions:*

$$\text{the Runge-Kutta method } (A, b^T, c) \text{ is } (k,l)\text{-algebraically stable with } 0 < k < 1; \tag{5.4a}$$

$$h(\alpha + \beta + \gamma L_K \tau\nu) < l. \tag{5.4b}$$

Applying Theorem 4.3 we obtain the following corollary.

Corollary 5.3. *Suppose the RKM (A, b^T, c) with $\det A \neq 0$ is algebraically stable for a positive diagonal matrix $D > 0$ and satisfies $|1 - b^T A^{-1} e| < 1$. Then the CQRKM (3.1)–(3.2) for the class $\mathscr{D}(\alpha, \beta, \gamma, L_K, 0)$ is asymptotically stable whenever*

$$\alpha + \beta + \gamma\tau\nu L_K < 0. \tag{5.5}$$

VDIDEs (1.4) as a special case of NVDIDEs, some authors have obtained some stability results of numerical solutions to this class of equations. For comparison purpose, we simply state these stability results.

For the problem class $\mathscr{D}(\alpha, \beta, \gamma, L_K, 0)$, Zhang and Vandewalle [44] proved that the CQRKM (3.1)–(3.2) is asymptotically stable under the assumptions (5.4a) and

$$h[2(\alpha + \beta) + \gamma(1 + L_K^2 \bar{\nu}^2)] < 2l, \tag{5.6}$$

where

$$\bar{\nu} = h\sqrt{(m+1)\sum_{i=0}^{m} |\nu_i|^2 + \epsilon}, \quad \epsilon > 0 \text{ is a suffieciently small real number.}$$

In [40], Yu and Li discussed the numerical stability of CQRKM for nonlinear NVDIDEs where the kernel $K(t, \theta, y)$ doesn't depend on y'. It can be viewed as a particular case of nonlinear NVDIDEs (1.6). As a corollary of the results obtained by them, it reports the similar result to Corollary 5.3, except for the fact that the condition (5.5) is replaced by the condition

$$\alpha + \beta + 2\gamma\tau\tilde{\nu} L_K < 0, \tag{5.7}$$

where $\tilde{\nu} = \max_{0 \le i \le m} \nu_i$.

In [28], Li investigated the contractivity and asymptotic stability of RKMs for more general Volterra functional differential equations (VFDEs). Applying his results to VDIDEs, he can obtain a result that an algebraically stable RKM (3.1)–(3.2) with CQ formula for the class $\mathscr{D}(\alpha, \beta, \gamma, L_K, 0)$ is asymptotically stable whenever

$$\alpha + q(\beta + \gamma)\nu \max\{1, \tau Ł_K\} < 0, \tag{5.8}$$

where $q \ge 1$ (see [28]).

Comparing our results with the results obtained by Zhang and Vandewalle [44], we find that conditions (5.6) is, generally, stronger than condition (5.4b). Comparing our results with the results obtained by Yu and Li [40] and Li [28], we find that conditions (5.7) and (5.8) are stronger than condition (5.5). Alternatively, we obtain the asymptotic stability results on RKMs with CQ formulae for nonlinear VDIDEs (1.4) under weaker conditions. More importantly, from our results, we can assert that the s stage Runge-Kutta method (A, b^T, c) of type Radau IA ($s \ge 1$), Radau IIA ($s \ge 1$) or Lobatto IIIC ($s \ge 2$) with the CT rule can preserve the asymptotic stability of the true solution to nonlinear VDIDEs (1.4). This result can not be obtained from the existed results in the literature.

5.3. Extension to NVDIDEs with weakly singular kernel

The sufficient conditions for the asymptotic stability of the true and numerical solutions are derived only for NVDIDEs with smooth kernel in the previous section. On the other hand, many researchers have investigated the numerical solution of VIDEs with weakly singular kernel (see, e.g., [8, 10, 12, 13, 22, 33]). The readers are referred to [10, 11] and the references therein for the regularity of solution to this class of equations. When the smooth kernel $K(t,\theta,y(\theta),y'(\theta))$ is replaced by the weakly singular kernel $\bar{K}_\sigma(t-\theta)K(t,\theta,y(\theta),y'(\theta))$, where

$$\bar{K}_\sigma(t-\theta)=(t-\theta)^{-\sigma},\ 0<\sigma<1, \tag{5.9}$$

we find that the results obtained for NVDIDEs (1.1)–(1.2) with smooth kernel are easily extended to NVDIDEs (1.1)–(1.2) with weakly singular kernel. In fact, we still have Theorems 2.1 and 2.2 with τ replacing by $\tau^{1-\sigma}/(1-\sigma)$. For the stability of numerical solution to this class of equations, we can obtain the same results as those presented in previous section. Of course, the weights v_i now depend on σ. For example, the weights v_i of the product trapezoidal quadrature (PT) formula are defined by (see, e.g., [15])

$$v_i=\begin{cases} h_\sigma, & i=0,\\ h_\sigma\left((i+1)^{2-\sigma}-2i^{2-\sigma}+(i-1)^{2-\sigma}\right), & 0<i<m,\\ h_\sigma\left(m^{1-\sigma}(1-\sigma-m+1)-(m-1)^{2-\sigma}\right), & i=m,\end{cases} \tag{5.10}$$

where $h_\sigma=h^{-\sigma}/[(1-\sigma)(2-\sigma)]$.

6. Numerical experiments

In this section, we report results of numerical experiments which confirm the theoretical analysis for RKM presented in this paper. We also give an example to illustrate the convergence of the methods which has been studied in [35] by the authors.

6.1. Solving the nonlinear equations

For solving the nonlinear equations (3.1), we consider the following iteration scheme where on iteration $\mathscr{N}$ we have

$$\begin{cases} Y_i^{(n,\mathscr{N})}=y_n+h\sum_{j=1}^{s}a_{ij}\tilde{Y}_j^{(n,\mathscr{N})}, & i=1,\cdots,s,\\ \tilde{Y}_j^{(n,\mathscr{N})}=f\left(t_{n,j},Y_j^{(n,l)},Y_j^{(n-m)},K_j^{(n,\mathscr{N}-1)}\right). \end{cases} \tag{6.1}$$

Here, $K_j^{(n,\mathscr{N}-1)}$ is defined by the following

$$K_j^{(n,\mathscr{N}-1)}=hv_0K\big(t_{n,j},t_{n,j},Y_j^{(n,\mathscr{N}-1)},\tilde{Y}_j^{(n,\mathscr{N}-1)}\big)+h\sum_{i=1}^{m}v_iK\big(t_{n,j},t_{n-i,j},Y_j^{(n-i)},\tilde{Y}_j^{(n-i)}\big),$$

Table 2: The differences e_n produced by PTRIIA2 and by PTGAUSS2 when applied to (6.2)-(6.3)-(6.4) and (6.2)-(6.3)-(6.5) with $G(u)=0.2u$, where $h=0.1$ and $h=0.01$.

	h	$t=1$	$t=10$	$t=50$	$t=100$
PTRIIA2	.1	$3.395191\cdot 10^{-3}$	$8.633201\cdot 10^{-6}$	$2.666452\cdot 10^{-17}$	$1.091782\cdot 10^{-31}$
	.01	$3.347977\cdot 10^{-3}$	$8.525687\cdot 10^{-6}$	$2.626066\cdot 10^{-17}$	$1.071583\cdot 10^{-31}$
PTGAUSS2	.1	$3.396375\cdot 10^{-3}$	$8.638383\cdot 10^{-6}$	$2.670791\cdot 10^{-17}$	$1.094963\cdot 10^{-31}$
	.01	$3.347996\cdot 10^{-3}$	$8.525759\cdot 10^{-6}$	$2.626118\cdot 10^{-17}$	$1.071620\cdot 10^{-31}$

and $Y_j^{(n,0)}$ and $\tilde{Y}_j^{(n,0)}$ are given initial values for iteration. Following the approach designed by Enright and Hu [16] for continuous RKMs, we can easily prove that the iteration (6.1) is convergent for sufficiently small h (see also [18]).

6.2. Example 1: Hammerstein-type NVDIDEs with weakly singular kernel

First, we apply CQRKMs to Hammerstein-type NVDIDEs with weakly singular kernel

$$y'(t)=f(t,y(t))+\int_{t-\tau}^{t}\bar{K}_{\sigma}(t-\theta)G(y'(\theta))d\theta,\quad t\geq 0 \tag{6.2}$$

subject to (1.2), where

$$f(t,y(t))=-y(t),\quad \tau=1,\quad \sigma=0.5,\quad G(u)=0.2u. \tag{6.3}$$

On the basis of the discussion in Subsection 5.3, we can assert that the solution to problem (6.2) with initial condition (1.2) is asymptotically stable.

Now we use 2-stage Radau IIA method with PT formula (5.10) (PTRIIA2) and 2-stage Gauss method with PT formula (PTGAUSS2) to solve the above problem. Define the differences of numerical solutions as $e_n=|y_{1,n}-y_{2,n}|$, where $y_{1,n}$ and $y_{2,n}$ are the numerical solutions approximating to the solutions of problems (6.2)–(6.3) with two different initial conditions

$$y(t)=\sin t,\quad t\in[-1,0] \tag{6.4}$$

and

$$y(t)=t,\quad t\in[-1,0], \tag{6.5}$$

at $t=t_n$, respectively. Table 2 shows the numerical results, where $h=0.1$ and $h=0.01$. These numerical results confirm our theoretical analysis that 2-stage Radau IIA method with PT rule (PTRIIA2) and 2-stage Gauss method with PT rule (PTGAUSS2) can preserve the asymptotic stability of the underlying system.

6.3. Example 2: A partial neutral functional differential equation

The next problem is defined for $t \geq 0$ and $x \in [0,1]$:

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{\pi}\frac{\partial^2}{\partial^2 x}u(x,t) + au(x,t-1) + b\int_{t-1}^{t} e^{-s}\sin u(x,s)\cos\frac{\partial}{\partial s}u(x,s)ds + g(x,t), \tag{6.6a}$$

with the initial and boundary conditions

$$u(x,t) = (x - x^2 + 1)e^{-t}, \qquad x \in [0,1], \quad t \in [-1,0], \tag{6.6b}$$

$$u(0,t) = u(1,t) = e^{-t}, \qquad t \geq 0. \tag{6.6c}$$

After application of the numerical method of lines, we obtain the following NVDIDEs of the form:

$$u_i'(t) = \frac{1}{\pi\Delta x^2}[u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)] + au_i(t-1) + b\int_{t-1}^{t} e^{-s}\sin u_i(s)\cos u_i'(s)ds + g_i(t), \qquad t \geq 0, \tag{6.7a}$$

$$u_0(t) = u_{N_x}(t) = e^{-t}, \qquad t \geq 0, \tag{6.7b}$$

$$u_i(t) = (i\Delta x - i^2\Delta x^2 + 1)e^{-t}, \qquad i = 1,\cdots,N_x - 1, \quad t \in [-1,0], \tag{6.7c}$$

where Δx is the spatial step, N_x is a natural number such that $N_x\Delta x = 1$, $x_i = i\Delta x$, $i = 1,\cdots,N_x - 1$, and $u_i(t) = u(x_i,t)$, $g_i(t) = g(x_i,t)$. Thus, we have

$$\alpha = -\frac{4N_x^2}{\pi}\sin^2\frac{\pi}{2N_x}, \qquad \beta = |a|, \qquad \gamma = |b|,$$

$$L_K = e + \frac{4eN_x^2}{\pi}, \qquad \mu = e,\ L_y = \frac{4N_x^2}{\pi}, \qquad L_\mu = e.$$

Observe that L_K, μ and L_μ will decrease with increasing values of t. As a result, the solution to problem (6.7) is asymptotically stable. The purpose of this numerical experiment is to illustrate the stability and convergence of CQRKMs.

First, we consider the stability of numerical methods. For this purpose, we set $g(x,t) \equiv 0$ and give the other initial condition

$$u(x,t) = (x - x^2)e^{-t}, \qquad x \in [0,1], \quad t \in [-1,0]. \tag{6.8}$$

After discretization this initial condition becomes

$$u_i(t) = (i\Delta x - i^2\Delta x^2)e^{-t}, \qquad i = 1,\cdots,N_x - 1, \quad t \in [-1,0]. \tag{6.9}$$

We take $\Delta x = 0.1$ for the numerical method of lines and use 2-stage Radau IIA method with compound trapezoidal (CT) rule (CTRIIA2) for the numerical integration of the problem (6.7). For solving the nonlinear algebraical equations, we consider the iteration scheme (6.1) with $\mathscr{N}=3$, $Y_j^{(n,0)}=Y_j^{(n-1,\mathscr{N})}$ and $\tilde{Y}_j^{(n,0)}=\tilde{Y}_j^{(n-1,\mathscr{N})}$. Define the differences of numerical solutions as

$$E_n=\max_{1\le i\le N_x-1}|U_{1,i}^n-U_{2,i}^n|,$$

where $U_{1,i}^n$ and $U_{2,i}^n$ are the numerical solutions approximating to the solutions of problems (6.7), and (6.7a), (6.7b) and (6.9), respectively. The numerical results of CTRIIA2 with $h=0.1$ and $h=0.01$ when applied to problems (6.7), and (6.7a), (6.7b) and (6.9) with $a=-e^{-1}$ and $b=0.01$ are listed in Table 3.

Table 3: The differences E_n produced by CTRIIA2 when applied to (6.7), and (6.7a), (6.7b), and (6.9) with $a=-e^{-1}$ and $b=0.01$, where $h=0.1$ and $h=0.01$.

h	$t=0.1$	$t=1$	$t=5$	$t=10$
.1	$9.275701\cdot10^{-1}$	$3.735039\cdot10^{-1}$	$6.314180\cdot10^{-3}$	$4.254367\cdot10^{-5}$
.01	$9.800226\cdot10^{-1}$	$3.545434\cdot10^{-1}$	$6.627136\cdot10^{-3}$	$4.465401\cdot10^{-5}$

In summary, we can conclude that under these sufficient conditions given in this paper CQRKMs are stable and asymptotically stable.

Now we consider the convergence of the numerical method. In the numerical experiment, the function $g(x,t)$ is selected in such a way that the true solution is

$$u(x,t)=(x-x^2+1)e^{-t}.$$

Let

$$\epsilon(T)=\max_{1\le i\le N_x-1}|U_i(T)-u(x_i,T)|$$

denote the error of a method when applied to problem (6.7), where $U_i(T)$ denotes the numerical solution which is produced by CTRIIA2 approximating $u(x_i,T)$. Table 4 shows the errors at $T=10$. These numerical results illustrate the convergence of CQRKMs for NVDIDEs.

Table 4: The errors $\epsilon(T)$ produced by CTRIIA2 when applied to (6.7) with $a=-e^{-1}$ and $b=0.01$, where $h=1/m$ and $T=10$.

$m=10$	$m=20$	$m=40$	$m=80$
1.295734×10^{-9}	1.654411×10^{-10}	2.090210×10^{-11}	2.628293×10^{-12}

7. Concluding remarks

We have given some sufficient conditions for the stability and asymptotic stability of the true solution to nonlinear NVDIDEs (1.1)–(1.2). This analysis is based on a Halanay inequality generalized by Liz and Trofimchuk [29] (see also [38]). The main purpose of this paper has been to obtain a comprehensive theory of the stability of RKM with the compound quadrature formulae for nonlinear NVDIDEs (1.1)–(1.2). The main results of the paper are given in Section 4. An extension of the stability results to NVDIDEs with weakly singular kernel was also discussed in this paper.

We have noted that the sufficient conditions for the stability of the numerical solution are slightly stronger than the sufficient conditions for the stability of the true solution. The difference is that there is a factor ν, which results from the compound quadrature formulae, in the sufficient conditions for the numerical stability. We also noted that ν approximate to 1 for some common quadrature formulas, especially, $\nu = 1$ for the CT rule, and that the Radau IA ($s \geq 1$), Radau IIA ($s \geq 1$) and Lobatto IIIC ($s \geq 2$) methods with the CT rule can completely preserve the asymptotic stability of the underlying system.

We remark that we have only considered the stability of the true solution and the numerical solution. The blowup properties of nonlinear Volterra equations have been investigated by many authors (see the recent survey papers by Bandle and Brunner [4] and Roberts [32]). Recently, Ma *et al.* [30] and Ma and Jiang [31] examined numerically the blowup solution of VIDEs and time fractional differential equations, which is equivalent to the Volterra integral equation with weakly singular kernel, by moving mesh method, respectively. To the best our knowledge, however, the blowup theory for nonlinear NVDIDEs is completely unknown. The blowup solution of nonlinear NVDIDEs is an interesting topic and will be our future work.

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