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EXTRAPOLATION FOR THE SECOND ORDER ELLIPTIC PROBLEMS BY MIXED FINITE ELEMENT METHODS IN THREE DIMENSIONS

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Abstract. In this paper we derive asymptotic error expansions for mixed finite element approximations of general second order elliptic problems in three dimensions. And extrapolation method is applied to improve the accuracy of the approximations with the help of the interpolation postprocessing technique. For the cubic domain and uniform partition, with the extrapolation, the accuracy of the mixed finite element approximations can be improved.

Key Words. Superconvergence, interpolation postprocessing, extrapolation, mixed finite element

1. Introduction

We are concerned with the approximations of the following system:

$$\left\{\begin{array}{rcll} \mathbf{u}+A\nabla p & = & 0 & \text{in } \Omega, \\ \nabla\cdot\mathbf{u}+cp & = & f & \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} & = & 0 & \text{on } \Gamma, \end{array}\right. \tag{1.1}$$

where ∇ and $\nabla\cdot$ are the gradient and divergence operators, $\Omega\subset\mathbf{R}^3$ is an open bounded cubic domain with boundary Γ, $\mathbf{n}$ indicates the outward unit normal vector along Γ, $A^{-1}=(\alpha_{ij})_{3\times3}$ is a full positive definite matrix uniformly in Ω.

Mixed finite element methods [1] should be employed to discretize the system (1.1).

The main content of this paper is to present an analysis for the extrapolation of the mixed finite elements in three dimensions. The application of this approach in finite element methods was first established by Q. Lin [12]. The extrapolation method relies heavily on the existence of an asymptotic expansion for the error. The extrapolation of mixed finite element approximation in two dimensions was studied in [5]. In this paper, we study the three dimensional case.

This paper is organized in the following way. In Section 2, we establish the approximation subspace and the variational formulation for the problem (1.1) and the Raviart-Thomas interpolation. The asymptotic expansion for the Raviart-Thomas interpolation is derived in Section 3. Section 4 is devoted to investigating the asymptotic expansion of the error between the mixed finite element solution and the Raviart-Thomas interpolation of the exact solution to (1.1). Based on the expansion, the asymptotic expansion of the mixed finite element approximation is demonstrated by an interpolation postprocessing method in Section 5. Hence, The extrapolation can be used to improve the accuracy of the mixed finite element solution. Some concluding remarks are given in the finial section.

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2. The mixed finite element method and Raviart-Thomas interpolation

In this section, we formulate the mixed finite element method for the second order elliptic differential equation (1.1).

Let

$$W := L^2(\Omega), \quad \mathbf{V} := \mathbf{H}(\mathrm{div}, \Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^3 : \ \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\}$$

be the standard L^2 space on Ω with the norm $||\cdot||_0$ and the Hilbert space equipped with the norm

$$\|\mathbf{v}\|_{\mathbf{V}} := \left(\|\mathbf{v}\|_0^2 + \|\nabla \cdot \mathbf{v}\|_0^2 \right)^{\frac{1}{2}},$$

respectively. In addition, set

$$\mathbf{V}_0 := \{ \mathbf{v} \in \mathbf{V} : \mathbf{v} \cdot \mathbf{n} = 0, x \in \Gamma \}.$$

So, the corresponding weak mixed formulation for (1.1) seeks $(\mathbf{u}, p) \in \mathbf{V}_0 \times W$ such that

$$a(\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_0, \tag{2.1}$$

$$(\nabla \cdot \mathbf{u}, \omega) + (cp, \omega) = (f, \omega) \quad \forall \omega \in W, \tag{2.2}$$

where $a(\cdot, \cdot)$ is a bilinear form defined by

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega A^{-1} \mathbf{u} \mathbf{v} dx dy dz$$

and $(\cdot, \cdot)$ denotes the standard L^2-inner product.

Let $\mathbf{T}_{h_1,h_2,h_3}$ be a finite element partition of Ω into uniform hexahedrons, $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ denote a pair of finite element spaces satisfying the LBB condition, h_1, h_2 and h_3 denote the mesh sizes in $x-$,$y-$ and $z-$ axis and

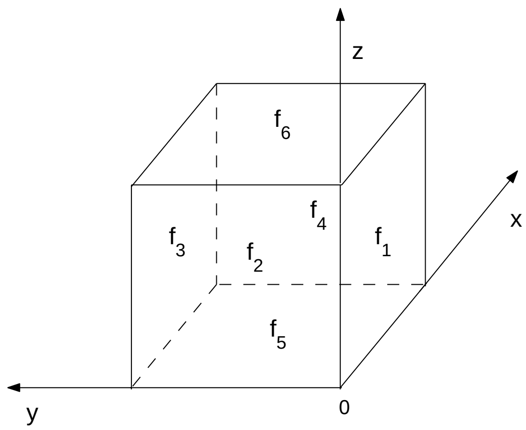
$$\mathbf{V}_h := \{ \mathbf{v}_h \in \mathbf{V}; \mathbf{v}_h|_e \in Q_{100} \times Q_{010} \times Q_{001}, \forall e \in \mathbf{T}_{h_1,h_2,h_3} \},$$
$$W_h := \{ w_h \in W : W_h|_e \in Q_{000}, \forall e \in \mathbf{T}_{h_1,h_2,h_3} \},$$

where Q_{ijk} denotes the space of polynomials of degree no more than i, j and k in x, y and z direction, respectively.

Let $\mathbf{V}_{0h} := \{ v \in \mathbf{V}_h : \mathbf{v} \cdot \mathbf{n} = 0, x \in \Gamma \}$. Hence, the corresponding discrete mixed finite element version of (1.1) is defined to seek $(\mathbf{u}_h, p_h) \in \mathbf{V}_{0h} \times W_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.3}$$

$$(\nabla \cdot \mathbf{u}_h, \omega) + (cp_h, \omega) = (f, \omega) \quad \forall \omega \in W_h. \tag{2.4}$$



The hexahedron element e and its six faces

Let $\mathbf{\Pi}_{h_1,h_2,h_3}$ and J_{h_1,h_2,h_3} be the corresponding Raviart-Thomas interpolation operators which are defined by the following conditions:

$$\int_{f_i}(\mathbf{u}-\mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u})\cdot\mathbf{n}_i ds=0,\quad i=1,2,3,4,5,6,\tag{2.5}$$

$$\int_e (p-J_{h_1,h_2,h_3}p)dxdydz=0.\tag{2.6}$$

Then the interpolation has the following properties:

$$||\mathbf{u}-\mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u}||_0 \le ch||\mathbf{u}||_1,\tag{2.7}$$

$$||\nabla\cdot(\mathbf{u}-\mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u})||_0 \le ch||\nabla\cdot\mathbf{u}||_1,\tag{2.8}$$

$$||p-J_{h_1,h_2,h_3}p||_0 \le ch||p||_1,\tag{2.9}$$

where $h:=\max\{h_1,h_2,h_3\}$.

3. The asymptotic expansion

The aim of this section is to give an asymptotic expansion for the Raviart-Thomas interpolation. For this aim, we first introduce some notations for the future use.

For any element $e\in\mathbf{T}_{h_1,h_2,h_3}$, let (x_e,y_e,z_e) stand for the center of e and $2h_1$, $2h_2$ and $2h_3$ be the side lengths of $x-$, $y-$ and $z-$ direction, respectively. We define the following three error functions for x,y and z :

$$\begin{aligned}E(x)&:=\frac{1}{2}((x-x_e)^2-h_1^2),\\ F(y)&:=\frac{1}{2}((y-y_e)^2-h_2^2),\\ G(z)&:=\frac{1}{2}((z-z_e)^2-h_3^2).\end{aligned}$$

Then, we have for $r\le m-1$

$$(E^m)^{(r)}|_{f_i}=0\quad (i=2,4),\tag{3.1}$$

$$(F^m)^{(r)}|_{f_i}=0\quad (i=1,3),\tag{3.2}$$

$$(G^m)^{(r)}|_{f_i}=0\quad (i=5,6).\tag{3.3}$$

In addition, it is easy to check

$$\begin{aligned}E&=\frac{1}{6}(E^2)_{xx}-\frac{1}{3}h_1^2,\\ (x-x_e)^2&=\frac{1}{3}(E^2)_{xx}+\frac{1}{3}h_1^2,\\ x-x_e&=\frac{1}{6}(E^2)_{xxx},\\ E^2&=\frac{1}{420}(E^4)_{xxxx}-\frac{2}{21}h_1^2(E^2)_{xx}+\frac{2}{15}h_1^4.\end{aligned}$$

Similarly, F and G have the same properties.

In the remainder of this section, we derive an asymptotic expansion for the Raviart-Thomas interpolation.

First, we study $(A^{-1}(\mathbf{u}-\mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u}),\mathbf{v})$, where $A^{-1}=(\alpha_{ij})_{3\times 3}$.

Lemma 3.1. *Assume* $\mathbf{u} \in \mathbf{V} \cap (H^4(\Omega))^3$ *and* $\alpha_{11} \in W^{3,\infty}(\Omega)$. *Then we have*

$$\begin{aligned}\int_\Omega \alpha_{11}(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1 dxdydz &= -\frac{h_1^2}{3}\int_\Omega \alpha_{11}(u_1)_{xx}v_1 dxdydz\\ &+\frac{h_2^2}{3}\int_\Omega (\alpha_{11})_y(u_1)_y v_1 dxdydz\\ &+\frac{h_3^2}{3}\int_\Omega (\alpha_{11})_z(u_1)_z v_1 dxdydz\\ &+O(h^4)||u_1||_4||v_1||_0. \qquad (3.4)\end{aligned}$$

Proof. For any $e \in \mathbf{T}_{h_1,h_2,h_3}$, by the Taylor expansion of α_{11} and (2.7), we have that

$$\begin{aligned} I_1 &:= \int_e \alpha_{11}(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1 dxdydz\\ &= \int_e [\alpha_{11}(x,y_e,z_e) + (y-y_e)(\alpha_{11})_y(x,y_e,z_e) + (z-z_e)(\alpha_{11})_z(x,y_e,z_e)\\ &+\frac{1}{2}(y-y_e)^2(\alpha_{11})_{yy}(x,y_e,z_e) + \frac{1}{2}(z-z_e)^2(\alpha_{11})_{zz}(x,y_e,z_e)\\ &+(y-y_e)(z-z_e)(\alpha_{11})_{yz}(x,y_e,z_e)](u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1 dxdydz\\ &+O(h^4)||u_1||_{1,e}||v_1||_{0,e}\\ &:= I_{11} + I_{12} + I_{13} +_{14} + I_{15} + I_{16} + O(h^4)||u_1||_{1,e}||v_1||_{0,e}. \qquad (3.5)\end{aligned}$$

Then, we shall deal with $I_{11}, I_{12}, I_{13}, I_{14}, I_{15}$ and I_{16} respectively.

For I_{11}, it follows from the Taylor expansion of v_1 that

$$\begin{aligned} I_{11} &= \int_e \alpha_{11}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1 dxdydz\\ &= \int_e \alpha_{11}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)[v_1(x_e,y,z) + (x-x_e)(v_1)_x]dxdydz\\ &:= I_{111} + I_{112}. \qquad (3.6)\end{aligned}$$

For I_{111}, we have

$$\begin{aligned} I_{111} &= \int_e \alpha_{11}(x,y_e,z_e)E_{xx}(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1(x_e,y,z)dxdydz\\ &= \int_e E[(u_1)_{xx}\alpha_{11}(x,y_e,z_e) + 2(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)_x(\alpha_{11})_x(x,y_e,z_e)]\\ &\quad v_1(x_e,y,z)dxdydz\\ &\quad + \int_e E(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)(\alpha_{11})_{xx}(x,y_e,z_e)v_1(x_e,y,z)dxdydz\\ &:= I_{111}^1 + I_{111}^2 + I_{111}^3. \qquad (3.7)\end{aligned}$$

$\alpha_{11}(x,y_e,z_e)$ has the following Taylor expansion

$$\alpha_{11}(x,y_e,z_e) = \alpha_{11}(x,y,z) - F_y(\alpha_{11})_y(x,y,z) - G_z(\alpha_{11})_z(x,y,z) + O(h^2),$$

It follows from integration by parts and the Taylor expansion of $\alpha_{11}(x, y_e, z_e)$ that

$$\begin{aligned}
I_{111}^1 &= \frac{1}{6}\int_e (E^2)_{xx}(u_1)_{xx}\alpha_{11}(x, y_e, z_e)v_1(x_e, y, z)dxdydz \\
&\quad -\frac{h_1^2}{3}\int_e (u_1)_{xx}\alpha_{11}v_1(x_e, y, z)dxdydz \\
&\quad -\frac{h_1^2}{3}\int_e (u_1)_{xxy}F(\alpha_{11})_y(x, y, z)v_1(x_e, y, z)dxdydz \\
&\quad -\frac{h_1^2}{3}\int_e (u_1)_{xxz}G(\alpha_{11})_z(x, y, z)v_1(x_e, y, z)dxdydz + O(h^4)||u_1||_{3,e}||v_1||_{0,e} \\
&= -\frac{h_1^2}{3}\int_e (u_1)_{xx}\alpha_{11}v_1(x_e, y, z)dxdydz + O(h^4)||u_1||_{4,e}||v_1||_{0,e} \\
&= -\frac{h_1^2}{3}\int_e (u_1)_{xx}\alpha_{11}v_1dxdydz - \frac{h_1^2}{3}\int_e E[(u_1)_{xx}\alpha_{11}]_x(v_1)_xdxdydz \\
&\quad +O(h^4)||u_1||_{4,e}||v_1||_{0,e} \\
&= -\frac{h_1^2}{3}\int_e (u_1)_{xx}\alpha_{11}v_1dxdydz + \frac{h_1^4}{9}\int_e [(u_1)_{xx}\alpha_{11}]_x(v_1)_xdxdydz \\
&\quad -\frac{h_1^2}{18}\int_e (E^2)_{xx}[(u_1)_{xx}\alpha_{11}]_x(v_1)_xdxdydz + O(h^4)||u_1||_{4,e}||v_1||_{0,e} \\
&= -\frac{h_1^2}{3}\int_e (u_1)_{xx}\alpha_{11}v_1dxdydz + \frac{h_1^4}{9}(\int_{f_2} - \int_{f_4})[(u_1)_{xx}\alpha_{11}]_xv_1ds \\
&\quad +O(h^4)||u_1||_{4,e}||v_1||_{0,e}. \qquad (3.8)
\end{aligned}$$

Here, we used the standard inverse inequality for finite element functions

$$h||v_1||_{1,e} \leq C||v_1||_{0,e}.$$

Similarly, we have for I_{111}^2 that

$$\begin{aligned}
I_{111}^2 &= -\frac{2}{3}h_1^2\int_e (u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)_x(\alpha_{11})_x(x, y_e, z_e)v_1(x_e, y, z)dxdydz \\
&\quad +\frac{1}{3}\int_e (E^2)_{xx}(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)_x(\alpha_{11})_x(x, y_e, z_e)v_1(x_e, y, z)dxdydz \\
&= -\frac{2}{3}h_1^2(\int_{f_4} - \int_{f_2})(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)(\alpha_{11})_x(x, y_e, z_e)v_1(x_e, y, z)ds \\
&\quad +\frac{2}{3}h_1^2\int_e (u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)(\alpha_{11})_{xx}(x, y_e, z_e)v_1(x_e, y, z)dxdydz \\
&\quad +O(h^4)||u_1||_{3,e}||v_1||_{0,e} \\
&= \frac{2}{3}h_1^2\int_e E_{xx}(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)(\alpha_{11})_{xx}(x, y_e, z_e)v_1(x_e, y, z)dxdydz \\
&\quad +O(h^4)||u_1||_{3,e}||v_1||_{0,e} \\
&= O(h^4)||u_1||_{3,e}||v_1||_{0,e}. \qquad (3.9)
\end{aligned}$$

Also, we obtain for I_{111}^3 that

$$I_{111}^3 = O(h^4)||u_1||_{2,e}||v_1||_{0,e}. \qquad (3.10)$$

Thus, from (3.7)-(3.10), we have

$$\begin{aligned} I_{111} &= -\frac{h_1^2}{3}\int_e \alpha_{11}(u_1)_{xx}v_1dxdydz + \frac{h_1^4}{9}(\int_{f_4} - \int_{f_2})[(u_1)_{xx}\alpha_{11}]_x ds \\ &\quad +O(h_1^4)||u_1||_{4,e}||v_1||_{0,e}. \end{aligned} \tag{3.11}$$

Then, we deal with I_{112} in (3.6).

$$\begin{aligned} I_{112} &= \frac{1}{6}\int_e (E^2)_{xxx}\alpha_{11}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)(v_1)_x dxdydz \\ &= -\frac{1}{6}\int_e E^2[\alpha_{11}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)]_{xxx}(v_1)_x dxdydz \\ &= -\frac{1}{6}\int_e [\frac{2}{15}h_1^4 + \frac{1}{420}(E^4)_{xxxx} - \frac{2}{21}h_1^2(E^2)_{xx}][\alpha_{11}(x,y_e,z_e) \\ &\quad (u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)]_{xxx}(v_1)_x dxdydz \\ &= -\frac{h_1^4}{45}\int_e [\alpha_{11}(x,y_e,z_e)(u_1)_{xxx} + 3(\alpha_{11})_x(x,y_e,z_e)(u_1)_{xx}](v_1)_x dxdydz \\ &\quad -\frac{h_1^4}{45}\int_e [3(\alpha_{11})_{xx}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)_x \\ &\quad +(\alpha_{11})_{xxx}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)](v_1)_x dxdydz \\ &\quad +O(h_1^4)||u_1||_{4,e}||v_1||_{0,e} \\ &= -\frac{h_1^4}{45}(\int_{f_4} - \int_{f_2})[\alpha_{11}(u_1)_{xxx} + 3(\alpha_{11})_x(u_1)_{xx}]v_1 ds \\ &\quad +\frac{h_1^4}{45}\int_e [\alpha_{11}(u_1)_{xxx} + 3(\alpha_{11})_x(u_1)_{xx}]_x v_1 dxdydz + O(h_1^4)||u_1||_{4,e}||v_1||_{0,e} \\ &= -\frac{h_1^4}{45}(\int_{f_4} - \int_{f_2})[\alpha_{11}(u_1)_{xxx} + 3(\alpha_{11})_x(u_1)_{xx}]v_1 ds + O(h_1^4)||u_1||_{4,e}||v_1||_{0,e}. \end{aligned} \tag{3.12}$$

Then, from (3.6), (3.11) and (3.12), we have

$$\begin{aligned} I_{11} &= -\frac{h_1^2}{3}\int_e \alpha_{11}(u_1)_{xx}v_1dxdydz + \frac{h_1^2}{9}(\int_{f_4} - \int_{f_2})[\alpha_{11}(u_1)_{xx}]_x v_1 ds \\ &\quad -\frac{h_1^4}{45}(\int_{f_4} - \int_{f_2})[\alpha_{11}(u_1)_{xxx} + 3(\alpha_{11})_x(u_1)_{xx}]v_1 ds \\ &\quad +O(h^4)||u_1||_{4,e}||v_1||_{0,e}. \end{aligned} \tag{3.13}$$

Now, we shall handle I_{12} as follows.

It follows form the definition of error function $F(y)$, integration by parts and the fact $v_1|_e \in Q_{100}(e)$ that

$$\begin{aligned} I_{12} &= -\int_e F(\alpha_{11})_y(x,y_e,z_e)(u_1)_y v_1 dxdydz \\ &= \frac{h_2^2}{3}\int_e (\alpha_{11})_y(x,y_e,z_e)(u_1)_y v_1 dxdydz + O(h^4)||u_1||_{3,e}||v_1||_{0,e}. \end{aligned} \tag{3.14}$$

Therefore, with the Taylor expansion of $\alpha_{11}(x,y_e,z_e)$,

$$\begin{aligned} \alpha_{11}(x,y_e,z_e) &= (\alpha_{11})_y(x,y,z) + (y_e - y)(\alpha_{11})_{yy}(x,y,z) \\ &\quad +(z_e - z)(\alpha_{11})_{yz}(x,y,z) + O(h^2), \end{aligned}$$

we have

$$I_{12} = \frac{h_2^2}{3}\int_e (\alpha_{11})_y (u_1)_y v_1 dxdydz + O(h^4)||u_1||_{3,e}||v_1||_{0,e}. \tag{3.15}$$

Similarly, we can obtain

$$I_{13} = \frac{h_3^2}{3}\int_e (\alpha_{11})_z (u_1)_z v_1 dxdydz + O(h^4)||u_1||_{3,e}||v_1||_{0,e}. \tag{3.16}$$

For I_{14}, from the Taylor expansion of v_1

$$v_1(x,y,z) = v(x_e,y,z) + (x-x_e)(v_1)_x,$$

we can obtain

$$\begin{aligned} I_{14} &= \frac{h_2^2}{6}\int_e (\alpha_{11})_{yy}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1(x_e,y,z)dxdydz \\ &\quad + \frac{h_2^2}{6}\int_e (\alpha_{11})_{yy}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)(x-x_e)(v_1)_x dxdydz \\ &\quad + \frac{1}{6}\int_e (F^2)_{yy}(\alpha_{11})_{yy}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1 dxdydz \\ &\quad + O(h_2^4)||u_1||_{2,e}||v_1||_{0,e} \\ &:= I_{14}^1 + I_{14}^2 + O(h_2^4)||u_1||_{2,e}||v_1||_{0,e}, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} I_{14}^1 &= \frac{h_2^2}{6}\left(\int_{f_4} - \int_{f_2}\right)E_x(\alpha_{11})_{yy}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1(x_e,y,z)ds \\ &\quad - \frac{h_2^2}{6}\int_e E_x[(\alpha_{11})_{yy}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)]_x v_1(x_e,y,z)dxdydz \\ &= O(h^4)||u_1||_{2,e}||v_1||_{0,e}, \end{aligned} \tag{3.18}$$

$$\begin{aligned} I_{14}^2 &= \frac{h_2^2}{36}\int_e (E^2)_{xxx}(\alpha_{11})_{yy}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)(v_1)_x dxdydz \\ &= \frac{h_2^2}{36}\int_e (E^2)_x[(\alpha_{11})_{yy}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)]_{xx}(v_1)_x dxdydz \\ &= O(h^4)||u_1||_{2,e}||v_1||_{0,e}. \end{aligned} \tag{3.19}$$

So, from (3.17)–(3.19), we have

$$I_{14} = O(h^4)||u_1||_{2,e}||v_1||_{0,e}. \tag{3.20}$$

Similarly, we have

$$I_{15} = O(h^4)||u_1||_{2,e}||v_1||_{0,e}. \tag{3.21}$$

For I_{16}, we have the similarly result

$$\begin{aligned} I_{16} &= \int_e F_y G_z (\alpha_{11})_{yz}(x,y_e,z_e)(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_1 dxdydz \\ &= O(h^4)||u_1||_{2,e}||v_1||_{0,e}. \end{aligned} \tag{3.22}$$

So, from (3.5), (3.13), (3.14), (3.16), (3.20), (3.21) and (3.22), we have

$$\begin{aligned} I_1 &= -\frac{h_1^2}{3}\int_e \alpha_{11}(u_1)_{xx}v_1 dxdydz + \frac{h_2^2}{3}\int_e (\alpha_{11})_y(u_1)_y v_1 dxdydz \\ &\quad +\frac{h_1^4}{45}(\int_{f_4} - \int_{f_2})[4\alpha_{11}(u_1)_{xxx} + 2(\alpha_{11})_x(u_1)_{xx}]v_1 ds \\ &\quad +\frac{h_3^2}{3}\int_e (\alpha_{11})_z(u_1)_z v_1 dxdydz + O(h^4)||u_1||_{4,e}||v_1||_{0,e}. \end{aligned} \tag{3.23}$$

The face integrals in (3.23) can be cancelled when we sum up I_1 over all the elements $e \in \mathbf{T}_{h_1,h_2,h_3}$. So, Lemma 3.1 can be obtained with summarize. □

Similarly with Lemma 3.1, we have the following Corollary.

Corollary 3.1. *Assume* $\mathbf{u} \in V \cap (H^4(\Omega))^3$ *and* $\alpha_{ii} \in W^{3,\infty}(\Omega)(i = 2,3)$*. Then we also have*

$$\begin{aligned} \int_\Omega \alpha_{22}(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_2 dxdydz &= -\frac{h_2^2}{3}\int_\Omega \alpha_{22}(u_2)_{yy}v_2 dxdydz \\ &\quad +\frac{h_1^2}{3}\int_\Omega (\alpha_{22})_x(u_2)_x v_2 dxdydz \\ &\quad +\frac{h_3^2}{3}\int_\Omega (\alpha_{22})_z(u_2)_z v_2 dxdydz \\ &\quad +O(h^4)||u_2||_4||v_2||_0, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \int_\Omega \alpha_{33}(u_3 - \mathbf{\Pi}_{3,h_1,h_2,h_3}u_3)v_3 dxdydz &= -\frac{h_3^2}{3}\int_\Omega \alpha_{33}(u_3)_{zz}v_3 dxdydz \\ &\quad +\frac{h_1^2}{3}\int_\Omega (\alpha_{33})_x(u_3)_x v_3 dxdydz \\ &\quad +\frac{h_2^2}{3}\int_\Omega (\alpha_{33})_y(u_3)_y v_3 dxdydz \\ &\quad +O(h^4)||u_3||_4||v_3||_0. \end{aligned} \tag{3.25}$$

Lemma 3.2. *Assume that the* $u_2 \in H^4(\Omega)$ *and* $\alpha_{12} \in W^{3,\infty}(\Omega)$*. Then we have*

$$\begin{aligned} \int_\Omega \alpha_{12}(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_1 dxdydz &= -\frac{h_1^2}{3}\int_\Omega \alpha_{12}(u_2)_{xx}v_1 dxdydz \\ &\quad -\frac{h_2^2}{3}\int_\Omega \alpha_{12}(u_2)_{yy}v_1 dxdydz \\ &\quad +O(h^4)||u_2||_4||v_1||_0. \end{aligned} \tag{3.26}$$

Proof. For any $e \in \mathbf{T}_{h_1,h_2,h_3}$, the Taylor expansions of v_1 leads to

$$\begin{aligned} I_2 &= \int_e \alpha_{12}(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)(v_1(x_e,y,z) + (x-x_e)(v_1)_x)dxdydz \\ &:= I_{21} + I_{22}. \end{aligned} \tag{3.27}$$

For I_{21}, from the Taylor expansion of α_{12}, we have

$$\begin{aligned}
I_{21} &= \int_e [\alpha_{12}(x_e,y,z)+(x-x_e)(\alpha_{12})_x(x_e,y,z)] \\
&\quad (u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_1(x_e,y,z)dxdydz \\
&\quad +\frac{1}{2}\int_e (x-x_e)^2(\alpha_{12})_{xx}(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_1(x_e,y,z)dxdydz \\
&\quad +O(h^4)||u_2||_{2,e}||v_1||_{0,e} \\
&:= I_{211}+I_{212}+I_{213}+O(h^4)||u_2||_{2,e}||v_1||_{0,e}. \qquad (3.28)
\end{aligned}$$

Then, we shall estimate I_{211}, I_{212} and I_{213} respectively.

First, for I_{211}, we have

$$\begin{aligned}
I_{211} &= \int_e F[\alpha_{12}(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)]_{yy}v_1(x_e,y,z)dxdydz \\
&= -\frac{h_2^2}{3}\int_e [\alpha_{12}(x_e,y,z)(u_2)_{yy}+2(\alpha_{12})_y(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)_y] \\
&\quad v_1(x_e,y,z)dxdydz \\
&\quad -\frac{h_2^2}{3}\int_e (\alpha_{12})_{yy}(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_1(x_e,y,z)dxdydz \\
&\quad +O(h_2^4)||u_2||_{4,e}||v_1||_{0,e} \\
&:= I_{211}^1+I_{211}^2+I_{211}^3+O(h_2^4)||u_2||_{4,e}||v_1||_{0,e}. \qquad (3.29)
\end{aligned}$$

From the Taylor expansion of $\alpha_{12}(x_e,y,z)$

$$\alpha_{12}(x_e,y,z)=\alpha_{12}(x,y,z)-E_x(\alpha_{12})_x(x,y,z)+O(h_1^2),$$

together with the Taylor expansion of $v_1(x_e,y,z)$ leads to

$$\begin{aligned}
I_{211}^1 &= -\frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}v_1dxdydz+\frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}E_x(v_1)_xdxdydz \\
&\quad -\frac{h_2^2}{3}\int_e E[(\alpha_{12})_x(x,y)(u_2)_{yy}]_xv_1(x_e,y,z)dxdydz+O(h^4)||u_2||_{2,e}||v_1||_{0,e} \\
&= -\frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}v_1dxdydz-\frac{h_2^2}{3}\int_e E[\alpha_{12}(u_2)_{yy}]_x(v_1)_xdxdydz \\
&\quad +O(h^4)||u_2||_{3,e}||v_1||_{0,e} \\
&= -\frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}v_1dxdydz+\frac{(h_2h_1)^2}{9}(\int_{f_4}-\int_{f_2})[\alpha_{12}(u_2)_{yy}]_xv_1ds \\
&\quad -\frac{(h_2h_1)^2}{9}\int_e [\alpha_{12}(u_2)_{yy}]_{xx}v_1dxdydz \\
&\quad +\frac{h_2^2}{18}\int_e (E^2)_x[\alpha_{12}(u_2)_{yy}]_{xx}(v_1)_xdxdydz+O(h^4)||u_2||_{3,e}||v_1||_{0,e} \\
&= -\frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}v_1dxdydz+\frac{(h_2h_1)^2}{9}(\int_{f_4}-\int_{f_2})[\alpha_{12}(u_2)_{yy}]_xv_1ds \\
&\quad +O(h^4)||u_2||_{4,e}||v_1||_{0,e}. \qquad (3.30)
\end{aligned}$$

Analogously to I_{111}^2, via the interpolation condition and integration by parts, we have

$$I_{211}^2=O(h_2^4)||u_2||_{2,e}||v_1||_{0,e}, \qquad (3.31)$$

$$I_{211}^3 = O(h_2^4)||u_2||_{2,e}||v_1||_{0,e}. \tag{3.32}$$

So, from (3.29)-(3.32), we have

$$\begin{aligned} I_{211} &= -\frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}v_1 de + \frac{(h_2h_1)^2}{9}(\int_{s_4} - \int_{s_2})[\alpha_{12}(u_2)_{yy}]_x v_1 ds \\ &\quad +O(h^4)||u_2||_{4,e}||v_1||_{0,e}. \end{aligned} \tag{3.33}$$

Then, for I_{212}, we have

$$\begin{aligned} I_{212} &= -\int_e E(\alpha_{12})_x(x_e,y,z)(u_2)_x v_1(x_e,y,z)dxdydz \\ &= \frac{h_1^2}{3}\int_e [(\alpha_{12})_x + (x_e - x)(\alpha_{12})_{xx}](u_2)_x v_1(x_e,y,z)dxdydz \\ &\quad +O(h_1^4)||u_2||_{3,e}||v_1||_{0,e} \\ &= \frac{h_1^2}{3}\int_e (\alpha_{12})_x(u_2)_x v_1 dxdydz - \frac{h_1^2}{3}\int_e E_x(\alpha_{12})_x(u_2)_x(v_1)_x dxdydz \\ &\quad +O(h_1^4)||u_2||_{3,e}||v_1||_{0,e} \\ &= \frac{h_1^2}{3}\int_e (\alpha_{12})_x(u_2)_x v_1 dxdydz - \frac{h_1^4}{9}\int_e [(\alpha_{12})_x(u_2)_x]_x(v_1)_x dxdydz \\ &\quad +\frac{h_1^2}{18}\int_e (E^2)_{xx}[(\alpha_{12})_x(u_2)_x]_x(v_1)_x de + O(h_1^4)||u_2||_{3,e}||v_1||_{0,e} \\ &= \frac{h_1^2}{3}\int_e (\alpha_{12})_x(u_2)_x v_1 dxdydz - \frac{h_1^4}{9}(\int_{f_4} - \int_{s_2})[(\alpha_{12})_x(u_2)_x]_x v_1 ds \\ &\quad +O(h_1^4)||u_2||_{3,e}||v_1||_{0,e}. \end{aligned} \tag{3.34}$$

Finally, I_{213} has the following estimate

$$\begin{aligned} I_{213} &= \frac{h_1^2}{6}\int_e (\alpha_{12})_{xx}(x_e,y,z)(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_1(x_e,y,z)dxdydz \\ &\quad +\frac{1}{6}\int_e (E^2)_{xx}(\alpha_{12})_{xx}(x_e,y,z)(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_1(x_e,y,z)dxdydz \\ &= \frac{h_1^2}{6}\int_e F_{yy}(\alpha_{12})_{xx}(x_e,y,z)(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_1(x_e,y,z)dxdydz \\ &\quad +O(h_1^4)||u_2||_{2,e}||v_1||_{0,e} \\ &= \frac{h_1^2}{6}\int_e F[(\alpha_{12})_{xx}(x_e,y,z)(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)]_{yy}v_1(x_e,y,z)dxdydz \\ &\quad +O(h_1^4)||u_2||_{2,e}||v_1||_{0,e} \\ &= O(h^4)||u_2||_{2,e}||v_1||_{0,e}. \end{aligned} \tag{3.35}$$

So, from (3.28), (3.33), (3.34) and (3.35), we have

$$\begin{aligned} I_{21} &= \frac{h_1^2}{3}\int_e (\alpha_{12})_x(u_2)_x v_1 dxdydz - \frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}v_1 dxdydz \\ &\quad -\frac{h_1^4}{9}(\int_{f_4} - \int_{f_2})[(\alpha_{12})_x(u_2)_x]_x v_1 ds + \frac{(h_1h_2)^2}{9}(\int_{f_4} - \int_{f_2})[\alpha_{12}(u_2)_{yy}]_x v_1 ds \\ &\quad +O(h^4)||u_2||_{4,e}||v_1||_{0,e}. \end{aligned} \tag{3.36}$$

For I_{22}, with the Taylor expansion of α_{12}, we have

$$\begin{aligned} I_{22} &= \int_e [\alpha_{12}(x_e,y,z)+(x-x_e)(\alpha_{12})_x(x_e,y,z)] \\ &\quad (u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)(x-x_e)(v_1)_x dxdydz \\ &\quad +\frac{1}{2}\int_e (x-x_e)^2(\alpha_{12})_{xx}(x_e,y,z) \\ &\quad (u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)(x-x_e)(v_1)_x dxdydz + O(h_1^4)||u_2||_{1,e}||v_1||_{0,e} \\ &:= I_{221}+I_{222}+I_{223}+O(h_1^4)||u_2||_{1,e}||v_1||_{0,e}. \end{aligned} \tag{3.37}$$

Together with integration by parts and the Taylor expansion of α_{12}, we have

$$\begin{aligned} I_{221} &= -\int_e E\alpha_{12}(x_e,y,z)(u_2)_x(v_1)_x dxdydz \\ &= \frac{h_1^2}{3}\int_e \alpha_{12}(u_2)_x(v_1)_x dxdydz - \frac{h_1^2}{3}\int_e E_x(\alpha_{12})_x(x,y,z)(u_2)_x(v_1)_x dxdydz \\ &\quad -\frac{1}{6}\int_e E^2\alpha_{12}(x_e,y,z)(u_2)_{xxx}(v_1)_x dxdydz + O(h_1^4)||u_2||_{2,e}||v_1||_{0,e} \\ &= \frac{h_1^2}{3}(\int_{f_4}-\int_{f_2})\alpha_{12}(u_2)_x v_1 ds - \frac{h_1^2}{3}\int_e [\alpha_{12}(u_2)_x]_x v_1 dxdydz \\ &\quad +\frac{h_1^2}{3}\int_e E[(\alpha_{12})_x(x,y,z)(u_2)_x]_x(v_1)_x dxdydz \\ &\quad -\frac{1}{6}\int_e [\frac{1}{420}(E^4)_{xxxx}-\frac{2}{21}h_1^2(E^2)_{xx}+\frac{2}{15}h_1^4]\alpha_{12}(x_e,y,z)(u_2)_{xxx}(v_1)_x dxdydz \\ &\quad +O(h_1^4)||u_2||_{2,e}||v_1||_{0,e} \\ &= -\frac{h_1^2}{3}\int_e [\alpha_{12}(u_2)_x]_x v_1 dxdydz + \frac{h_1^2}{3}(\int_{f_4}-\int_{f_2})\alpha_{12}(u_2)_x v_1 ds \\ &\quad -\frac{h_1^4}{9}(\int_{f_4}-\int_{f_2})[(\alpha_{12})_x(u_2)_x]_x v_1 ds - \frac{h_1^4}{45}(\int_{f_4}-\int_{f_2})\alpha_{12}(u_2)_{xxx}v_1 ds \\ &\quad +O(h_1^4)||u_2||_{4,e}||v_1||_{0,e}. \end{aligned} \tag{3.38}$$

Also, we have

$$\begin{aligned} I_{222} &= \frac{h_1^2}{3}\int_e F_{yy}(\alpha_{12})_x(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)(v_1)_x dxdydz \\ &\quad +\frac{1}{3}\int_e E^2(\alpha_{12})_x(x_e,y,z)(u_2)_{xx}(v_1)_x dxdydz \\ &= \frac{h_1^2}{3}(\int_{f_3}-\int_{f_1})F_y(\alpha_{12})_x(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)(v_1)_x ds \\ &\quad -\frac{h_1^2}{3}\int_e F_y[(\alpha_{12})_x(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)]_y(v_1)_x dxdydz \\ &\quad +\frac{1}{3}\int_e [\frac{2}{15}h_1^4+\frac{1}{420}(E^4)_{xxxx}-\frac{2}{21}h_1^2(E^2)_{xx}](\alpha_{12})_x(x_e,y,z)(u_2)_{xx}(v_1)_x dxdydz \end{aligned}$$

$$
\begin{aligned}
&= -\frac{(h_1h_2)^2}{9}\int_e (\alpha_{12})_x(u_2)_{yy}(v_1)_x dxdydz \\
&\quad -\frac{2(h_1h_2)^2}{9}(\int_{f_3}-\int_{f_1})(\alpha_{12})_{xy}(x_e,y,z)(u_2-\mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)(v_1)_x ds \\
&\quad +\frac{2}{45}h_1^4(\int_{f_4}-\int_{f_2})(\alpha_{12})_x(u_2)_{xx}v_1 ds + O(h^4)||u_2||_{3,e}||v_1||_{0,e} \\
&= -\frac{(h_1h_2)^2}{9}(\int_{f_4}-\int_{f_2})(\alpha_{12})_x(u_2)_{yy}v_1 ds + \frac{2}{45}h_1^4(\int_{s_4}-\int_{s_2})(\alpha_{12})_x(u_2)_{xx}v_1 ds \\
&\quad +O(h^4)||u_2||_{3,e}||v_1||_{0,e}. \qquad (3.39)
\end{aligned}
$$

For I_{223}, since

$$(x-x_e)^3 = \frac{1}{15}(E^3)_{xxx} + \frac{9}{15}h_1^2 E_x,$$

together with integration by parts, we have

$$
\begin{aligned}
I_{223} &= -\frac{3h_1^2}{10}\int_e E(\alpha_{12})_{xx}(x_e,y,z)(u_2)_x(v_1)_x dxdydz + O(h_1^4)||u_2||_{2,e}||v_1||_{0,e} \\
&= \frac{h_1^4}{10}\int_e (\alpha_{12})_{xx}(u_2)_x(v_1)_x de + O(h_1^4)||u_2||_{2,e}||v_1||_{0,e} \\
&= \frac{h_1^4}{10}(\int_{f_4}-\int_{f_2})(\alpha_{12})_{xx}(u_2)_x v_1 ds + O(h_1^4)||u_2||_{2,e}||v_1||_{0,e}. \qquad (3.40)
\end{aligned}
$$

So, from (3.37), (3.38), (3.39) and (3.40), we have

$$
\begin{aligned}
I_{22} &= -\frac{h_1^2}{3}\int_e [\alpha_{12}(u_2)_x]_x v_1 de + \frac{h_1^2}{3}(\int_{f_4}-\int_{f_2})\alpha_{12}(u_2)_x v_1 ds \\
&\quad -\frac{h_1^4}{9}(\int_{f_4}-\int_{f_2})[\alpha_{12}(u_2)_x]_x v_1 ds - \frac{h_1^4}{45}(\int_{f_4}-\int_{f_2})\alpha_{12}(u_2)_{xxx}v_1 ds \\
&\quad -\frac{(h_1h_2)^2}{9}(\int_{f_4}-\int_{f_2})(\alpha_{12})_x(u_2)_{yy}v_1 ds - \frac{2h_1^4}{45}(\int_{f_4}-\int_{f_2})(\alpha_{12})_x(u_2)_{xx}v_1 ds \\
&\quad +\frac{h_1^4}{10}(\int_{f_4}-\int_{f_2})(\alpha_{12})_{xx}(u_2)_x v_1 ds + O(h^4)||u_2||_{4,e}||v_1||_{0,e}. \qquad (3.41)
\end{aligned}
$$

Thus, from (3.27), (3.36) and (3.41), we can obtain

$$
\begin{aligned}
I_2 &= -\frac{h_1^2}{3}\int_e \alpha_{12}(u_2)_{xx}v_1 dxdydz - \frac{h_2^2}{3}\int_e \alpha_{12}(u_2)_{yy}v_1 dxdydz \\
&\quad +\frac{h_2^2}{3}(\int_{f_4}-\int_{f_2})\alpha_{12}(u_2)_x v_1 ds \\
&\quad -\frac{h_1^4}{9}(\int_{f_4}-\int_{f_2})\{[(\alpha_{12})_x(u_2)_x]_x + [\alpha_{12}(u_2)_x]_x + \frac{1}{5}\alpha_{12}(u_2)_{xxx}\}v_1 ds \\
&\quad +\frac{(h_1h_2)^2}{9}(\int_{f_4}-\int_{f_2})\alpha_{12}(u_2)_{yyx}v_1 ds + O(h^4)||u_2||_{4,e}||v_1||_{0,e}. \qquad (3.42)
\end{aligned}
$$

Furthermore, summing up I_2 over all the elements $e \in \mathbf{T}_{h_1,h_2,h_3}$ yield the desired result (3.26). □

Similarly, we can get the following corollary.

Corollary 3.2. *Assume* $\mathbf{u} \in \mathbf{V} \cap (H^4(\Omega))^3$ *and* $\alpha_{ij} \in W^{3,\infty}(\Omega)(1 \le i,j \le 3)$. *Then we have*

$$\begin{aligned}\int_\Omega \alpha_{13}(u_3 - \mathbf{\Pi}_{3,h_1,h_2,h_3}u_3)v_1dxdydz &= -\frac{h_1^2}{3}\int_\Omega \alpha_{13}(u_3)_{xx}v_1dxdydz\\ &\quad -\frac{h_3^2}{3}\int_\Omega \alpha_{13}(u_3)_{zz}v_1dxdydz\\ &\quad +O(h^4)||u_3||_4||v_1||_0,\end{aligned}\tag{3.43}$$

$$\begin{aligned}\int_\Omega \alpha_{23}(u_3 - \mathbf{\Pi}_{3,h_1,h_2,h_3}u_3)v_2dxdydz &= -\frac{h_2^2}{3}\int_\Omega \alpha_{23}(u_3)_{yy}v_2dxdydz\\ &\quad -\frac{h_3^2}{3}\int_\Omega \alpha_{23}(u_3)_{zz}v_2dxdydz\\ &\quad +O(h^4)||u_3||_4||v_2||_0,\end{aligned}\tag{3.44}$$

$$\begin{aligned}\int_\Omega \alpha_{13}(u_3 - \mathbf{\Pi}_{3,h_1,h_2,h_3}u_3)v_1dxdydz &= -\frac{h_1^2}{3}\int_\Omega \alpha_{13}(u_3)_{xx}v_1dxdydz\\ &\quad -\frac{h_3^2}{3}\int_\Omega \alpha_{13}(u_3)_{zz}v_1dxdydz\\ &\quad +O(h^4)||u_3||_4||v_1||_0,\end{aligned}\tag{3.45}$$

$$\begin{aligned}\int_\Omega \alpha_{21}(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_2dxdydz &= -\frac{h_1^2}{3}\int_\Omega \alpha_{21}(u_1)_{xx}v_2dxdydz\\ &\quad -\frac{h_2^2}{3}\int_\Omega \alpha_{21}(u_1)_{yy}v_2dxdydz\\ &\quad +O(h^4)||u_3||_4||v_1||_0,\end{aligned}\tag{3.46}$$

$$\begin{aligned}\int_\Omega \alpha_{31}(u_1 - \mathbf{\Pi}_{1,h_1,h_2,h_3}u_1)v_3dxdydz &= -\frac{h_1^2}{3}\int_\Omega \alpha_{31}(u_1)_{xx}v_3dxdydz\\ &\quad -\frac{h_3^2}{3}\int_\Omega \alpha_{31}(u_1)_{zz}v_3dxdydz\\ &\quad +O(h^4)||u_3||_4||v_1||_0,\end{aligned}\tag{3.47}$$

and

$$\begin{aligned}\int_\Omega \alpha_{32}(u_2 - \mathbf{\Pi}_{2,h_1,h_2,h_3}u_2)v_3dxdydz &= -\frac{h_2^2}{3}\int_\Omega \alpha_{32}(u_2)_{yy}v_3dxdydz\\ &\quad -\frac{h_3^2}{3}\int_\Omega \alpha_{32}(u_2)_{zz}v_3dxdydz\\ &\quad +O(h^4)||u_3||_4||v_1||_0.\end{aligned}\tag{3.48}$$

Then, we can obtain the following theorem.

Theorem 3.1. *Assume* $\mathbf{u} \in (H^4(\Omega))^3$ *and* $\alpha_{ij} \in W^{3,\infty}(\Omega)(1 \le i,j \le 3)$. *Then we have*

$$a(\mathbf{u} - \mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u}, \mathbf{v}) = h^2 S_h + O(h^4)||\mathbf{u}||_4||\mathbf{v}||_0,\tag{3.49}$$

where

$$
\begin{aligned}
S_h \quad = \quad & -\frac{h_1^2}{3h^2}\int_\Omega \alpha_{11}(u_1)_{xx}v_1dxdydz + \frac{h_2^2}{3h^2}\int_\Omega (\alpha_{11})_y(u_1)_y v_1dxdydz \\
& +\frac{h_3^2}{3h^2}\int_\Omega (\alpha_{11})_z(u_1)_z v_1dxdydz - \frac{h_2^2}{3h^2}\int_\Omega \alpha_{22}(u_2)_{yy}v_2dxdydz \\
& +\frac{h_1^2}{3}\int_\Omega (\alpha_{22})_x(u_2)_x v_2dxdydz + \frac{h_3^2}{3h^2}\int_\Omega (\alpha_{22})_z(u_2)_z v_2dxdydz \\
& -\frac{h_3^2}{3h^2}\int_\Omega \alpha_{33}(u_3)_{zz}v_3dxdydz + \frac{h_1^2}{3h^2}\int_\Omega (\alpha_{33})_x(u_3)_x v_3dxdydz \\
& +\frac{h_2^2}{3h^2}\int_\Omega (\alpha_{33})_y(u_3)_y v_3dxdydz - \frac{h_1^2}{3h^2}\int_\Omega \alpha_{12}(u_2)_{xx}v_1dxdydz \\
& -\frac{h_2^2}{3h^2}\int_\Omega \alpha_{12}(u_2)_{yy}v_1dxdydz - \frac{h_1^2}{3h^2}\int_\Omega \alpha_{13}(u_3)_{xx}v_1dxdydz \\
& -\frac{h_3^2}{3h^2}\int_\Omega \alpha_{13}(u_3)_{zz}v_1dxdydz - \frac{h_2^2}{3h^2}\int_\Omega \alpha_{23}(u_3)_{yy}v_2dxdydz \\
& -\frac{h_3^2}{3h^2}\int_\Omega \alpha_{23}(u_3)_{zz}v_2dxdydz - \frac{h_1^2}{3h^2}\int_\Omega \alpha_{21}(u_1)_{xx}v_2dxdydz \\
& -\frac{h_2^2}{3h^2}\int_\Omega \alpha_{21}(u_1)_{yy}v_2dxdydz - \frac{h_1^2}{3h^2}\int_\Omega \alpha_{31}(u_1)_{xx}v_3dxdydz \\
& -\frac{h_3^2}{3h^2}\int_\Omega \alpha_{31}(u_1)_{zz}v_3dxdydz - \frac{h_2^2}{3h^2}\int_\Omega \alpha_{32}(u_2)_{yy}v_3dxdydz \\
& -\frac{h_3^2}{3h^2}\int_\Omega \alpha_{32}(u_2)_{zz}v_3dxdydz. \qquad (3.50)
\end{aligned}
$$

Theorem 3.2. *Assume $p \in H^3(\Omega)$ and $c \in W^{3,\infty}(\Omega)$. Then there holds the asymptotic expansion*

$$
\int_\Omega c(p - J_{h_1,h_2,h_3}p)wdxdydz \quad = \quad h^2R_h + O(h^4)||p||_3||w||_0, \qquad (3.51)
$$

where

$$
\begin{aligned}
R_h \quad = \quad & \frac{h_1^2}{3h^2}\int_e c_xp_xwdxdydz + \frac{h_2^2}{3h^2}\int_e c_yp_ywdxdydz \\
& +\frac{h_3^2}{3h^2}\int_e c_zp_zwdxdydz. \qquad (3.52)
\end{aligned}
$$

Proof. For any $e \in \mathbf{T}_{h_1,h_2,h_3}$, it follows from the Taylor expansion of $c(x, y, z)$ at x_e and (2.9) that

$$
\begin{aligned}
\int_e c(x,y,z)(p - J_{h_1,h_2,h_3}p)wdxdydz \quad = \quad & \int_e [c(x_e,y,z) + (x - x_e)c_x(x_e,y,z)] \\
& (p - J_{h_1,h_2,h_3}p)wdxdydz \\
& +\frac{1}{2}\int_e (x - x_e)^2 c_{xx}(x_e,y,z) \\
& (p - J_{h_1,h_2,h_3}p)wdxdydz \\
& +O(h_1^4)||p||_{1,e}||w||_{0,e} \\
:= \quad & II_1 + II_2 + II_3 + O(h_1^4)||p||_{1,e}||w||_{0,e}. \qquad (3.53)
\end{aligned}
$$

For II_1, with the same way, we have

$$\begin{aligned} II_1 &= \int_e (c(x_e,y_e,z)+(y-y_e)c_y(x_e,y_e,z))(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &+\frac{1}{2}\int_e (y-y_e)^2 c_{yy}(x_e,y_e,z)(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &+O(h_2^4)||p||_{1,e}||w||_{0,e} \\ &:= II_{11}+II_{12}+II_{13}+O(h_2^4)||p||_{1,e}||w||_{0,e}. \end{aligned} \tag{3.54}$$

It follow from the Taylor expansion of c, integration by parts and (2.9) that

$$\begin{aligned} II_{11} &= \int_e (c(x_e,y_e,z_e)+(z-z_e)c_z(x_e,y_e,z_e))(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &+\frac{1}{2}\int_e (z-z_e)^2 c_{zz}(x_e,y_e,z_e)(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &+O(h_3^4)||p||_{1,e}||w||_{0,e} \\ &= -\int_e Gc_z(x_e,y_e,z_e)p_z wdxdydz \\ &+\frac{h_3^2}{6}\int_e c_{zz}(x_e,y_e,z_e)(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &+\frac{1}{6}\int_e (G^2)_{zz}c_{zz}(x_e,y_e,z_e)(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &+O(h_3^4)||p||_{1,e}||w||_{0,e} \\ &= \frac{h_3^2}{3}\int_e c_z(x_e,y_e,z_e)p_z wdxdydz-\frac{1}{6}\int_e (G^2)_{zz}c_z(x_e,y_e,z_e)p_z wdxdydz \\ &+O(h_2^4)||p||_{2,e}||w||_{0,e} \\ &= \frac{h_3^2}{3}\int_e c_z p_z wdxdydz-\frac{h_3^2}{3}\int_e E_x c_{xz}p_z wdxdydz \\ &-\frac{h_3^2}{3}\int_e F_y c_{yz}p_z wdxdydz-\frac{h_3^2}{3}\int_e G_z c_{zz}p_z wdxdydz \\ &+O(h_3^4)||p||_{3,e}||w||_{0,e} \\ &= \frac{h_3^2}{3}\int_e c_z u_z wdxdydz+O(h^4)||p||_{3,e}||w||_{0,e}. \end{aligned} \tag{3.55}$$

Similarly, we have

$$\begin{aligned} II_{12} &= -\int_e Fc_y(x_e,y_e,z)p_y wdxdydz \\ &= \frac{h_2^2}{3}\int_e (c_y-E_x c_{xy}-F_y c_{yy})p_y wdxdydz+O(h_2^4)||p||_{3,e}||w||_{0,e} \\ &= \frac{h_2^2}{3}\int_e c_y p_y wdxdydz+O(h_2^4)||p||_{3,e}||w||_{0,e} \end{aligned} \tag{3.56}$$

and

$$\begin{aligned} II_{13} &= \frac{h_2^2}{6}\int_e c_{yy}(x_e,y_e,z)(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &+\frac{1}{6}\int_e (F^2)_{yy}(p-J_{h_1,h_2,h_3}p)wdxdydz \end{aligned}$$

$$\begin{aligned} &= \frac{h_2^2}{6}\int_e c_{yyz}(x_e,y_e,z_e)G_z(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &\quad +O(h_2^4)||p||_{2,e}||w||_{0,e} \\ &= O(h^4)||p||_{2,e}||w||_{0,e}. \end{aligned} \tag{3.57}$$

So, from (3.54)-(3.57), we obtain

$$\begin{aligned} II_1 &= \frac{h_2^2}{3}\int_e c_y p_y wdxdydz + \frac{h_3^2}{3}\int_e c_z p_z wdxdydz \\ &\quad +O(h^4)||p||_{3,e}||w||_{0,e}. \end{aligned} \tag{3.58}$$

Analogously, we have

$$\begin{aligned} II_2 &= = -\int_e Ec_x(x_e,y,z)p_x wde \\ &= \frac{h_1^2}{3}\int_e (c_x - E_x c_{xx})p_x wdxdydz + O(h_1^4)||p||_{3,e}||w||_{0,e} \\ &= \frac{h_1^2}{3}\int_e c_x p_x wdxdydz + O(h_1^4)||p||_{3,e}||w||_{0,e} \end{aligned} \tag{3.59}$$

and

$$\begin{aligned} II_3 &= \frac{h_1^2}{6}\int_e c_{xx}(x_e,y,z)(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &\quad +\frac{1}{6}\int_e c_{xx}(x_e,y,z)(E^2)_{xx}(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &= \frac{h_1^2}{6}\int_e c_{xx}(x_e,y_e,z)(p-J_{h_1,h_2,h_3}p)wdxdydz \\ &\quad +\frac{h_1^2}{6}\int_e c_{xxy}(x_e,y_e,z)F_y(p-J_{h_1,h_2,h_3}p)wdxdydz + O(h^4)||p||_{2,e}||w||_{0,e} \\ &= \frac{h_1^2}{6}\int_e c_{xxz}(x_e,y_e,z_e)G_z(p-J_{h_1,h_2,h_3}p)wdxdydz + O(h^4)||p||_{2,e}||w||_{0,e} \\ &= O(h^4)||p||_{2,e}||w||_{0,e}. \end{aligned} \tag{3.60}$$

Thus, from (3.53), (3.58), (3.59) and (3.60)

$$\begin{aligned} \int_e c(p-J_{h_1,h_2,h_3}p)wdxdydz &= \frac{h_1^2}{3}\int_e c_x p_x wdxdydz + \frac{h_2^2}{3}\int_e c_y p_y wdxdydz \\ &\quad +\frac{h_3^2}{3}\int_e c_z p_z wdxdydz + O(h^4)||p||_3||w||_0. \end{aligned} \tag{3.61}$$

Summing over all the elements $e \in \mathbf{T}_{h_1,h_2,h_3}$, lemma is obtained. □

From Theorem 3.1 and 3.2, we can obtain the following corollary.

Corollary 3.3. *Assume* $\mathbf{u} \in (H^2(\Omega))^3, p \in H^1(\Omega)$ *and* $\alpha_{ij} \in W^{3,\infty}(\Omega)(1 \le i,j \le 3)$. *Then we have the following estimates*

$$||\mathbf{u}_h - \mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u}||_0 \le Ch^2(||\mathbf{u}||_2 + ||p||_1), \tag{3.62}$$

$$||p_h - J_{h_1,h_2,h_3}p||_0 \le Ch^2(||\mathbf{u}||_2 + ||p||_1). \tag{3.63}$$

4. A certain error expansion

In order to obtain asymptotic error expansion, we need to construct the following auxiliary equations

$$a(\widetilde{\mathbf{u}}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \widetilde{p}) = S_h(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_0, \tag{4.1}$$

$$(\nabla \cdot \widetilde{\mathbf{u}}, \omega) + (c\widetilde{p}, \omega) = R_h(\omega) \quad \forall \omega \in W, \tag{4.2}$$

where S_h and R_h are defined as (3.50) and (3.52).

Obviously, S_h and R_h have the important property

$$S_h = S_{h/2}, \qquad R_h = R_{h/2}. \tag{4.3}$$

We know, for equation (4.1)-(4.2), there exists a unique solution$(\widetilde{\mathbf{u}}, \widetilde{p}) \in \mathbf{V}_0 \times W$.

The corresponding finite element equation is defined to seek $(\widetilde{\mathbf{u}}_h, \widetilde{p}_h) \in \mathbf{V}_{0h} \times W_{0h}$ such that

$$a(\widetilde{\mathbf{u}}_h, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \widetilde{p}_h) = S_h(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_{0h}, \tag{4.4}$$

$$(\nabla \cdot \widetilde{\mathbf{u}}_h, \omega) + (c\widetilde{p}_h, \omega) = R_h(\omega) \quad \forall \omega \in W_h. \tag{4.5}$$

Then we have the following error expansions.

Theorem 4.1. *Assume* $\mathbf{u} \in (H^4(\Omega))^3$*,* $p \in H^3(\Omega)$*,* $\alpha_{ij} \in W^{3,\infty}(\Omega)(1 \le i, j \le 3)$ *and* $c \in W^{3,\infty}(\Omega)$*,* $\mathbf{u}$ *and* p *be the exact solution of (2.1)-(2.2),* $\mathbf{u}_h$ *and* p_h *be the finite element solution of (2.3)-(2.4) and* $\widetilde{\mathbf{u}}_h$ *and* $\widetilde{p}_h$ *be the mixed finite element solution of (4.4)-(4.5). Then we have the following expansions in the sense of* $(L^2(\Omega))^3$*-norm and* $L^2(\Omega)$*-norm, respectively:*

$$\mathbf{u}_h - \mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u} = h^2\widetilde{\mathbf{u}}_h + O(h^4), \tag{4.6}$$

$$p_h - J_{h_1,h_2,h_3}p = h^2\widetilde{p}_h + O(h^4). \tag{4.7}$$

Proof. Let $\Theta = \mathbf{u}_h - \mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u} - h^2\widetilde{\mathbf{u}}_h, \eta = p_h - J_{h_1,h_2,h_3}p - h^2\widetilde{p}_h$ and define

$$D((\mathbf{u}, p), (\mathbf{v}, w)) = a(\mathbf{u}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) + (\nabla \cdot \mathbf{u}, w) + (cp, w).$$

Then the following two properties for D are also satisfied

$$\begin{array}{ll} (1) & D((\mathbf{u}, p), (\mathbf{u}, p)) = a(\mathbf{u}, \mathbf{u}) + (cp, p), \\ (2) & D((\mathbf{u} - \mathbf{u}_h, p - p_h), (\mathbf{v}, w)) = 0 \quad \forall (\mathbf{v}, w) \in \mathbf{V}_h \times W_h. \end{array}$$

Then with Theorem 3.1 and 3.2, we know there exists a constant C independent of mesh-size h such that

$$\begin{aligned} a(\mathbf{\Theta}, \mathbf{\Theta}) + (c\eta, \eta) &= D((\mathbf{\Theta}, \eta), (\mathbf{\Theta}, \eta)) \\ &= D((\mathbf{u} - \mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u} - h^2\widetilde{\mathbf{u}}_h, p - J_{h_1,h_2,h_3}p - h^2\widetilde{p}_h), (\mathbf{\Theta}, \eta)) \\ &\le Ch^4(||\mathbf{u}||_4||\mathbf{\Theta}||_0 + ||p||_3||\eta||_0) \\ &\le Ch^4(||\mathbf{u}||_4 + ||p||_3)(||\mathbf{\Theta}||_0 + ||\eta||_0). \end{aligned}$$

By the property of A, we know there exist a constant $c > 0$ independent of h such that

$$\begin{aligned} c(||\mathbf{\Theta}||_0^2 + ||\eta||_0^2) &\le D((\mathbf{\Theta}, \eta), (\mathbf{\Theta}, \eta)) \\ &\le Ch^4(||\mathbf{u}||_4 + ||p||_3)(||\mathbf{\Theta}||_0 + ||\eta||_0). \end{aligned}$$

So

$$||\mathbf{\Theta}||_0 + ||\eta||_0 = \frac{C}{c}h^4(||\mathbf{\Theta}||_4 + ||p||_3). \tag{4.8}$$

Then, we obtain (4.6) and (4.7). □

From (3.62) and (3.63), we have the following corollary.

Corollary 4.1. *For $\widetilde{\mathbf{u}}_h$ and $\widetilde{p}_h$, we have the following estimates in the sense of $(L^2(\Omega))^3$-norm and $L^2(\Omega)$-norm, respectively:*

$$\widetilde{\mathbf{u}}_h - \mathbf{\Pi}_{h_1,h_2,h_3}\widetilde{\mathbf{u}} = O(h^2)(||\widetilde{\mathbf{u}}||_2 + ||\widetilde{p}||_1), \tag{4.9}$$

$$\widetilde{p}_h - J_{h_1,h_2,h_3}\widetilde{p} = O(h^2)(||\widetilde{\mathbf{u}}||_2 + ||\widetilde{p}||_1). \tag{4.10}$$

5. Interpolation postprocessing and extrapolation

In this section, we introduce the interpolation postprocessing operator and based on the postprocessing, extrapolation is adopted to generate higher order approximations to the exact solution of (1.1).

Let $\tau = \bigcup_{i=1}^{64} e_i$ with $e_i \in \mathbf{T}_{h_1,h_2,h_3}$ and $\mathbf{\Pi}^3_{4h}$ and J^3_{4h} are defined as follows

$$\begin{cases} \mathbf{\Pi}^3_{4h} \in Q_{433} \times Q_{343} \times Q_{334}, & J^3_{4h} \in Q_{333}, \\ \int_{f_i}(\mathbf{u} - \mathbf{\Pi}^3_{4h}) \cdot \mathbf{n} ds = 0 & i = 1, \cdots, 240, \\ \int_{e_i}(H - J^3_{4h}H) = 0 & i = 1, \cdots, 64. \end{cases} \tag{5.1}$$

The interpolation postprocessing (5.1) have the following properties

$$\begin{aligned} &\mathbf{\Pi}^3_{4h}\mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u} = \mathbf{\Pi}^3_{4h}\mathbf{u}, \\ &||\mathbf{\Pi}^3_{4h}\mathbf{v}||_0 \le C||\mathbf{v}||_0 \quad \forall \mathbf{v} \in \mathbf{V}_h, \\ &||\mathbf{\Pi}^3_{4h}\mathbf{u} - \mathbf{u}||_0 \le Ch^4||\mathbf{u}||_4 \quad \forall \mathbf{u} \in (H^4(\Omega))^3, \\ &J^3_{4h}J_{h_1,h_2,h_3}p = J^3_{4h}p, \\ &||J^3_{4h}w||_0 \le C||w||_0 \quad \forall w \in W_h, \\ &||J^3_{4h}p - p||_0 \le Ch^4||p||_4 \quad \forall p \in H^4(\Omega). \end{aligned}$$

Theorem 5.1. *Under the condition of Theorem 4.1, we have the following expansions in the sense of $(L^2(\Omega))^3$-norm and $L^2(\Omega)$-norm, respectively:*

$$\mathbf{\Pi}^3_{4h}\mathbf{u}_h - \mathbf{u} = h^2\widetilde{\mathbf{u}} + O(h^4), \tag{5.2}$$

$$J^3_{4h}p_h - p = h^2\widetilde{p} + O(h^4). \tag{5.3}$$

Proof. By (4.11) and the properties of interpolation $\mathbf{\Pi}^3_{4h}$, we have

$$\begin{aligned} \mathbf{\Pi}^3_{4h}\mathbf{u}_h - \mathbf{u} - h^2\widetilde{\mathbf{u}} &= \mathbf{\Pi}^3_{4h}(\mathbf{u}_h - \mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u} - h^2\widetilde{\mathbf{u}}_h) + (\mathbf{\Pi}^3_{4h}\mathbf{\Pi}_{h_1,h_2,h_3}\mathbf{u} - \mathbf{u}) \\ &\quad + h^2(\mathbf{\Pi}^3_{4h}\widetilde{\mathbf{u}}_h - \widetilde{\mathbf{u}}) \\ &= (\mathbf{\Pi}^3_{4h}\mathbf{u} - \mathbf{u}) + h^2(\mathbf{\Pi}^3_{4h}(\widetilde{\mathbf{u}}_h - \mathbf{\Pi}_h\widetilde{\mathbf{u}}) + (\mathbf{\Pi}^3_{4h}\mathbf{\Pi}_h\widetilde{\mathbf{u}} - \widetilde{\mathbf{u}})) \\ &\quad + O(h^4). \end{aligned} \tag{5.4}$$

Since $||\widetilde{\mathbf{u}}_h - \mathbf{\Pi}_{h_1,h_2,h_3}\widetilde{\mathbf{u}}|| = O(h^2)$, we can obtain

$$\mathbf{\Pi}^3_{4h}\mathbf{u}_h - \mathbf{u} - h^2\widetilde{\mathbf{u}} = O(h^4).$$

Similarly, we can prove (5.3). □

In order to use extrapolation, we divide each element $e_i \in \mathbf{T}_{h_1,h_2,h_3}$ into eight small congruent elements $e_{i,j} \in T_{h/2}(j = 1, 2, \cdots, 8)$ and the corresponding Raviart-Thomas mixed finite element space is denoted by $\mathbf{V}_{h/2} \times W_{h/2}$. Let $(\mathbf{u}_{h/2}, p_{h/2}) \in \mathbf{V}_{h/2} \times W_{h/2}$ and $(\mathbf{\Pi}^3_{2h}, J^3_{2h})$ be the mixed finite element approximation and interpolation operator with respect to this new partition.

With the formula (5.2) and (5.3), we can improve the accuracy by applying extrapolation:

Compute $(\mathbf{u}_{\text{extra}}, p_{\text{extra}})$ by the following formula

$$\mathbf{u}_{\text{extra}} := \frac{4\mathbf{\Pi}_{2h}^3\mathbf{u}_{h/2} - \mathbf{\Pi}_{4h}^3\mathbf{u}_h}{3}, \tag{5.5}$$

$$p_{\text{extra}} := \frac{4J_{2h}^3 p_{h/2} - J_{4h}^3 p_h}{3}. \tag{5.6}$$

Theorem 5.2. *Under the condition of Theorem 5.1, we have the following estimates in the sense of $(L^2(\Omega))^3$-norm and $L^2(\Omega)$-norm, respectively:*

$$\mathbf{u}_{\text{extra}} - \mathbf{u} = O(h^4), \tag{5.7}$$

$$p_{\text{extra}} - p = O(h^4). \tag{5.8}$$

Proof. First, we prove (5.7), by Theorem 5.1, we have

$$\begin{aligned} 4\mathbf{\Pi}_{2h}^3\mathbf{u}_{h/2} - \mathbf{\Pi}_{4h}^3\mathbf{u}_h - 3\mathbf{u} &= 4(\mathbf{\Pi}_{2h}^3\mathbf{u}_{h/2} - \mathbf{u}) - (\mathbf{\Pi}_{4h}^3\mathbf{u}_h - \mathbf{u}) \\ &= 4(\mathbf{\Pi}_{2h}^3\mathbf{u}_{h/2} - \mathbf{u} - (\frac{h}{2})^2\widetilde{\mathbf{u}}) - (\mathbf{\Pi}_{4h}^3\mathbf{u}_h - \mathbf{u} - h^2\widetilde{\mathbf{u}}) \\ &= O(h^4). \end{aligned}$$

Then (5.7) is proved, and (5.8) can be proved similarly. □

Remark. We can also use the extrapolation in one direction and parallel method. Hence we can save the computation and storage. It is more efficient than the normal extrapolation method described in (5.7) and (5.8). For the details, please read [5].

6. Concluding remarks

Practically the interpolation postprocessing and extrapolation method may give "good" results even though the true solution does not satisfy regularity assumptions guaranteeing superconvergence theoretically.

As a by-product, we can use the approximations of higher accuracy to form a class of a posterior error estimators ([5] and [13]) for the mixed finite element approximations.

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