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Relativistic paramagnetism of a weakly interacting Fermi gas

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Abstract. By considering strict relativistic energy of single particle and using the methods of quantum statisticsąAthe relativistic paramagnetism of a weakly interacting Fermi gas in a weak magnetic filed is studied, and the relativistic most probable paramagnetic susceptibility as well as the average magnetic susceptibility of the system are solved. On the basis, the influences of relativistic effect on the most probable paramagnetic susceptibility of the system are discussed, and the relativistic critical value of particle number is given. It is shown that, comparing with nonrelativistic situationąAwhen, the relativistic most probable magnetic susceptibility and the relativistic critical value of particle number have not changed. When ąAthe relativistic effects make the system display paramagnetism easily and susceptibility increase,but the relativistic effects also amplified the impact of the interaction on the magnetic susceptibility.

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Key words: Fermi gas, the relativistic effect, paramagnetism

1 Introduction

In recent years, the discovering of neutrino mass interests one in theoretical research of the relativistic Fermi gas. The rest mass of a neutrino is one of the million less than that of an electron. For a system which consists of these small particles, the relativistic effect on the statistic properties of the system needs to be considered even if under the condition of low temperatures. However, the most studies are thermodynamic properties of Fermi systems [1–5], the studies on magnetism of Fermi systems for finite number of particles are much few, and the paramagnetic researches of considering the relativistic effects have not been

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reported. Xiong *et al.* [6] and Dong [7] have respectively investigated Pauli's paramagnetism of an ideal and a weakly interacting Fermi gas with finite number of particles trapped in a weakly magnetic field, and have given the respective critical value of magnetic field. Men *et al.* [8, 9] respectively investigated relativistic thermodynamic properties and stability of a weakly interacting Fermi gas in a weak magnetic filed. For the Fermi system, these researches are important to deep understand magnetic filed and interparticle interactions.

In the present paper, by considering the relativistic effects and using the probability distribution function, we will continually study the paramagnetism of a weakly interacting Fermi system for finite number of particles, and give the relativistic most probable susceptibility as well as the average magnetic susceptibility of system. At the same time, we also give the relativistic critical values of particle number of the system.

2 The relativistic most probable magnetic susceptibility of the system

We study an imperfect gas of spin-1/2 fermions, with volume V and particle number N, confined in an homogenous weak magnetic field $B=B_Z$. It has been derived that, if s-wave considered only, the energy eigenvalue of the system can be written as [10]

$$E=\sum_p (n_p^+ + n_p^-)\varepsilon_p + \frac{\alpha}{V}N^+N^- - (N^+ - N^-)\mu B, \tag{1}$$

where μ is μ_B, the magnetic moment of fermions, $\alpha = 4\pi a\hbar^2/m$ is the parameter of interactions, a is s-wave scattering length, $n_p^+(n_p^-)$ is the number of particles in the states of momentum p with spin-up(spin-down), $N^+(N^-)$ is the total number of particles with spin-up(spin-down). Considering the relativistic effect, the energy of a single particle is expressed as

$$\varepsilon_p = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} - mc^2. \tag{2}$$

Based on the means which deal with nonideal Bose gases proposed by Huang, Yang *et al.* [11, 12], we introduce $A_0(\xi)$ and consider it as the relativistic free energy of the system of ξ spinless and noninteracting fermions without external potential in volume V

$$A_0(\xi) = -\frac{1}{\beta}\ln\sum_{\{\xi_p\}}\exp\left(-\beta\sum_p \xi_p\varepsilon_p\right), \tag{3}$$

followed by the method of Ref. [6], through calculation, the probability distribution function of N^+ consideration the relativistic effect can be given as

$$G(\beta,N^+) = A_n\exp\int_{\overline{N}^+}^{N^+}\delta(\beta,N^+)dN^+ = A_n\exp\left(-\left(\frac{\beta\sigma}{N}-\frac{2a\lambda^2}{V}\right)\left(N^+ - \overline{N}^+\right)^2\right), \tag{4}$$

and the relativistic most probable magnetic susceptibility of system can be written as

$$\chi_p = \frac{M_p}{VB} = \frac{2n\mu^2}{\sigma}\left(1+\frac{\alpha n}{\sigma}\right), \tag{5}$$

where A_n is a normalized constant, M_p is the most probable magnetization, $\sigma = \partial u_0(xN)/\partial x|_{x=1/2}$, u_0 is the relativistic chemical potential of system of spinless and free fermions corresponding to A_0, n is the particle-number density, , $\lambda = h/\sqrt{2\pi mkT}$, $\beta = 1/(kT)$, $\overline{N}^+ = (1+r)N/2$, $r \approx 2\mu B/(\sigma - \alpha n)$, k is the Boltzmann constant, and h is the Planck constant.

2.1 The case of $T \gg T_F$

In the case of high temperatures, i.e., $T \gg T_F$, the Fermi gas evolves into classical ideal gas, then we have [13]

$$u_0 = u_0(T,V,xN) = -kT\ln\left(\frac{4\pi}{xn}\left(\frac{mc}{h}\right)^3\frac{K_2(\phi)}{\phi}\right) - mc^2, \tag{6}$$

where $K_n(\phi)$ is the correctional Bessel function, $\phi = \beta mc^2$, by substituting Eq. (6) into Eq. (5), we will get the relativistic most probable magnetic susceptibility of system at high temperatures as

$$\chi_p = \frac{n\mu^2}{KT}\left(1+\frac{1}{\pi}\frac{T_F}{T}\frac{a}{\lambda_0}\right). \tag{7}$$

When $\phi \gg 1$, i.e., under the nonrelativistic condition, there is

$$u_0(xN,V,T) \approx kT\ln(xn\lambda^3), \tag{8}$$

via similar calculations, we get

$$\chi_p = \frac{n\mu^2}{KT}\left(1+\frac{1}{\pi}\frac{T_F}{T}\frac{a}{\lambda_0}\right), \tag{9}$$

$T_F = (3n/8\pi)^{2/3}h^2/(2mk)$ is nonrelativistic Fermi temperature, and $\lambda_0 = (8\pi n^{1/2}/3)^{-2/3}$. Especially, when $\phi \ll 1$, i.e., under the ultrarelativistic situation, we get the same analytical expression with Eq. (9).

Therefore, we may educe conclusion that the relativistic effect does not influence on the most probable magnetic susceptibility of system in the case of $T \gg T_F$.

2.2 The case of $T \ll T_F$

It can be known from Ref. [14] that, the chemical potential of consideration the relativistic effect satisfies

$$N = -\frac{\partial\Omega}{\partial\mu} = \frac{V^*_{n'}}{\lambda^{n'}}\left(\frac{2}{\pi}\beta mc^2\right)^{1/2}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}e^{j\beta\mu}}{j^{D+\eta-1}}K_D(j\beta mc^2). \tag{10}$$

where

$$V_{n'}^{*}=\prod_{i=1}^{n'}\frac{2L_i}{(\beta\varepsilon_i)^{1/t_i}}\Gamma\left(\frac{1}{t_i}+1\right),\quad D=\frac{n'+1}{2},\quad \eta=\sum_{i=1}^{n'}\frac{1}{t_i},$$

n' is space dimension. Let $n'=3$, $\eta=0$. Under the condition of $T\ll T_F$, considering relativistic effects, we get the chemical potential via calculations

$$u_0(xN,V,T)\approx E_F\left(\left(1-\frac{7}{30}\frac{E_F}{mc^2}\right)-\frac{\pi^2}{12}\left(\frac{kT}{E_F}\right)^2\left(1+\frac{7}{20}\frac{E_F}{mc^2}\right)\right), \tag{11}$$

where $E_f=\frac{h^2}{2m}(\frac{3n}{4\pi})^{2/3}$. By Eq. (5), we calculated the most probable magnetic susceptibility of the system as

$$\chi_p=\frac{3n\mu^2}{2\varepsilon_F\left(g+\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)}\left(1+\frac{3\alpha n}{4\varepsilon_F\left(g+\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right)}\right), \tag{12}$$

where $g=1-\frac{7}{15}\frac{\varepsilon_F}{mc^2}$, $\varepsilon_F=kT_F$ is nonrelativistic Fermi energy. From this we can see that, comparing with nonrelativistic situation,the relativistic effect increased susceptibility, but also amplified the impact of the interaction.

2.3 The case of $T\to 0$

We know that, for a system which consists of very small static mass fermions and is of a high particle number density, the average kinetic energy of the particles is close to or more than mc^2 even if the temperature tends to 0.

At extremely low temperatures, i.e., $T\to 0$, we get [13]

$$u_0(xN,V,T)=mc^2\sqrt{1+\left(\frac{3h^3xN}{4\pi V}\right)^{2/3}\frac{1}{(mc)^2}}-mc^2. \tag{13}$$

By substituting Eq. (13) into Eq. (5), we will get the relativistic most probable magnetic susceptibility of the system of at 0 temperatures as

$$x_p=\frac{3mc^2}{\varepsilon_F^{0\,2}}n\mu^2\sqrt{1+\left(\frac{\varepsilon_F^0}{mc^2}\right)^2}\left(1+\frac{3mc^2}{2\varepsilon_F^{0\,2}}\alpha n\sqrt{1+\left(\frac{\varepsilon_F^0}{mc^2}\right)^2}\right), \tag{14}$$

where $\varepsilon_F^0=hc(3n/8\pi)^{1/3}$ is ultrarelativistic Fermi energy. Especially, under the ultrarelativistic case, i.e., $\varepsilon_F^0\gg mc^2$, there is

$$\chi_p=\frac{3n\mu^2}{\varepsilon_F^0}\left(1+\frac{3\alpha n}{2\varepsilon_F^0}\right). \tag{15}$$

Under the nonrelativistic case, i.e., $\varepsilon_F^0 \ll mc^2$, there is

$$\chi_p = \frac{3}{2}\frac{n\mu^2}{\varepsilon_F}\left(1+\frac{3}{4}\frac{\alpha n}{\varepsilon_F}\right), \tag{16}$$

where $\varepsilon_F = kT_F$ is nonrelativistic Fermi energy.

Thereout it may be known that the relativistic effect has evident influence on the most probable magnetic susceptibility of system at 0 temperatures, which mostly shows that the effect of particle-number density on the most probable magnetic susceptibility of system changed.

3 The relativistic average magnetic susceptibility of the system

We normalize $G(\beta,N^+)$ as

$$\begin{aligned}
&\int_0^N A_n \exp\left(-\left(\frac{\beta\sigma}{N}-\frac{2a\lambda^2}{V}\right)(N^+-\overline{N}^+)^2\right)dN^+\\
&=A_n\int_{-\overline{N}^+}^{N-\overline{N}^+} e^{-x_0 y^2}dy = A_n\int_{-N/2-\phi}^{N/2-\phi} e^{-x_0 y^2}dy\\
&\approx 2A_n\int_0^\infty e^{-x_0 z^2}dz = A_n\sqrt{\pi/x_0}=1,
\end{aligned}$$

so $A_n=\sqrt{x_0/\pi}$, where $x_0=\frac{\beta\sigma}{N}-\frac{2a\lambda^2}{V}$, $\phi=\frac{2\mu B}{\sigma-\alpha n}$. The average value of N^+ is

$$\begin{aligned}
\langle N^+\rangle &= \sum_{N^+=0}^{N} N^+ G(\beta,N^+) \approx \int_0^N A_n N^+ \exp\left(-x_0(N^+-\overline{N}^+)^2\right)dN^+\\
&=\overline{N}^+ - \frac{A_n}{2x_0}\times\exp\left(-x_0\frac{N^2}{4}\right)\left(1-\exp(-x_0 N^2 r)\right),
\end{aligned}$$

we get the average magnetic susceptibility

$$\langle\chi\rangle = \left(\langle 2N^+\rangle - N\right)\frac{\mu}{VB} = \chi_p - \frac{A_n\mu}{x_0 VB}\times\exp\left(-x_0\frac{N^2}{4}\right)\left(1-\exp(-x_0N^2 r)\right). \tag{17}$$

When $T \gg T_F$, $\sigma = 2kT$, then we obtain

$$\langle\chi\rangle = \chi_p - \sqrt{\frac{1}{2\pi}}\frac{n\mu}{B\sqrt{N}}\left(1+\frac{\alpha n}{4kT}\right)\times\exp\left(-\frac{N}{2}\right)\left(1-\exp\left(-\frac{N\mu B}{kT}\right)\right). \tag{18}$$

When $T\to 0$,

$$\sigma = \frac{2(\varepsilon_F^0)^2}{3mc^2}\left(1+\left(\frac{\varepsilon_F^0}{mc^2}\right)^2\right)^{-1/2},$$

then $\langle\chi\rangle=\chi_p$, i.e., when $T\to 0$, the effects of finite number of particles disappear, where $\chi_p=(2\overline{N}^+-N)\mu/(VB)$ is the relativistic most probable magnetic susceptibility of system, the second term on the right-hand side of Eq. (18) is an amendatory term, which will be close to zero when the particle number N is maximum, is caused by effects of finite number of particles. From Eq. (18), it is obvious to find that, in the case of $T\gg T_F$, the average magnetic susceptibility of the system is less than most probable magnetic susceptibility whether the interactions are repulsive or attractive, and the average magnetic susceptibility $\langle\chi\rangle$ relate to magnetic field B, the amendatory term decreases with temperature raising.

4 The relativistic critical values of particle number

The most probable magnetic susceptibility χ_p in the case of thermodynamic limit only mirrors the effects of particle-number density and interactions, but it is independent of the total particle number and magnetic field. For the system with finite particle numbers, the situation is not so. By considering the relativistic effects and using the methods of Ref. [7], we obtain

$$\sqrt{\langle\partial^2 N^+\rangle}=\sqrt{\langle(N^+)^2\rangle-\langle N^+\rangle^2}\approx\frac{1}{\sqrt{2\left(\frac{\beta\sigma}{N}-\frac{2a\lambda^2}{V}\right)}},\tag{19}$$

where it should be emphasized that the chemical potential in Eq. (19) is relativistic. If the system displays paramagnetism,

$$\sqrt{\langle\partial^2 N^+\rangle}<(\overline{N}^+-\overline{N}^-)\tag{20}$$

will be satisfied. From Eq. (20) we will get

$$N>\frac{1}{8}\frac{KT}{(\mu B)^2}(\sigma-\alpha n).\tag{21}$$

Eq. (21) is the relativistic condition which particle number of the system will satisfy to observe the phenomenon of paramagnetism. Under the condition of $T\gg T_F$, by substituting Eq. (6) into Eq. (21), we will get the relativistic condition at high temperatures

$$N>\frac{1}{8}\frac{KT}{(\mu B)^2}(2KT-\alpha n)=N_0,\tag{22}$$

where N_0 is the relativistic critical value of particle number when $T\gg T_F$, i.e., for the system with finite number of particles, the particle number must be larger than the critical value to observe paramagnetism. We can also conclude from the critical value of particle number that, the inter-particle repulsions make the system display paramagnetism easily but the inter-particle attractions make the system display paramagnetism difficultly.

Similarly, by substituting Eq. (8) into Eq. (21), we will get the same result with Eq. (22). It shows that, under the condition of $T\gg T_F$, the relativistic effect does not influence on the critical value of particle number.

By substituting Eq. (11) into Eq. (21), we get

$$N > \frac{kT\varepsilon_F}{6(\mu B)^2}\left(g+\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2-\frac{3\alpha n}{4\varepsilon_F}\right). \tag{23}$$

From this we can know that, under the condition of $T \ll T_F$ to consider the relativistic effects, the critical value of the number of particles become smaller, this indicates that the system easy show paramagnetism to consider the relativistic effects. When $T \to 0$, whether the situations are relativistic or nonrelativistic, one can get $N_0 = 0$, i.e., the paramagnetism of the system require nothing for number of particles. From Eq. (20), we can also discuss the relativistic critical value of magnetic field to observe paramagnetism.

5 Conclusions

We have studied relativistic paramagnetism of a weakly interacting Fermi gas for finite number of particles, and have gained the analytical expression of the relativistic most probable magnetic susceptibility as well as the average magnetic susceptibility of system. At the same time, we have given the relativistic critical values of particle number. It is shown that, comparing with nonrelativistic situation, when $T \gg T_F$, the relativistic most probable magnetic susceptibility and the relativistic critical value of particle number have not changed. When $T \ll T_F$, the relativistic effects make the system display paramagnetism easily and susceptibility increase,but the relativistic effects also amplified the impact of the interaction on the magnetic susceptibility. When $T \to 0$, whether the situations are relativistic or nonrelativistic, the critical value of particle number is all zero. But, comparison between the ultrarelativistic case and the nonrelativistic case, the most probable magnetic susceptibility is different, that chiefly shows the influence of particle-number density on the most probable magnetic susceptibility is more evident in the ultrarelativistic case.

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