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A Priori and A Posteriori Error Estimates of Streamline Diffusion Finite Element Method for Optimal Control Problem Governed by Convection Dominated Diffusion Equation

Ningning Yan* and Zhaojie Zhou

LSEC, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100080, China.

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Abstract. In this paper, we investigate a streamline diffusion finite element approximation scheme for the constrained optimal control problem governed by linear convection dominated diffusion equations. We prove the existence and uniqueness of the discretized scheme. Then a priori and a posteriori error estimates are derived for the state, the co-state and the control. Three numerical examples are presented to illustrate our theoretical results.

AMS subject classifications: 65N30

Key words: Constrained optimal control problem, convection dominated diffusion equation, streamline diffusion finite element method, a priori error estimate, a posteriori error estimate.

1. Introduction

Optimal control problem governed by advection dominated diffusion equations arises in many science and engineering applications (see., e.g., [9, 24, 30]). Recently, extensive research has been carried out on various theoretical aspects of the optimal control problems governed by advection diffusion and convection dominated equations, see, e.g., [2–4, 9, 27]. Most of them have been concerned with the unconstrained optimal control problem.

In this paper, we consider the following constrained optimal control problem:

$$\min_{u\in K\subset U}\{g(y)+j(u)\} \tag{1.1}$$

subject to

$$-\varepsilon\triangle y+\vec{b}\cdot\nabla y+ay=f+Bu \quad \text{in } \Omega, \tag{1.2a}$$

$$y=0 \quad \text{on } \partial\Omega, \tag{1.2b}$$

*Corresponding author. *Email addresses:* `ynn@amss.ac.cn` (N. Yan), `zzj534@amss.ac.cn` (Z. Zhou)

where the bounded open sets Ω, $\Omega_U \subset R^2$ with boundary $\partial\Omega$ and $\partial\Omega_U$,

$$K \subset U = L^2(\Omega_U)$$

is a bounded convex set, $g(\cdot)$ and $j(\cdot)$ are convex functionals. The details will be specified in the next section.

It is well known that the standard finite element discretizations applied to convection dominated diffusion problems lead to strongly oscillation. Some effective discrete schemes are investigated to improve the approximation properties of the standard Galerkin method and to reduce the oscillatory behavior, see, e.g., [11–13]. In [11], Hughes and Brooks first propose the streamline diffusion finite element method (also names SUPG), attracting more and more attentions because of its advantages of numerical stability and high order accurate. In [27], the authors apply SUPG method to the *unconstrained* optimal control problem governed by convection diffusion equation. They consider two approaches: optimize-then-discretize and discretize-then-optimize. A priori error estimates were proved for both the state, the co-state and the control. In [3], authors propose another stabilized finite element method for the discretization of the optimal control problem governed by convection diffusion equation with the constrained and unconstrained control.

In this paper, we apply the streamline diffusion finite element method to approximate the *constrained* optimal control problem (1.1)-(1.2a). We first derive the continuous optimality condition, which contains the state equation, the co-state equation and the optimal inequality. Then we use the streamline diffusion finite element methods to discretize the state equation and the co-state equation, and use the standard Galerkin method to approximate the optimal inequality directly. We prove the existence and the uniqueness for the approach. Moreover, a priori and a posteriori error estimates are obtained for both the state, the co-state and the control. The numerical examples are presented to illustrate our theoretical results.

Although a priori error estimates of *unconstrained* optimal control problem have been discussed in [27], there are new difficulties in our approach. Firstly, because the authors only consider the unconstrained problem in [27], the optimal inequality can be replaced by equality. Therefore the existence and the uniqueness of the problem become trivial and there is no need to prove them. While for the *constrained* problem, the existence and the uniqueness of our discrete SUPG scheme must be proved because it is not equivalent to a discrete optimal control problem. Moreover, the proof of a priori error estimate for the constrained control problems is more complicated than the one for the unconstrained control problems. Furthermore, we provide a posteriori error estimates of SUPG scheme for the optimal control problems, which is unavailable in literatures to our knowledge.

The outline of the paper is as follows: In Section 2, we present the streamline diffusion scheme for constrained optimal control problem governed by convection dominated diffusion equations. In Section 3, we prove the existence and uniqueness of the approach. In Sections 4 and 5, a priori and a posteriori error estimates are derived, respectively. In Section 6, we present three numerical examples to illustrate the theoretical results. In the last section, we briefly summarize the method used, results obtained and possible future extensions and challenges.

2. The streamline diffusion finite element scheme

Consider following constrained optimal control problem governed by convection dominated diffusion equations:

$$\min_{u\in K\subset U}\{g(y)+j(u)\} \tag{2.1}$$

subject to

$$-\varepsilon\triangle y+\vec{b}\cdot\nabla y+ay=f+Bu \quad \text{in } \Omega, \tag{2.2a}$$

$$y=0 \quad \text{on } \partial\Omega, \tag{2.2b}$$

where g and j are given convex functionals, B is a linear operator from $L^2(\Omega_U)$ to $L^2(\Omega)$, $f\in L^2(\Omega)$, $a>0$ is a constant, $0<\varepsilon\ll 1$ is a small constant, $\vec{b}$ is constant vector, $\Omega\subset R^2$ and $\Omega_U\subset R^2$ are bounded domains with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$, $K\subset U=L^2(\Omega_U)$ is a convex set. In this paper, we consider the constrained problem with the constrain set

$$K=\{v\in U:\ v\geq 0\}. \tag{2.3}$$

We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev space on Ω with a norm $\|\cdot\|_{m,q,\Omega}$ and a semi-norm $|\cdot|_{m,q,\Omega}$. We set

$$W_0^{m,q}(\Omega)=\{v\in W^{m,q}(\Omega):\ v\,|_{\partial\Omega}=0\}.$$

For $q=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$ and $\|\cdot\|_{m,\Omega}=\|\cdot\|_{m,2,\Omega}$. Especially, we denote the state space $Y=H_0^1(\Omega)$, and the control space $U=L^2(\Omega_U)$. The inner products in $L^2(\Omega_U)$ and $L^2(\Omega)$ are indicated by $(\cdot,\cdot)_U$ and $(\cdot,\cdot)$, respectively. In addition, c and C denote general constants.

Note that Eq. (2.2a) is the convection dominated diffusion equation when ε is very small. It is well known that the standard finite element method can not be used well for solving this kind of problems. Stabilized method should be adopted in order to improve the computation stability. The streamline diffusion finite element scheme (see, e.g., [11]) has been proved to be an efficient scheme for the state equation (2.2a). In this paper, we use the streamline diffusion finite element scheme to deal with the optimal control problem (2.1)-(2.2a).

To consider the streamline diffusion finite element approximation of above optimal control problem, we first derive a weak formula for the state equation. The weak formulation of the state equation (2.2a) is to find $y(u)\in Y=H_0^1(\Omega)$, such that

$$(\varepsilon\nabla y,\nabla w)+(\vec{b}\cdot\nabla y,w)+(ay,w)=(f+Bu,w) \quad \forall w\in Y.$$

Therefore, the optimal control problem (2.1)-(2.2a) can be rewritten as

$$\min_{u\in K\subset U}\{g(y)+j(u)\}, \tag{2.4}$$

subject to

$$(\varepsilon\nabla y,\nabla w)+(\vec{b}\cdot\nabla y,w)+(ay,w)=(f+Bu,w) \quad \forall w\in Y. \tag{2.5}$$

It is known (see, e.g., [7, 18]) that the control problem (2.4)-(2.5) has a unique solution (y,u), and that a pair (y,u) is the solution of (2.4)-(2.5) if and only if there is a co-state $p\in Y$, such that (y,p,u) satisfies the following optimality conditions:

$$(\varepsilon\nabla y,\nabla w)+(\vec{b}\cdot\nabla y,w)+(ay,w)=(f+Bu,w)\quad\forall w\in Y, \tag{2.6}$$

$$(\varepsilon\nabla p,\nabla q)-(\vec{b}\cdot\nabla p,q)+(ap,q)=(g'(y),q)\quad\forall q\in Y, \tag{2.7}$$

$$(j'(u)+B^*p,v-u)_U\geq 0\quad\forall v\in K\subset U, \tag{2.8}$$

where B^* is the adjoint operator of B.

Next, let us consider the streamline diffusion finite element approximation of (2.6)-(2.8). Let T^h and T^h_U be regular triangulations of Ω and Ω_U, respectively, so that $\bar{\Omega}=\cup_{\tau\in T^h}\bar{\tau}$, $\bar{\Omega}_U=\cup_{\tau_U\in T^h_U}\bar{\tau}_U$. Let $h=\max_{\tau\in T^h}h_\tau, h_U=\max_{\tau_U\in T^h_U}h_{\tau_U}$, where h_τ and h_{τ_U} denote the diameter of the element τ and τ_U , respectively.

Associated with T^h is a finite dimensional subspace W^h of $C(\bar{\Omega})$, such that $\phi|_\tau$ are polynomials of k-order ($k\geq 1$), $\forall\phi\in W^h$. Set $Y^h=W^h\cap H^1_0(\Omega)$. Then it is easy to see that $Y^h\subset Y=H^1_0(\Omega)$.

Associated with T^h_U is another finite dimensional subspace U^h of $U=L^2(\Omega_U)$, such that $\chi|_{\tau_U}$ are polynomials of m-order ($m\geq 0$), $\forall\chi\in U^h$. Set $K^h=U^h\cap K$. Then we have $K^h\subset K$. In this paper we consider the case $k=1$ and $m=0$.

We use the streamline diffusion finite element scheme to approximate the state equation (2.6) and the co-state equation (2.7), and use the standard finite element scheme to approximate the inequality (2.8). Then the approximation scheme of the optimality conditions (2.6)-(2.8) is formulated as follow: find $(y_h,p_h,u_h)\in Y^h\times Y^h\times K^h$ such that

$$A^s_h(y_h,w_h)=(f+Bu_h,w_h+\delta\vec{b}\cdot\nabla w_h)\quad\forall w_h\in Y^h, \tag{2.9}$$

$$A^a_h(p_h,q_h)=(g'(y_h),q_h-\delta\vec{b}\cdot\nabla q_h)\quad\forall q_h\in Y^h, \tag{2.10}$$

$$(j'(u_h)+B^*p_h,v_h-u_h)_U\geq 0\quad\forall v_h\in K^h\subset U^h, \tag{2.11}$$

where

$$A^s_h(y_h,w_h)=(\varepsilon\nabla y_h,\nabla w_h)+(\vec{b}\cdot\nabla y_h+ay_h,w_h+\delta\vec{b}\cdot\nabla w_h), \tag{2.12}$$

$$A^a_h(p_h,q_h)=(\varepsilon\nabla p_h,\nabla q_h)-(\vec{b}\cdot\nabla p_h-ap_h,q_h-\delta\vec{b}\cdot\nabla q_h), \tag{2.13}$$

δ is a stabilized parameter, which can be chosen as

$$\delta|_\tau=\begin{cases}\tau_1 h_\tau^2/\varepsilon & P\leq 1,\\ \tau_2 h_\tau & P>1,\end{cases} \tag{2.14}$$

where $P=\|\vec{b}\|_{0,\infty,\tau}h_\tau/(2\varepsilon)$ is P'eclet number, τ_1 and τ_2 are constants. Note that in our paper, $\|\vec{b}\|_{0,\infty,\tau}$ is a constant since $\vec{b}$ is a constant vector, and $\varepsilon\ll h$. We set $\tau_1=\tau_2=1$, and hence $\delta|_\tau=h_\tau$.

3. Existence and uniqueness of the discrete solution

In the last section, the streamline diffusion finite element scheme for approximating the control problem (2.4)-(2.5) is provided. It should be pointed out that we derive the approximation scheme (2.9)-(2.11) by using the streamline diffusion finite element method for the continuous optimality condition (2.6)-(2.8) directly, instead of approximating the optimal control problem (2.4)-(2.5) and then deriving the discrete optimality condition as the standard finite element method for optimal control problem governed by elliptic partial differential equations (see, e.g., [19] and [20]). Then it is clear that although (y_h, p_h, u_h) is the approximate solution of the optimal control problem (2.1)-(2.2a), it is not equivalent to the approximation solution (y_h^*, p_h^*, u_h^*) which satisfies

$$\min_{u_h^* \in K^h \subset U^h} \{g(y_h^*) + j(u_h^*)\}, \tag{3.1}$$

subject to

$$A_h^s(y_h^*, w_h) = (f + Bu_h^*, w_h + \delta \vec{b} \cdot \nabla w_h) \quad \forall w_h \in Y^h, \tag{3.2}$$

where $A_h^s(\cdot,\cdot)$ is defined by (2.12). Thus the proof of the existence of (y_h, p_h, u_h) is not trivial, although the existence and uniqueness of the solution (y_h^*, p_h^*, u_h^*) is obvious provided that the functional $J(y(u), u) = g(y(u)) + j(u)$ is convex.

In the following we shall provide the proof of the existence and the uniqueness of the solution for the scheme (2.9)-(2.11). Similar technique have been used in [21] for the optimal control problem governed by elliptic partial differential equations.

Theorem 3.1. *Suppose the functional $j(\cdot)$ is uniformly convex, $g(\cdot)$ is convex, $j'(\cdot)$ and $g'(\cdot)$ are Lipschitz continuous, and the operator B is bounded. Then if δ is a sufficient small positive number, the solution of the scheme* (2.9)-(2.11) *is existent and unique.*

Proof. Let

$$\begin{aligned}
&(f + Bu_h, w_h)_h^s = (f + Bu_h, w_h + \delta \vec{b} \cdot \nabla w_h),\\
&(g'(y_h), q_h)_h^a = (g'(y_h), q_h - \delta \vec{b} \cdot \nabla q_h).
\end{aligned}$$

Then the scheme (2.9)-(2.11) can be rewritten as

$$A_h^s(y_h, w_h) = (f + Bu_h, w_h)_h^s \quad \forall w_h \in Y^h, \tag{3.3}$$

$$A_h^a(p_h, q_h) = (g'(y_h), q_h)_h^a \quad \forall q_h \in Y^h, \tag{3.4}$$

$$(j'(u_h) + B^* p_h, v_h - u_h)_U \geq 0 \quad \forall v_h \in K^h. \tag{3.5}$$

Introduce two operators Q_U: $L^2(\Omega_U) \to U^h$ and P_K: $L^2(\Omega_U) \to K^h$ such that for all $z \in L^2(\Omega_U)$, $Q_U z \in U^h$, $P_K z \in K^h$, and

$$\begin{aligned}
&(Q_U z, z_h)_U = (z, z_h)_U \quad \forall\, z_h \in U^h,\\
&\| z - P_K z \|_{0,\Omega_U} = \min_{z_h \in K^h} \| z - z_h \|_{0,\Omega_U}.
\end{aligned}$$

It is easy to be verified that the operator P_K satisfies

$$\| P_K(z') - P_K(z'') \|_{0,\Omega_U} \leq \| z' - z'' \|_{0,\Omega_U} \quad \forall z',\, z'' \in L^2(\Omega_U). \tag{3.6}$$

For $v_h \in K^h$, introduce $(y_h(v_h), p_h(v_h))$ to be the solution of the following auxiliary equations:

$$A_h^s(y_h(v_h), w_h) = (f + Bv_h, w_h)_h^s \quad \forall w_h \in Y^h, \tag{3.7}$$

$$A_h^a(p_h(v_h), q_h) = (g'(y_h(v_h)), q_h)_h^a \quad \forall q_h \in Y^h. \tag{3.8}$$

Set mapping Φ: $U^h \to U^h$, such that

$$\Phi(z_h) = z_h - \rho(j'(z_h) + Q_U(B^* p_h(z_h))) \quad \forall\, z_h \in U_h, \quad 0 < \rho < 1.$$

Let $\hat{z}_h = T(z_h) = P_k \Phi(z_h)$. Then $\hat{z}_h$ satisfies (see, [22]),

$$(\hat{z}_h, z_h' - \hat{z}_h) \geq (\Phi(z_h), z_h' - \hat{z}_h), \quad \forall z_h' \in K^h.$$

Then the key problem for the proof of the existence and the uniqueness of (2.9)-(2.11) is to show that $T(z_h)$ is a contractive mapping. It follows from (3.6) that for all $z_h', z_h'' \in U^h$,

$$\begin{aligned} &\| T(z_h') - T(z_h'') \|_{0,\Omega_U}^2 = \| P_k(\Phi(z_h')) - P_k(\Phi(z_h'')) \|_{0,\Omega_U}^2 \\ \leq &\| \Phi(z_h') - \Phi(z_h'') \|_{0,\Omega_U}^2 = (\Phi(z_h') - \Phi(z_h''), \Phi(z_h') - \Phi(z_h''))_U. \end{aligned}$$

Note that

$$\begin{aligned} &(\Phi(z_h') - \Phi(z_h''), \Phi(z_h') - \Phi(z_h''))_U \\ =&\| z_h' - z_h'' \|_{0,\Omega_U}^2 - 2\rho(z_h' - z_h'', j'(z_h') - j'(z_h''))_U \\ &- 2\rho(z_h' - z_h'', Q_U(B^* p_h(z_h') - B^* p_h(z_h'')))_U \\ &+ \rho^2 \| j'(z_h') - j'(z_h'') + Q_U(B^* p_h(z_h') - B^* p_h(z_h'')) \|_{0,\Omega_U}^2, \end{aligned}$$

and

$$\begin{aligned} &(z_h' - z_h'', Q_U(B^* p_h(z_h') - B^* p_h(z_h'')))_U \\ =& (z_h' - z_h'', B^* p_h(z_h') - B^* p_h(z_h''))_U = (B(z_h' - z_h''), p_h(z_h') - p_h(z_h'')). \end{aligned}$$

We have

$$\begin{aligned} &\| T(z_h') - T(z_h'') \|_{0,\Omega_U}^2 \\ \leq &\| z_h' - z_h'' \|_{0,\Omega_U}^2 - 2\rho(z_h' - z_h'', j'(z_h') - j'(z_h''))_U - 2\rho(B(z_h' - z_h''), p_h(z_h') - p_h(z_h'')) \\ &+ \rho^2 \| j'(z_h') - j'(z_h'') + Q_U(B^* p_h(z_h') - B^* p_h(z_h'')) \|_{0,\Omega_U}^2 . \end{aligned} \tag{3.9}$$

Note that $j(\cdot)$ is uniformly convex. It is easy to see that

$$(z_h' - z_h'', j'(z_h') - j'(z_h''))_U \geq C_1 \| z_h' - z_h'' \|_{0,\Omega_U}^2, \tag{3.10}$$

where C_1 is the constant dependent on the uniformly convexity of $j(\cdot)$. For $z_h', z_h'' \in K^h$, it follows from (3.7)-(3.8) that

$$\left\{\begin{array}{l} A_h^s(y_h(z_h') - y_h(z_h''), w_h) = (B(z_h' - z_h''), w_h)_h^s, \\ A_h^a(p_h(z_h') - p_h(z_h''), q_h) = (g'(y_h(z_h')) - g'(y_h(z_h'')), q_h)_h^a. \end{array}\right.$$

Setting $w_h = p_h(z_h') - p_h(z_h'')$ and $q_h = y_h(z_h') - y_h(z_h'')$, we deduce that

$$\begin{aligned} &(B(z_h' - z_h''), p_h(z_h') - p_h(z_h''))_h^s \\ =& A_h^s(y_h(z_h') - y_h(z_h''), p_h(z_h') - p_h(z_h'')) - A_h^a(p_h(z_h') - p_h(z_h''), y_h(z_h') - y_h(z_h'')) \\ &+ (g'(y_h(z_h')) - g'(y_h(z_h'')), y_h(z_h') - y_h(z_h''))_h^a. \end{aligned}$$

Therefore,

$$\begin{aligned} &(B(z_h' - z_h''), p_h(z_h') - p_h(z_h'')) \\ =& A_h^s(y_h(z_h') - y_h(z_h''), p_h(z_h') - p_h(z_h'')) - A_h^a(p_h(z_h') - p_h(z_h''), y_h(z_h') - y_h(z_h'')) \\ &+ (g'(y_h(z_h')) - g'(y_h(z_h'')), y_h(z_h') - y_h(z_h''))_h^a \\ &- \delta(B(z_h' - z_h''), \vec{b} \cdot \nabla(p_h(z_h') - p_h(z_h''))). \end{aligned} \tag{3.11}$$

Let $Y_h = y_h(z_h') - y_h(z_h'')$, $P_h = p_h(z_h') - p_h(z_h'')$. By the definitions of $A_h^s(Y_h, P_h)$, $A_h^a(P_h, Y_h)$ and integrating by parts, we have

$$\begin{aligned} &A_h^s(Y_h, P_h) - A_h^a(P_h, Y_h) \\ =& (\vec{b} \cdot \nabla Y_h + aY_h, P_h + \delta \vec{b} \cdot \nabla P_h) + (\vec{b} \cdot \nabla P_h - aP_h, Y_h - \delta \vec{b} \cdot \nabla Y_h) = 0. \end{aligned} \tag{3.12}$$

Note that $g(\cdot)$ is convex. We have

$$\begin{aligned} &(g'(y_h(z_h')) - g'(y_h(z_h'')), y_h(z_h') - y_h(z_h''))_h^a \\ =& (g'(y_h(z_h')) - g'(y_h(z_h'')), y_h(z_h') - y_h(z_h'')) \\ &+ \delta(g'(y_h(z_h')) - g'(y_h(z_h'')), \vec{b}\nabla(y_h(z_h') - y_h(z_h''))) \\ \geq& 0 - C_2\delta\frac{1}{\gamma} \| y_h(z_h') - y_h(z_h'') \|_{0,\Omega}^2 - \delta\gamma \|\vec{b}\nabla(y_h(z_h') - y_h(z_h''))\|_{0,\Omega}^2, \end{aligned} \tag{3.13}$$

where γ is an arbitrary positive number, C_2 is the constant dependent on the bound of $g'(\cdot)$. Moreover, it is easy to see that

$$\begin{aligned} &\delta(B(z_h' - z_h''), \vec{b}\nabla(p_h(z_h') - p_h(z_h''))) \\ \leq& C_3\delta\frac{1}{\gamma} \| z_h' - z_h'' \|_{0,\Omega_U}^2 + \delta\gamma \| \vec{b}\nabla(p_h(z_h') - p_h(z_h'')) \|_{0,\Omega}^2, \end{aligned} \tag{3.14}$$

again, where γ is an arbitrary positive number, and C_3 is the constant dependent on the bound of the operator B. By the property of $A_h^s(.,.)$ and $A_h^a(.,.)$ (see, e.g., [1] and [25]), we have

$$\|y_h(z_h') - y_h(z_h'')\|_{0,\Omega}^2 + \delta \| \vec{b} \cdot \nabla(y_h(z_h') - y_h(z_h'')) \|_{0,\Omega}^2 \leq C_4 \| z_h' - z_h'' \|_{0,\Omega_U}^2, \tag{3.15}$$

$$\|p_h(z_h') - p_h(z_h'')\|_{0,\Omega}^2 + \delta \| \vec{b}\nabla \cdot (p_h(z_h') - p_h(z_h'')) \|_{0,\Omega}^2 \leq C_5 \| z_h' - z_h'' \|_{0,\Omega_U}^2 . \tag{3.16}$$

Combining (3.11)-(3.16), we deduce that

$$\begin{aligned}
&(B(z_h' - z_h''), p_h(z_h') - p_h(z_h'')) \\
\geq & -C_2\delta\frac{1}{\gamma} \| y_h(z_h') - y_h(z_h'') \|_{0,\Omega}^2 - \delta\gamma \| \vec{b}\nabla(y_h(z_h') - y_h(z_h''))\|_{0,\Omega}^2 \\
& - C_3\delta\frac{1}{\gamma} \| z_h' - z_h'' \|_{0,\Omega_U}^2 - \delta\gamma \| \vec{b}\nabla(p_h(z_h') - p_h(z_h'')) \|_{0,\Omega}^2 \\
\geq & -((C_2C_4 + C_3)\delta\frac{1}{\gamma} + (C_4 + C_5)\gamma) \| z_h' - z_h'' \|_{0,\Omega_U}^2 .
\end{aligned} \tag{3.17}$$

Moreover, it is easy to see that

$$\| j'(z_h') - j'(z_h'') + Q_U(B^*p_h(z_h') - B^*p_h(z_h'')) \|_{0,\Omega_U}^2 \leq C_6 \| z_h' - z_h'' \|_{0,\Omega_U}^2 . \tag{3.18}$$

Then it follows from (3.9), (3.10), (3.17) and (3.18) that

$$\| T(z_h') - T(z_h'') \|_{0,\Omega_U}^2 \leq C^* \| z_h' - z_h'' \|_{0,\Omega_U}^2, \tag{3.19}$$

where

$$C^* = 1 - 2\rho C_1 + 2\rho((C_2C_4 + C_3)\delta\frac{1}{\gamma} + (C_4 + C_5)\gamma) + \rho^2 C_6.$$

Set $\gamma = C_1/2(C_4 + C_5)$. We have

$$C^* = 1 - \rho C_1 + \frac{4\rho\delta}{C_1}(C_2C_4 + C_3)(C_4 + C_5) + \rho^2 C_6 = 1 - \rho C_7 + \rho^2 C_6,$$

where $C_7 = C_1 - 4\delta(C_2C_4 + C_3)(C_4 + C_5)/C_1 > 0$, when δ is small enough, which is the reasonable assumption because that $\delta = O(h)$ generally in our case. It follows that $C^* < 1$ if $\rho < C_7/C_6$. Therefore $T(z_h)$ is the contractive mapping and hence the existence and the uniqueness of (2.9)-(2.11) are direct conclusions. □

4. A priori error estimates

In this section, we consider a priori error estimate for the optimal control problem (2.6)-(2.8) and its streamline diffusion finite element approximation (2.9)-(2.11).

Lemma 4.1. *Let* (y, p, u) *and* (y_h, p_h, u_h) *be the solutions of* (2.6)-(2.8) *and* (2.9)-(2.11)*, respectively. Assume that* $j(\cdot)$ *is uniformly convex such that*

$$(j'(v), v - u)_U - (j'(u), v - u)_U \geq c \| v - u \|_{0,\Omega_U}^2, \tag{4.1}$$

$g(\cdot)$ *is convex,* $j'(\cdot)$ *and* $g'(\cdot)$ *are Lipschitz continuous, the operator* B *is bounded,* $K^h \subset K$*,* $u \in H^1(\Omega_U)$*. Then we have that*

$$\| u - u_h \|_{0,\Omega_U} \leq C(h_U + \| p(u_h) - p_h \|_{0,\Omega}), \tag{4.2}$$

where $(y(u_h), p(u_h)) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *is the solution of the following equation:*

$$(\varepsilon\nabla y(u_h), \nabla w) + (\vec{b}\cdot\nabla y(u_h), w) + (ay(u_h), w) = (f + Bu_h, w) \qquad \forall w \in Y \tag{4.3}$$

$$(\varepsilon\nabla p(u_h), \nabla q) - (\vec{b}\cdot\nabla p(u_h), q) + (ap(u_h), q) = (g'(y(u_h)), q) \qquad \forall q \in Y. \tag{4.4}$$

Proof. Set $J(u) = j(u) + g(y(u))$ as in the problem (2.4). Then,

$$(J'(u), v) = (j'(u) + B^* p, v)_U, \tag{4.5}$$

$$(J'(u_h), v) = (j'(u_h) + B^* p(u_h), v)_U, \tag{4.6}$$

where p and $p(u_h)$ are the solutions of (2.6)-(2.8) and (4.3)-(4.4), respectively. It follows from (4.1) and (4.5)-(4.6) that

$$\begin{aligned}&(J'(u), u-u_h) - (J'(u_h), u-u_h)\\ \ge\ & c \parallel u-u_h \parallel_{0,\Omega_U}^2 + (B^*p - B^*p(u_h), u-u_h)_U.\end{aligned} \tag{4.7}$$

Moreover, it can be deduced from (2.6)-(2.7) and (4.3)-(4.4) that

$$\begin{aligned}&(B^*p - B^*p(u_h), u-u_h)_U = (B(u-u_h), p-p(u_h))\\
=\ &(\varepsilon\nabla(y-y(u_h)), \nabla(p-p(u_h))) + (\vec{b}\cdot\nabla(y-y(u_h)), p-p(u_h))\\
&+ (a(y-y(u_h)), p-p(u_h))\\
=\ &(\varepsilon\nabla(p-p(u_h)), \nabla(y-y(u_h))) - (\vec{b}\cdot\nabla(p-p(u_h)), y-y(u_h))\\
&+ (a(p-p(u_h)), y-y(u_h))\\
=\ &(g'(y) - g'(y(u_h)), y-y(u_h)) \ge 0.\end{aligned} \tag{4.8}$$

Thus, (4.7) and (4.8) imply that

$$(J'(u), u-u_h) - (J'(u_h), u-u_h) \ge c \parallel u-u_h \parallel_{0,\Omega_U}^2. \tag{4.9}$$

Let $u_I \in K^h$ be the L^2-projection of u. Then it follows from (4.9),(2.8) and (2.11) that

$$\begin{aligned}&c \parallel u-u_h \parallel_{0,\Omega_U}^2 \le (J'(u) - J'(u_h), u-u_h)\\
=\ &(j'(u) + B^*p, u-u_h)_U - (j'(u_h) + B^*p(u_h), u-u_h)_U\\
=\ &(j'(u) + B^*p, u-u_h)_U + (B^*(p(u_h) - p_h), u_h-u)_U\\
&+ (j'(u_h) + B^*p_h, u_h-u_I)_U + (j'(u_h) + B^*p_h, u_I-u)_U\\
\le\ &0 + (B^*(p(u_h) - p_h), u_h-u)_U + 0 + (j'(u_h) + B^*p_h, u_I-u)_U\\
=\ &(j'(u_h) + B^*p_h - j'(u) - B^*p, u_I-u)_U + (j'(u) + B^*p, u_I-u)_U\\
&+ (B^*(p(u_h) - p_h), u_h-u)_U\\
\le\ &(j'(u_h) - j'(u), u_I-u)_U + (B^*(p_h - p(u_h)), u_I-u)_U + (B^*(p(u_h) - p), u_I-u)_U\\
&+ (j'(u) + B^*p, u_I-u)_U + (B^*(p(u_h) - p_h), u_h-u)_U\\
\le\ &(j'(u) + B^*p, u_I-u)_U + C(\gamma) \parallel u-u_I \parallel_{0,\Omega_U}^2 + C(\gamma) \parallel B^*(p(u_h) - p_h) \parallel_{0,\Omega}^2\\
&+ C\gamma \parallel B^*(p(u_h) - p) \parallel_{0,\Omega}^2 + C\gamma \parallel j'(u_h) - j'(u) \parallel_{0,\Omega_U}^2 + C\gamma \parallel u-u_h \parallel_{0,\Omega_U}^2,\end{aligned} \tag{4.10}$$

where γ is an arbitrary positive number. Note that $u_I \in U^h$ be the integral average of u on each element such that

$$u_I|_{\tau_U} = \pi^a u|_{\tau_U} = \frac{\int_{\tau_U} u}{\int_{\tau_U} 1}.$$

Then it is easy to prove that (see, e.g., [5]) if $u \in H^1(\Omega_U)$,

$$\| u - u_I \| \leq C h_U \| u \|_{1,\Omega_U} . \tag{4.11}$$

Moreover, if $u \in H^1(\Omega_U)$ and $p \in H^1(\Omega)$, we have

$$\begin{aligned}
&(j'(u) + B^* p, u_I - u)_U = (j'(u) + B^* p, \pi^a u - u)_U \\
&= \sum_{\tau \in T_U^h} \int_{\tau_U} (j'(u) + B^* p - \pi^a(j'(u) + B^* p))(\pi^a u - u) \\
&\leq \| j'(u) + B^* p - \pi^a(j'(u) + B^* p) \|_{0,\Omega_U} \| \pi^a u - u \|_{0,\Omega_U} \\
&\leq C h_U^2 \mid j'(u) + B^* p \mid_{1,\Omega_U} \mid u \mid_{1,\Omega_U} \\
&\leq C h_U^2 (\mid u \mid_{1,\Omega_U}^2 + \mid p \mid_{1,\Omega}^2) \leq C h_U^2 .
\end{aligned} \tag{4.12}$$

Note that B^* and $j'(\cdot)$ are bounded. Then it follows from (4.10)-(4.12) that

$$\begin{aligned}
\| u - u_h \|_{0,\Omega_U}^2 \leq\ & C(\gamma) h_U^2 + C(\gamma) \| p(u_h) - p_h \|_{0,\Omega}^2 \\
& + C\gamma \| p(u_h) - p \|_{0,\Omega}^2 + C\gamma \| u - u_h \|_{0,\Omega_U}^2 .
\end{aligned} \tag{4.13}$$

From (2.7) and (4.4) we can deduce that

$$A(q, p - p(u_h)) = (g'(y) - g'(y(u_h)), q).$$

Setting $q = p - p(u_h)$ and $||| v ||| = \varepsilon \| \nabla v \|^2 + a \| v \|^2$, we have that

$$||| p - p(u_h) ||| \leq C \| g'(y) - g'(y(u_h)) \|_{0,\Omega} \leq C ||| y - y(u_h) ||| .$$

Similarly, it can be proved that

$$||| y - y(u_h) ||| \leq C \| u - u_h \|_{0,\Omega_U} .$$

Therefore, we obtain that

$$\| p - p(u_h) \|_{0,\Omega} \leq ||| p - p(u_h) ||| \leq C \| u - u_h \|_{0,\Omega_U} . \tag{4.14}$$

Thus (4.13) and (4.14) imply that

$$\| u - u_h \|_{0,\Omega_U}^2 \leq C(\gamma) h_U^2 + C(\gamma) \| p_h - p(u_h) \|_{0,\Omega}^2 + C\gamma \| u - u_h \|_{0,\Omega_U}^2 .$$

This proves (4.2) by setting $\gamma = \frac{1}{2C}$. □

Theorem 4.2. *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of* (2.6)-(2.8) *and* (2.9)-(2.11)*, respectively. Assume that all conditions of Lemma* 4.1 *are valid. Moreover, assume that* $y,\ p,\ y(u_h),\ p(y_h)\in H^2(\Omega)$*, where* $y(u_h)$ *is the solution of the system* (4.3)-(4.4)*,* $p(y_h)$ *is the solution of the equation:*

$$(\varepsilon\nabla p(y_h),\nabla q)-(\vec{b}\cdot\nabla p(y_h),q)+(ap(y_h),q)=(g'(y_h),q)\quad \forall q\in H_0^1(\Omega). \tag{4.15}$$

Let δ *be defined by* (2.14)*. Then we have*

$$\| y-y_h\|_{SD}+\| p-p_h\|_{SD}+\| u-u_h\|_{0,\Omega_U}\le C(h_U+h^{3/2}+h\varepsilon^{1/2}), \tag{4.16}$$

where

$$\| w\|_{SD}^2=\varepsilon\|\nabla w\|_{0,\Omega}^2+\| a^{\frac{1}{2}}w\|_{0,\Omega}^2+\delta\|\vec{b}\cdot\nabla w\|_{0,\Omega}^2 .$$

Proof. Let $p(y_h)$ be the solution of Eq. (4.15). Then it is easy to see that p_h is the streamline diffusion finite element solution of $p(y_h)$, and

$$\|p(u_h)-p(y_h)\|_{0,\Omega}\le C\|g'(y(u_h))-g'(y_h)\|_{0,\Omega}\le C\|y(u_h)-y_h\|_{0,\Omega}. \tag{4.17}$$

Using the results of [1] and [25], we obtain the following error estimate:

$$\| p(y_h)-p_h\|_{SD}\le C(h^{3/2}+h\varepsilon^{1/2}). \tag{4.18}$$

Similarly, we have

$$\| y(u_h)-y_h\|_{SD}\le C(h^{3/2}+h\varepsilon^{1/2}). \tag{4.19}$$

Then it follows from (4.17)-(4.19) that

$$\|p(u_h)-p_h\|_{0,\Omega}\le\|p(u_h)-p(y_h)\|_{0,\Omega}+\|p(y_h)-p_h\|_{0,\Omega}\le C(h^{3/2}+h\varepsilon^{1/2}). \tag{4.20}$$

Recall Lemma 4.1, it can be deduced from (4.2) and (4.20) that

$$\| u-u_h\|_{0,\Omega_U}\le C(h_U+h^{3/2}+h\varepsilon^{1/2}). \tag{4.21}$$

Let $\tilde{y}_h$ be the solution of the following equation

$$A_h^s(\tilde{y}_h,w_h)=(f+Bu,w_h)_h^s\quad \forall w_h\in V^h. \tag{4.22}$$

We can find that $\tilde{y}_h$ is the streamline diffusion finite element solution of y, and

$$\|\tilde{y}_h-y_h\|_{SD}\le C\| u-u_h\|_{0,\Omega_U} . \tag{4.23}$$

Again, it can be deduced as in [1] and [25] that

$$\| y-\tilde{y}_h\|_{SD}\le Ch(h^{1/2}+\varepsilon^{1/2}). \tag{4.24}$$

Consequently, (4.21) and (4.23)-(4.24) imply that

$$\| y-y_h\|_{SD}\le Ch(h^{1/2}+\varepsilon^{1/2})+C\| u-u_h\|_{0,\Omega_U}\le C(h_U+h^{3/2}+h\varepsilon^{1/2}). \tag{4.25}$$

Similarly, let $\tilde{p}_h$ be the solution of the following equation

$$A_h^a(\tilde{p}_h, q_h) = (g'(y), q_h)_h^a \quad \forall q_h \in V^h. \tag{4.26}$$

We have

$$\| p - \tilde{p}_h \|_{SD} \leq Ch(h^{1/2} + \varepsilon^{1/2}), \tag{4.27}$$

and

$$\| \tilde{p}_h - p_h \|_{SD} \leq C \| g'(y) - g'(y_h) \|_{0,\Omega} \leq C \| y - y_h \|_{0,\Omega} . \tag{4.28}$$

Then, from (4.25)-(4.27) we obtain

$$\| p - p_h \|_{SD} \leq Ch(h^{1/2} + \varepsilon^{1/2}) + C \| y - y_h \|_{0,\Omega} \leq C(h_U + h\varepsilon^{1/2} + h^{3/2}). \tag{4.29}$$

Summing up, (4.16) is proved from (4.21), (4.25) and (4.29). □

Remark 4.3. In this section, we discussed a priori error estimate. It is proven that the accuracy order of the discrete scheme is $\mathscr{O}(h_U + h^{3/2})$ when the solution is smooth enough and $\epsilon \leq h$. It is not optimal for the state and the co-state in L^2-norm, and it is well known that the optimal L^2-error estimate is not available for the advection dominated diffusion problems generally. Using the superconvergence analysis technique, the optimal L^2-error estimate can be obtained, but some strong conditions are required (see, e.g., [14, 16, 17, 28, 29], for more details).

5. A posteriori error estimates

In the following we shall derive a posteriori error estimates in L^2-norm for the problem (2.6)-(2.8) and its streamline diffusion finite element approximation (2.9)-(2.11).

Recall that we consider the optima control problem with constrained control set (see (2.3)) in this paper, i.e., $K = \{v \in U, v \geq 0\}$. In this section, let the functional $j(u) = \int_{\Omega_U} \tilde{j}(u)$, where $\tilde{j}(\cdot)$ is a function. Hence $j'(\cdot)(v) = (\tilde{j}'(\cdot), v)_U$ for all $v \in L^2(\Omega_U)$. For simplicity, we denote $\tilde{j}(\cdot)$ still by $j(\cdot)$ in the following. In order to construct a posteriori error estimates, we divide the domain Ω_U into three subdomains:

$$\begin{aligned}
&\Omega^- = \{x \in \Omega_U : \ (B^* p_h)(x) + j'(0) \leq 0\}, \\
&\Omega^0 = \{x \in \Omega_U : \ (B^* p_h)(x) + j'(0) > 0, \ u_h(x) = 0\}, \\
&\Omega^+ = \{x \in \Omega_U : \ (B^* p_h)(x) + j'(0) > 0, \ u_h(x) > 0\}.
\end{aligned}$$

Then it is easy to see that above three subsets are not intersected each other, and

$$\bar{\Omega}_U = \bar{\Omega}^- \cup \bar{\Omega}^0 \cup \bar{\Omega}^+.$$

Moreover, it is clear that Ω^- should be the active set, while Ω^0 should be the inactive one.

To derive a posteriori error estimates, we need the following lemmas. The proof of Lemma 5.1 can be find in [26]. Lemma 5.2 is a well known result; its proof can be find in, e.g., [5].

Lemma 5.1. *Let π_h be the average interpolation operator defined in* [26]. *For $m = 0$ or 1, $1 \leq q \leq \infty$ and $v \in W^{1,q}(\Omega)$, we have $\pi_h v \in V^h$, and*

$$\|v - \pi_h v\|_{m,q,\tau} \leq C \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} h_{\tau}^{1-m} |v|_{1,q,\tau'}.$$

Lemma 5.2. *Assume that the element τ is regular. Then for $v \in W^{1,q}(\tau)$, $1 \leq q < \infty$, we have*

$$\|v\|_{0,q,\partial\tau} \leq C(h_{\tau}^{-\frac{1}{q}} \|v\|_{0,q,\tau} + h_{\tau}^{1-\frac{1}{q}} |v|_{1,q,\tau}).$$

Firstly, let us consider the a posteriori error estimate for the control u.

Lemma 5.3. *Let $(y,p,u),(y_h,p_h,u_h)$ be the solution of* (2.6)-(2.8) *and* (2.9)-(2.11), *respectively. Assume that $j(\cdot)$ is uniformly convex (see* (4.1) *in Lemma* 4.1*), $g(\cdot)$ is convex, $j'(\cdot)$ is Lipschitz continuous, the operator B is bounded, $K^h \subset K$. Then we have*

$$\| u - u_h \|_{0,\Omega_U}^2 \leq C(\eta_1^2 + \| p(u_h) - p_h \|_{0,\Omega}^2), \tag{5.1}$$

where $(y(u_h), p(u_h))$ is the solution of the system (4.3)-(4.4), *and*

$$\eta_1^2 = \int_{\Omega^- \cup \Omega^+} (j'(u_h) + B^* p_h)^2.$$

Proof. Let $J(u) = j(u) + g'(y(u))$. It has been proved in Lemma 4.1 (see (4.9)) that

$$(J'(u), u - u_h) - (J'(u_h), u - u_h) \geq c \| u - u_h \|_{0,\Omega_U}^2 . \tag{5.2}$$

Note that $u_h \in K$. Similar to Lemma 4.1, it follows from (5.2) and (2.8) that

$$\begin{aligned} c\|u - u_h\|_{0,\Omega_U}^2 &\leq (J'(u), u - u_h)_U - (J'(u_h), u - u_h)_U \\ &= (j'(u) + B^* p, u - u_h)_U - (j'(u_h) + B^* p(u_h), u - u_h)_U \\ &\leq -(B^* p(u_h) + j'(u_h), u - u_h)_U \\ &= (j'(u_h) + B^* p_h, u_h - u)_U + (B^*(p_h - p(u_h)), u - u_h)_U. \end{aligned} \tag{5.3}$$

It is easy to see that

$$\begin{aligned} &(j'(u_h) + B^* p_h, u_h - u)_U \\ =& \int_{\Omega^- \cup \Omega^+} (j'(u_h) + B^* p_h)(u_h - u) + \int_{\Omega^0} (j'(u_h) + B^* p_h)(u_h - u), \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\int_{\Omega^- \cup \Omega^+} (j'(u_h) + B^* p_h)(u_h - u) \\ \leq& C(\gamma) \int_{\Omega^- \cup \Omega^+} (j'(u_h) + B^* p_h)^2 + C\gamma \|u_h - u\|_{0,\Omega_U}^2 \\ =& C(\gamma)\eta_1^2 + C\gamma \|u_h - u\|_{0,\Omega_U}^2, \end{aligned} \tag{5.5}$$

where γ is an arbitrary positive number. Note that $j'(u_h)+B^*p_h \geq j'(0)+B^*p_h > 0$ and $u_h - u = 0 - u \leq 0$ on the domain Ω^0. This leads to

$$\int_{\Omega^0}(j'(u_h)+B^*p_h)(u_h-u) \leq 0. \tag{5.6}$$

Then (5.4)-(5.6) imply that

$$(j'(u_h)+B^*p_h, u_h-u)_U \leq C(\gamma)\eta_1^2 + C\gamma\|u_h-u\|^2_{0,\Omega_U}. \tag{5.7}$$

Moreover, Schwartz inequality implies that

$$\begin{aligned}(B^*(p_h-p(u_h)), u-u_h)_U &\leq C(\gamma)\|B^*(p_h-p(u_h))\|^2_{0,\Omega_U} + C\gamma\|u-u_h\|^2_{0,\Omega_U}\\ &\leq C(\gamma)\|p_h-p(u_h)\|^2_{0,\Omega} + C\gamma\|u-u_h\|^2_{0,\Omega_U}.\end{aligned} \tag{5.8}$$

Summing up, (5.1) follows from (5.3), (5.7) and (5.8). This proves the theorem. □

In order to obtain the a posteriori error estimate of $\| y_h - y(u_h)\|_{0,\Omega}$ and $\| p_h - p(u_h)\|_{0,\Omega}$, we introduce the following auxiliary dual problems:

$$\begin{cases} -\varepsilon\triangle\phi_1 - \vec{b}\cdot\nabla\phi_1 + \alpha\phi_1 = f_1 & \text{in} \quad \Omega,\\ \phi_1 = 0 & \text{on} \quad \partial\Omega,\end{cases} \tag{5.9}$$

and

$$\begin{cases} -\varepsilon\triangle\phi_2 + \vec{b}\cdot\nabla\phi_2 + \alpha\phi_2 = f_2 & \text{in} \quad \Omega,\\ \phi_2 = 0 & \text{on} \quad \partial\Omega.\end{cases} \tag{5.10}$$

For above dual problems, we have the following stability estimates (see, e.g., [23]).

Lemma 5.4. *Let ϕ_i be the solution of* (5.9) *or* (5.10)*. If Ω is convex polygon or smooth, then for $i = 1$ or 2, we have*

$$\varepsilon^{3/2} \| \phi_i \|_{2,\Omega} + \varepsilon^{1/2} \| \phi_i \|_{1,\Omega} + \| \phi_i \|_{0,\Omega} \leq C \| f_i \|_{0,\Omega}.$$

Now we are in the position to prove the a posteriori error estimate for the problem (2.6)-(2.8) and its streamline diffusion finite element scheme (2.9)-(2.11).

Theorem 5.5. *Let $(y,p,u),(y_h,p_h,u_h)$ be the solution of* (2.6)-(2.8) *and* (2.9)-(2.11)*, respectively. Assume that $j(\cdot)$ is uniformly convex (see* (4.1) *in Lemma* 4.1*), $g(\cdot)$ is convex, $j'(\cdot)$ and $g'(\cdot)$ are Lipschitz continuous, the operator B is bounded, $K^h \subset K$, Ω is convex polygon or smooth. Then we have*

$$\| u - u_h \|^2_{0,\Omega_U} + \| y - y_h \|^2_{0,\Omega} + \| p - p_h \|^2_{0,\Omega} \leq C\sum_{i=1}^{5}\eta_i^2, \tag{5.11}$$

where η_1 is defined in Lemma 5.3,

$$\eta_2^2 = \sum_{\tau \in T^h} \frac{h_\tau^2}{\varepsilon} \int_\tau (f + Bu_h - \vec{b} \cdot \nabla y_h - \alpha y_h)^2,$$

$$\eta_3^2 = \sum_{l \in \partial T^h, l \cap \partial\Omega = \emptyset} \frac{h_l}{\varepsilon} \int_l [\varepsilon \nabla y_h \cdot \vec{n}]^2,$$

$$\eta_4^2 = \sum_{\tau \in T^h} \frac{h_\tau^2}{\varepsilon} \int_\tau (\vec{b} \cdot \nabla p_h - \alpha p_h + g'(y_h))^2,$$

$$\eta_5^2 = \sum_{l \in \partial T^h, l \cap \partial\Omega = \emptyset} \frac{h_l}{\varepsilon} \int_l [\varepsilon \nabla p_h \cdot \vec{n}]^2.$$

Here $l = \bar{\tau}_l^1 \cap \bar{\tau}_l^2$ is the edge of the element, h_l is the length of the edge l, $[v]_l$ is the jump of v over the edge l:

$$[v(x)]_{x \in l} = \lim_{s \to 0^+} \Big(v(x + sn) - v(x - sn) \Big),$$

and $\vec{n}$ is the unit normal vector on l outward $\partial \tau_l^1$.

Proof. Let $f_2 = p(u_h) - p_h$ in (5.10). We obtain

$$\begin{aligned}\| p(u_h) - p_h \|_{0,\Omega}^2 &= (-\varepsilon \triangle \phi_2 + \vec{b} \cdot \nabla \phi_2 + a\phi_2, p(u_h) - p_h) \\ &= (g'(y(u_h)), \phi_2) - (\varepsilon \nabla \phi_2, \nabla p_h) + (\phi_2, \vec{b} \cdot \nabla p_h) - (\alpha \phi_2, p_h).\end{aligned}$$

Note that

$$(\varepsilon \nabla p_h, \nabla q_h) - (\vec{b} \cdot \nabla p_h - \alpha p_h, q_h - \delta \vec{b} \cdot \nabla q_h) = (g'(y_h), q_h - \delta \vec{b} \cdot \nabla q_h) \quad \forall q_h \in V^h.$$

Then

$$\begin{aligned}&\| p(u_h) - p_h \|_{0,\Omega}^2 = (g'(y(u_h)), \phi_2) - (\varepsilon \nabla \phi_2, \nabla p_h) + (\phi_2, \vec{b} \cdot \nabla p_h) - (\alpha \phi_2, p_h) \\ &\quad + (\varepsilon \nabla p_h, \nabla q_h) - (\vec{b} \cdot \nabla p_h - \alpha p_h, q_h - \delta \vec{b} \cdot \nabla q_h) - (g'(y_h), q_h - \delta \vec{b} \cdot \nabla q_h) \\ &= (g'(y_h), \phi_2 - q_h + \delta \vec{b} \cdot \nabla q_h) + (\varepsilon \nabla p_h, \nabla(q_h - \phi_2)) + (\vec{b} \cdot \nabla p_h, \phi_2 - q_h + \delta \vec{b} \cdot \nabla q_h) \\ &\quad - (\alpha p_h, \phi_2 - q_h + \delta \vec{b} \cdot \nabla q_h) + (g'(y(u_h)) - g'(y_h), \phi_2) \\ &= \sum_{\tau \in T^h} \int_\tau (\vec{b} \cdot \nabla p_h - \alpha p_h + g'(y_h))(\phi_2 - q_h + \delta_2 \vec{b} \cdot \nabla q_h) \\ &\quad + \sum_{l \in \partial T^h} \int_l [\varepsilon \nabla p_h \cdot \vec{n}](q_h - \phi_2) + (g'(y(u_h)) - g'(y_h), \phi_2) \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{5.12}$$

Let $q_h = \pi_h \phi_2$, where π_h is defined in Lemma 5.1. Then we have

$$
\begin{aligned}
&\| \phi_2 - q_h + \delta_2 \vec{b} \cdot \nabla q_h \|_{0,\tau} \\
=&\| \phi_2 - \pi_h \phi_2 + \delta \vec{b} \cdot \nabla(\pi_h \phi_2) \|_{0,\tau} \leq \| \phi_2 - \pi_h \phi_2 \|_{0,\tau} + \delta \| \vec{b} \cdot \nabla(\pi_h \phi_2) \|_{0,\tau} \\
\leq& Ch_\tau \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} \| \nabla \phi_2 \|_{0,\tau'} + C\delta \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} \| \vec{b} \|_{L^\infty(\tau)} \| \nabla \phi_2 \|_{0,\tau'} \leq Ch_\tau \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} \| \nabla \phi_2 \|_{0,\tau'},
\end{aligned}
$$

where we used the fact that $\delta \leq Ch_\tau$ which can be derived from (2.14). Thus, we obtain

$$
\begin{aligned}
I_1 =& \sum_{\tau \in T^h} \int_\tau (\vec{b} \cdot \nabla p_h - \alpha p_h + g'(y_h))(\phi_2 - q_h + \delta_2 \vec{b} \cdot \nabla q_h) \\
\leq& C \sum_{\tau \in T^h} h_\tau \| \vec{b} \cdot \nabla p_h - \alpha p_h + g'(y_h) \|_{0,\tau} \sum_{\bar{\tau}' \cap \bar{\tau} \neq \emptyset} \| \nabla \phi_2 \|_{0,\tau'} \\
\leq& C(\gamma) \sum_{\tau \in T^h} \frac{h_\tau^2}{\varepsilon} \int_\tau (\vec{b} \cdot \nabla p_h - \alpha p_h + g'(y_h))^2 + C\gamma\varepsilon \|\phi_2\|_{1,\Omega}^2 \\
\leq& C(\gamma)\eta_4^2 + C\gamma \| p(u_h) - p_h \|_{0,\Omega}^2,
\end{aligned}
\tag{5.13}
$$

where γ is an arbitrary positive number. In the last step of (5.13), Lemma 5.4 is used to obtain

$$
\varepsilon \|\phi_2\|_{1,\Omega}^2 \leq C \| p(u_h) - p_h \|_{0,\Omega}^2 .
$$

Similarly, it follows from Lemmas 5.1 and 5.2 that when $l \subset \partial\tau$,

$$
\begin{aligned}
\| \pi_h \phi_2 - \phi_2 \|_{0,l} \leq& C \sum_{l \subset \partial\tau} (h_l^{-1/2} \| \pi_h \phi_2 - \phi_2 \|_{0,\tau} + h_l^{1/2} \| \pi_h \phi_2 - \phi_2 \|_{1,\tau}) \\
\leq& Ch_l^{1/2} \sum_{l \subset \partial\tau, \bar{\tau}' \cap \bar{\tau} \neq \emptyset} \| \nabla \phi_2 \|_{0,\tau'}.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 =& \sum_{l \in \partial T^h} \int_l [\varepsilon \nabla p_h \cdot \vec{n}](q_h - \phi_2) = \sum_{l \in \partial T^h, l \cap \partial\Omega = \emptyset} \int_l [\varepsilon \nabla p_h \cdot \vec{n}](q_h - \phi_2) \\
\leq& C \sum_{l \in \partial T^h, l \cap \partial\Omega = \emptyset} h_l^{1/2} (\int_l [\varepsilon \nabla p_h \cdot \vec{n}]^2)^{\frac{1}{2}} \sum_{\sum_{l \subset \partial\tau}, \bar{\tau}' \cap \bar{\tau} \neq \emptyset} \| \nabla \phi_2 \|_{0,\tau'} \\
\leq& C(\gamma) \sum_{l \in \partial T^h, l \cap \partial\Omega = \emptyset} \frac{h_l}{\varepsilon} \int_l [\varepsilon \nabla p_h \cdot \vec{n}]^2 + C\gamma\varepsilon \|\phi_2\|_{1,\Omega}^2 \\
\leq& C(\gamma)\eta_5^2 + C\gamma \| p(u_h) - p_h \|_{0,\Omega}^2 .
\end{aligned}
\tag{5.14}
$$

Moreover, it is easy to see that

$$
\begin{aligned}
I_3 =& (g'(y(u_h)) - g'(y_h), \phi_2) \leq \|g'(y(u_h)) - g'(y_h) \|_{0,\Omega} \| \phi_2 \|_{0,\Omega} \\
\leq& C(\gamma) \| y(u_h) - y_h \|_{0,\Omega}^2 + C\gamma \|p(u_h) - p_h \|_{0,\Omega}^2 .
\end{aligned}
\tag{5.15}
$$

Summing up, it follows from (5.12)-(5.15) that

$$\| p(u_h) - p_h \|_{0,\Omega}^2 \leq C \| y(u_h) - y_h \|_{0,\Omega}^2 + C\eta_4^2 + C\eta_5^2. \tag{5.16}$$

Similarly, setting $f_1 = y(u_h) - y_h$ in (5.9), we have

$$\begin{aligned}
\| y(u_h) - y_h \|_{0,\Omega}^2 &= (-\varepsilon\triangle\phi_1 - \vec{b}\cdot\nabla\phi_1 + \alpha\phi_1, y(u_h) - y_h) \\
&= (\varepsilon\nabla y(u_h), \nabla\phi_1) + (\vec{b}\cdot\nabla y(u_h), \phi_1) + (\alpha y(u_h), \phi_1) \\
&\quad - (\varepsilon\nabla\phi_1, \nabla y_h) - (\phi_1, \vec{b}\cdot\nabla y_h) - (a\phi_1, y_h) \\
&= (f + Bu_h, \phi_1) - (\varepsilon\nabla y_h, \nabla\phi_1) - (\vec{b}\cdot\nabla y_h, \phi_1) - (ay_h, \phi_1) \\
&\quad - (f + Bu_h, \pi_h\phi_1 + \delta\vec{b}\cdot\nabla\pi_h\phi_1) \\
&\quad + (\varepsilon\nabla y_h, \nabla\pi_h\phi_1) + (\vec{b}\cdot\nabla y_h + ay_h, \pi_h\phi_1 + \delta\vec{b}\cdot\nabla\pi_h\phi_1) \\
&= (f + Bu_h, \phi_1 - \pi_h\phi_1 - \delta\vec{b}\cdot\nabla\pi_h\phi_1) + (\varepsilon\nabla y_h, \nabla(\pi_h\phi_1 - \phi_1)) \\
&\quad + (\vec{b}\cdot\nabla y_h + ay_h, \pi_h\phi_1 + \delta\vec{b}\cdot\nabla\pi_h\phi_1 - \phi_1) \\
&= \sum_{\tau\in T^h}\int_\tau (f + Bu_h - \vec{b}\cdot\nabla y_h - \alpha y_h)(\phi_1 - \pi_h\phi_1 - \delta_1\vec{b}\cdot\nabla\pi_h\phi_1) \\
&\quad + \sum_{l\in\partial T^h}\int_l [\varepsilon\nabla y_h\cdot\vec{n}](\pi_h\phi_1 - \phi_1) \\
&\leq C(\gamma)\sum_{\tau\in T^h}\frac{h_\tau^2}{\varepsilon}\int_\tau (f + Bu_h - \vec{b}\cdot\nabla y_h - \alpha y_h)^2 \\
&\quad + C(\gamma)\sum_{l\in\partial T^h, l\cap\partial\Omega=\emptyset}\frac{h_l}{\varepsilon}\int_l [\varepsilon\nabla y_h\cdot\vec{n}]^2 + C\gamma\varepsilon\|\phi_1\|_{1,\Omega}^2 \\
&\leq C(\gamma)(\eta_2^2 + \eta_3^2) + C\gamma \| y(u_h) - y_h \|_{0,\Omega}^2 .
\end{aligned}$$

Therefore,

$$\| y(u_h) - y_h \|_{0,\Omega}^2 \leq C(\eta_2^2 + \eta_3^2). \tag{5.17}$$

Combining (5.16), (5.17) and Lemma 5.3, we obtain that

$$\|u - u_h\|_{0,\Omega_U}^2 \leq C\sum_{i=1}^{5}\eta_i^2. \tag{5.18}$$

Moreover, it is easy to see that

$$\|y - y(u_h)\|_{0,\Omega} \leq C\|B(u - u_h)\|_{0,\Omega} \leq C\|u - u_h\|_{0,\Omega_U} \tag{5.19}$$

and

$$\begin{aligned}
\|p - p(u_h)\|_{0,\Omega} &\leq C\|g'(y) - g'(y(u_h))\|_{0,\Omega} \\
&\leq C\|y - y(u_h)\|_{0,\Omega} \leq C\|u - u_h\|_{0,\Omega_U}.
\end{aligned} \tag{5.20}$$

Thus, it can be deduced from (5.16)-(5.20) that

$$\|y-y_h\|_{0,\Omega}^2 \le C\|y-y(u_h)\|_{0,\Omega}^2 + C\|y(u_h)-y_h\|_{0,\Omega}^2 \le C\sum_{i=1}^{5}\eta_i^2, \tag{5.21}$$

$$\|p-p_h\|_{0,\Omega}^2 \le C\|p-p(u_h)\|_{0,\Omega}^2 + C\|p(u_h)-p_h\|_{0,\Omega}^2 \le C\sum_{i=1}^{5}\eta_i^2. \tag{5.22}$$

Then (5.11) follows from (5.18), (5.21) and (5.22). This proves the theorem. □

6. Numerical examples

In this section we will present several numerical examples to illustrate our theoretical results obtained in the earlier sections.

Consider the problem:

$$\min_{u\in K\subset U}\left\{\frac{1}{2}\int_\Omega (y-y_0)^2 + \int_{\Omega_U}(u-u_0)^2\right\} \tag{6.1}$$

subject to

$$\begin{aligned} &-\varepsilon\triangle y + \vec{b}\cdot\nabla y + ay = f + Bu \quad \text{in } \Omega,\\ &y = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{6.2}$$

with $\Omega=[0,1]\times[0,1]$, $B=I$. For simplicity, we use the same mesh for T^h and T_U^h.

Example 6.1. *Let $\varepsilon=10^{-5}$, $\vec{b}=(1,1)$, $a=10$, y_0, u_0, f be as following:*

$$\begin{aligned} y_0 &= \pi(\cos\pi x_1\sin 2\pi x_2 + 2\cos 2\pi x_2\sin\pi x_1)(1+e^{-(x_1+x_2)/\varepsilon}),\\ u_0 &= 1-\sin(\pi x_1/2)-\sin(\pi x_2/2)-\sin(\pi x_1)\sin(\pi x_2)\\ &\quad +(1-e^{-(x_1+x_2)/\varepsilon})\sin(\pi x_1)\sin(2\pi x_2),\\ f &= (5\pi^2\varepsilon+10)^2(1-e^{-(x_1+x_2)/\varepsilon})\sin(\pi x_1)\sin(2\pi x_2)\\ &\quad +4(5\pi^2\varepsilon+10)/\varepsilon e^{-(x_1+x_2)/\varepsilon}\sin(\pi x_1)\sin(2\pi x_2)\\ &\quad +(5\pi^2\varepsilon+10)\pi\varepsilon(-3e^{-(x_1+x_2)/\varepsilon}+1)(\sin(2\pi x_2)\cos(\pi x_1)+2\sin(\pi x_1)\cos(2\pi x_2))\\ &\quad -\max\{0,1-\sin(\pi x_1/2)-\sin(\pi x_2/2)-\sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

Then the exact solution of problem (6.1)-(6.2) *is*

$$\begin{aligned} y &= (5\pi^2\varepsilon+10)(1-e^{-(x_1+x_2)/\varepsilon})\sin(\pi x_1)\sin(2\pi x_2),\\ p &= (1-e^{-(x_1+x_2)/\varepsilon})\sin(\pi x_1)\sin(2\pi x_2),\\ u &= \max\{0,1-\sin(\pi x_1/2)-\sin(\pi x_2/2)-\sin(\pi x_1)\sin(\pi x_2)\}. \end{aligned}$$

Table 1: Convergence results on uniform mesh for Example 6.1.

Dofs	$\| y-y_h \|_{SD}$	order	$\| p-p_h \|_{SD}$	order	$\| u-u_h \|_0$	order
41	5.265111e+0		5.344924e-1		5.26e-2	
145	2.658064e+0	0.98	2.722418e-1	0.97	2.57e-2	1.03
545	1.352481e+0	0.98	1.409325e-1	0.95	1.31e-2	0.97
2113	6.999507e-1	0.95	7.019852e-2	1.01	6.57e-3	1.00

Table 1 presents the errors and convergence orders of the streamline diffusion finite element approximation on the uniform mesh, where DoFs denotes the numbers of the degree of freedoms. It is shown that the numerical results are accordant with our theoretical results. Noting that we choose $h=h_U$ in our example, the convergence order is

$$\begin{aligned}&\|y-y_h\|_{SD}+\|p-p_h\|_{SD}+\|u-u_h\|_{0,\Omega_U}\\ =&\mathcal{O}(h^{1.5}+h_U)=\mathcal{O}(h_U)=\mathcal{O}(h).\end{aligned}$$

Fig. 1(a) shows the surface of control u, and Fig. 1(b) is the adaptive mesh for control u obtained by using the indicator η_1^2. It is shown that a higher density of node points are distributed along the free boundary. We obtain an error

$$\| u-u_h \|_{0,\Omega_U}=6.339274\times 10^{-3}$$

using 716 nodes on the adaptive mesh shown in Fig. 1(b). However, it is shown in Table 1 that we need 2113 nodes on uniform mesh to obtain a similar error. Thus it is evident that our adaptive mesh indeed saves substantial computing work.

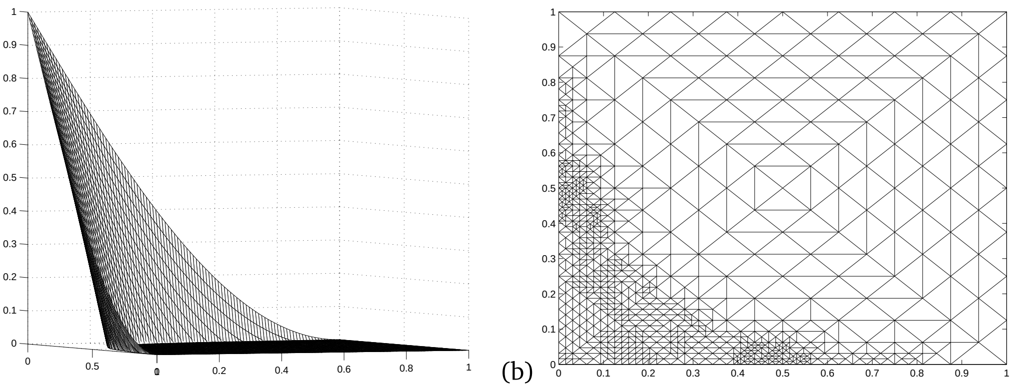
(a)


Figure 1: Example 6.1: (a) the surface of u; (b) the adaptive mesh T_U^h.

Example 6.2. *Let* $\varepsilon=10^{-4}$, $\vec{b}=(-1,-1)$, $a=10$. *Set* $M=e^{-((x_1-1/2)^2+(x_2-1/2)^2)/\varepsilon}$. *Assume* y_0, u_0, f *are of the following forms:*

$$\begin{aligned}y_0=&\ M\sin(\pi x_1)\sin(\pi x_2)(x_1(2x_1-1)/\varepsilon+y(2x_2-1)/\varepsilon)\\ &+\pi M\sin(\pi x_2)\cos(\pi x_1)(1/2-2x_1)+\pi M\sin(\pi x_1)\cos(\pi x_2)(1/2-2x_2),\end{aligned}$$

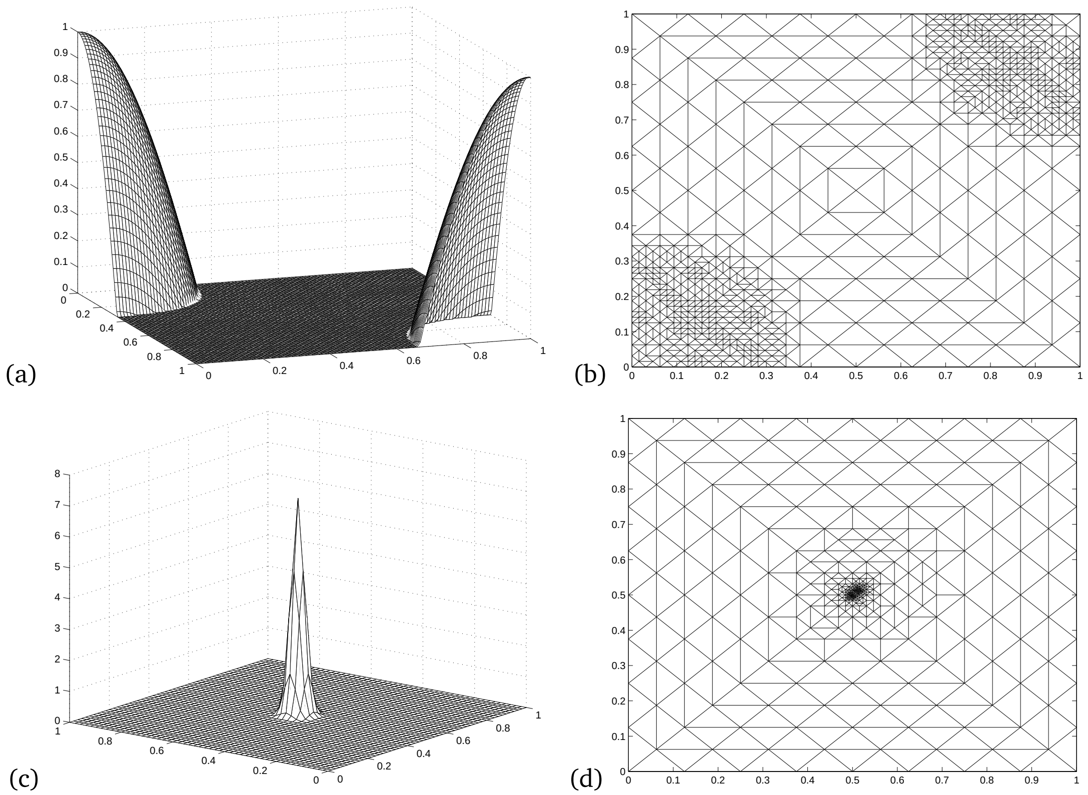


Figure 2: Example 6.2: (a) the surface of u; (b) the adaptive mesh T_U^h; (c) the surface of y; (d) the adaptive mesh T^h.

$$
\begin{aligned}
u_0 &= 2\cos(\pi x_1)\cos(\pi x_2) - 1 + M/2,\\
f &= 2(\pi^2\varepsilon+7)^2 M\sin(\pi x_1)\sin(\pi x_2) + 2(\pi^2\varepsilon+7)\pi M\cos(\pi x_1)\sin(\pi x_2)(2x_1-3/2)\\
&\quad + 2(\pi^2\varepsilon+7)\pi M\cos(\pi x_2)\sin(\pi x_1)(2x_2-3/2)\\
&\quad - 2(\pi^2\varepsilon+7)M\sin(\pi x_1)\sin(\pi x_2)(x_1-1)(2x_1-1)/\varepsilon\\
&\quad - 2(\pi^2\varepsilon+7)M\sin(\pi x_1)\sin(\pi x_2)(x_2-1)(2x_2-1)/\varepsilon - \max\{u_0 - M/2, 0\}.
\end{aligned}
$$

The exact solutions is

$$
\begin{aligned}
y &= (\pi^2\varepsilon+7)M\sin(\pi x_1)\sin(\pi x_2),\\
p &= M/2,\\
u &= \max\{0, 2\cos(\pi x_1)\cos(\pi x_2) - 1\}.
\end{aligned}
$$

We use η_1^2 as the indicator to construct the adaptive finite element mesh T_U^h for the control u, and use $\eta_2^2+\eta_3^2$ as the indicator to construct the adaptive finite element mesh T^h for the state y and the co-state p. Figs. 2(a) and 2(c) are surfaces of u and y. Figs. 2(b) and 2(d) are adaptive meshes obtained by the indicators η_1^2 and $\eta_2^2+\eta_3^2$, respectively.

Table 2 presents the error of the state y and the co-state p on the uniform mesh and the adaptive mesh, respectively. It is shown that the errors of $y - y_h$ and $p - p_h$ on the

Table 2: Error of y and p on uniform and adaptive meshes for Example 6.2.

	unform mesh, nodes=2113	adaptive mesh, nodes=852
$\| y - y_h \|_{0,\Omega}$	5.456995e-002	5.341020e-002
$\| y - y_h \|_{1,\Omega}$	2.175102e+000	1.925758e+000
$\| p - p_h \|_{0,\Omega}$	2.774408e-003	2.872870e-003
$\| p - p_h \|_{1,\Omega}$	1.453691e-001	1.282954e-001

adaptive mesh with nodes of 852 are similar to those on the uniform mesh with nodes of 2113. Again, the substantial computing work can be saved by using efficient adaptive mesh.

Example 6.3. *Let* $\varepsilon = 10^{-4}$, $\vec{b} = (2,3)$, $a = 1$. *The exact solution of problem* (6.1)-(6.2) *is*

$$\begin{aligned} &y = \sin(x_1)(1 - e^{(-2+2x_1)/\varepsilon})x_2^2(1 - e^{(-3+3x_2)/\varepsilon}), \\ &p = \sin(\pi x_1)\sin(\pi x_2), \\ &u = \max\{\sin(2\pi x_1) + \sin(2\pi x_2), 0\}. \end{aligned}$$

The right hand side f and the Dirichlet boundary data on $\partial\Omega$ are chosen accordingly.

The solution y shows exponential boundary layers along the boundary at $x_1 = 1$ and $x_2 = 1$.

We again use η_1^2 as the indicator to construct the adaptive finite element mesh T_U^h for the control u, and use $\eta_2^2 + \eta_3^2$ as the indicator to construct the adaptive finite element mesh T^h for the state y. Fig. 3(a) shows the contour-line of the control u; and Fig. 3(b) presents the adaptive mesh obtained by the indicators η_1^2 for the control u. Moreover, Fig. 4(a) shows the exact solution of the state y and Fig. 4(b) presents the adaptive mesh obtained by the indicators $\eta_2^2 + \eta_3^2$ for the state y.

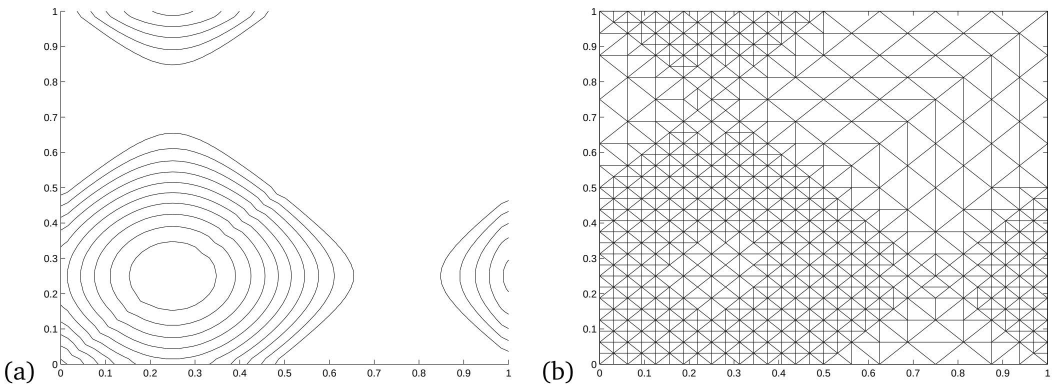
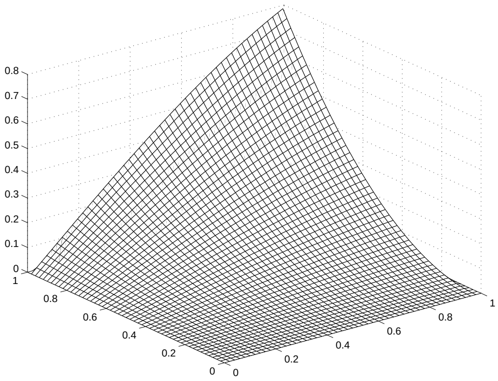


Figure 3: Example 6.3: (a) contour-line of control u; (b) the adaptive mesh of control u.

The errors of y and u on uniform and adaptive meshes are presented in Table 3. Again, a substantial computing work can be saved by using efficient adaptive mesh.

(a) 

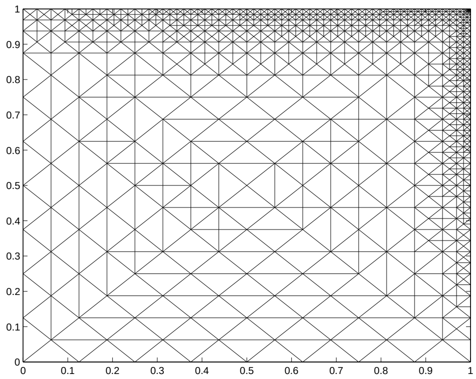
(b) 

Figure 4: Example 6.3: (a) the exact solution of state y; (b) the adaptive mesh of state y.

Table 3: Error of y and u on uniform and adaptive meshes.

Mesh	y-dof	u-dof	$\| y - y_h \|_{0,\Omega}$	$\| u - u_h \|_{0,\Omega_U}$
Uniform	2113	2113	6.85e-002	2.32e-002
Adaptive	833	930	5.37e-002	2.73e-002

7. Discussions

In this paper, we discussed the streamline diffusion finite element method for the constrained optimal control problem governed by convection dominated diffusion problems. The existence and uniqueness of the scheme are discussed. The a priori and a posteriori error estimates are provided. The numerical examples are presented to demonstrate our theoretical results. For simplicity, we only discuss the convection diffusion problem with the constant coefficients in this paper. The basic idea and technique used in this paper can be extended to variable coefficient problems.

There are many important issues still to be addressed in this area. Firstly, it should be pointed out that our streamline diffusion finite element scheme is only the approximation of the continuous optimality condition. It is not equivalent to the discrete optimal control problem. In the coming work, we will construct and analyze another streamline diffusion finite element scheme, which is derived from discrete optimal control problem. Moreover, it is important and challenging to investigate the control problem governed by evolution convection dominated diffusion problems.

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