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LOW ORDER CROUZEIX-RAVIART TYPE NONCONFORMING FINITE ELEMENT METHODS FOR APPROXIMATING MAXWELL'S EQUATIONS

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Abstract. The aim of this paper is to study the convergence analysis of three low order Crouzeix-Raviart type nonconforming rectangular finite elements to Maxwell's equations, on a mixed finite element scheme and a finite element scheme, respectively. The error estimates are obtained for one of above elements with regular meshes and the other two under anisotropic meshes, which are as same as those in the previous literature for conforming elements under regular meshes.

Key Words. Maxwell's equations, low order nonconforming finite elements, error estimates, anisotropic meshes.

1. Introduction

It is well-known that Maxwell's equations are very important equations in the electric-magnetic fields and are usually solved with finite element methods(see [1-8]). P.Monk [2-4] described a mixed finite element scheme and a finite element scheme, respectively, and provided convergence and superconvergence analysis for smooth solutions for three-dimensional Maxwell's equations. Lin and Yan [5] improved Monk's results by means of the technique of integral identity. The similar results were proved for two-dimensional Maxwell's equations by Lin [6,8] and Brandts [7].

However, there are still some defects in the work mentioned above. On the one hand, all of previous analysis only concentrated on conforming finite elements, for examples, ECHL element, MECHL element, N*édé*lec's element [1] and so on. Whether those results hold for nonconforming ones or not is still an open problem. On the other hand, to our best knowledge, almost all the convergence analysis in the literature on this aspect are based on the classical regularity assumption or quasi-uniform assumption on the meshes [9], i.e., $\frac{h_K}{\rho_K} \leq C$ or $\frac{h}{h_{min}} \leq C$, $\forall K \in T_h$, where T_h is a family of meshes of Ω, h_K and ρ_K are the diameter of K and the biggest circle contained in the element K, respectively, $h = \max\limits_{K \in T_h} h_K, h_{min} = \min\limits_{K \in T_h} h_K$ and C is a positive constant which is independent of h_K and the function considered. However, in many cases, the above regular assumptions on meshes are great deficient in applications of finite element methods. For example, the solutions of some elliptic problems may have anisotropic behavior in parts of the defined domain. This means that the solution only varies significantly in certain directions. It is an

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obvious idea to reflect this anisotropy in the discretion by using anisotropic meshes with a finer mesh size in the direction of the rapid variation of the solution and a coarser mesh size in the perpendicular direction. Besides, some problems may be defined in narrow domain, for example, in modeling a gap between rotor and stator in an electrical machine, if we employ the regular partition of the domain, the cost of calculation will be very high. Therefore, to employ anisotropic meshes with fewer degrees of freedom is a better choice to overcome these difficulties. However, anisotropic elements K are characterized by $\frac{h_K}{\rho_K} \to \infty$, where the limit can be considered as $h \to \infty$. In this case, the Bramble-Hilbert Lemma can not be used directly in the estimate of the interpolation error, and the consistency error estimate. The key of the nonconforming finite element analysis, will become very difficult to be dealt with, because there will appear a factor $\frac{|F|}{|K|}$ which tends to ∞ when the estimate is made on the longer sides F of the element K, which means that the traditional finite element analysis techniques are no longer valid. Zenisek [10,11] and Apel [12,13] published a series of papers concentrating on the interpolation error estimates of some Lagrange type conforming elements, and [13] represented an anisotropic interpolation theorem, but it is very difficult to be verified for some elements. Chen and Shi [14] generalized Apel's results and studied many problems, including anisotropic nonconforming elements, and obtained a lot of valuable results [14-19]. Although anisotropic finite element methods have such obvious advantages over conventional ones, it seems that there are few studies focusing on Maxwell's equations of the finite element formulations, especially the nonconforming ones.

In this paper, we will apply three Crouzeix-Raviart type nonconforming finite elements (one is the so-called five-node nonconforming element[15,20], another is similar to the so-called P_1 nonconforming finite element discussed in [21] and the last one is a new element constructed in this paper) to Maxwell's equations on a mixed finite element scheme and a finite element scheme, respectively. The plan of this paper as follows: in section 2, we will give the constructions of the three Crouzeix-Raviart type nonconforming finite elements, analyze the mixed finite element scheme and the finite element scheme for the time-dependent Maxwell's system in two dimensions and prove some important Lemmas. In section 3, the so-called five-node nonconforming element is applied to Maxwell's equations on the finite element scheme, meanwhile, the other two elements are applied to approximating Maxwell's equations on the mixed finite element scheme and the finite element scheme, respectively. Based on some novel approaches and elements' properties, the convergence analysis and error estimates are obtained for two elements under anisotropic meshes and the other one with regular meshes, respectively.

2. Constructions of nonconforming finite element schemes

Let $\hat{K} = [-1,1]\times[-1,1]$ be the reference element on $\xi-\eta$ plane, the four vertices of $\hat{K}$ are $\hat{d}_1 = (-1,-1), \hat{d}_2 = (1,-1), \hat{d}_3 = (1,1)$ and $\hat{d}_4 = (-1,1)$, the four edges of $\hat{K}$ are $\hat{l}_1 = \overline{\hat{d}_1\hat{d}_2}$, $\hat{l}_2 = \overline{\hat{d}_2\hat{d}_3}$, $\hat{l}_3 = \overline{\hat{d}_3\hat{d}_4}$ and $\hat{l}_4 = \overline{\hat{d}_4\hat{d}_1}$.

The shape function spaces and the interpolation operators of the finite elements on $\hat{K}$ are defined by

(2.1)
$$\hat{P}^1 = span\{1,\xi,\eta,\varphi(\xi),\varphi(\eta)\}, \frac{1}{|\hat{K}|}\int_{\hat{K}}(\hat{v}-\hat{I}^1\hat{v})d\xi d\eta = 0, \frac{1}{|\hat{l}_k|}\int_{\hat{l}_k}(\hat{v}-\hat{I}^1\hat{v})d\hat{s} = 0,$$

$$\hat{P}^2 = span\{1,\xi,\eta\}, \frac{1}{|\hat{l}_k|}\int_{\hat{l}_k}\hat{I}^2\hat{v}d\hat{s} = \frac{1}{2}(\hat{v}(\hat{d}_k)+\hat{v}(\hat{d}_{k+1})), \tag{2.2}$$

(2.3) $$\hat{P}^3 = span\{1,\xi,\eta,\xi^2\}, \hat{P}^4 = span\{1,\xi,\eta,\eta^2\}, \frac{1}{|\hat{l}_k|}\int_{\hat{l}_k}(\hat{v}-\hat{I}^j\hat{v})d\hat{s} = 0,$$

where $\varphi(\xi) = \frac{1}{2}(3\xi^2-1),\ \varphi(\eta) = \frac{1}{2}(3\eta^2-1),\ k = 1,2,3,4,\ \hat{d}_5 = \hat{d}_1,\ j = 3,4.$
It can be easily checked that interpolations defined above are well-posed. If we denote $\hat{v}^k = \frac{1}{|\hat{l}_k|}\int_{\hat{l}_k}\hat{v}d\hat{s},\ \hat{v}^5 = \frac{1}{|\hat{K}|}\int_{\hat{K}}\hat{v}d\xi d\eta,\ \hat{v}^{k+5} = \hat{v}(\hat{d}_k)$ $(k = 1,2,3,4)$, the interpolation functions $\hat{I}^i\hat{v}(i = 1,2,3,4)$ can be expressed as

(2.4)
$$\hat{I}^1\hat{v} = \hat{v}^5+\frac{1}{2}(\hat{v}^2-\hat{v}^4)\xi+\frac{1}{2}(\hat{v}^3-\hat{v}^1)\eta+\frac{1}{2}(\hat{v}^2+\hat{v}^4-2\hat{v}^5)\varphi(\xi)+\frac{1}{2}(\hat{v}^3+\hat{v}^1-2\hat{v}^5)\varphi(\eta),$$

(2.5) $$\hat{I}^2\hat{v} = \frac{1}{4}(\hat{v}^6+\hat{v}^7+\hat{v}^8+\hat{v}^9)+\frac{1}{4}(\hat{v}^7+\hat{v}^8-\hat{v}^6-\hat{v}^9)\xi+\frac{1}{4}(\hat{v}^8+\hat{v}^9-\hat{v}^6-\hat{v}^7)\eta,$$

(2.6)
$$\hat{I}^3\hat{v} = \frac{3(\hat{v}^1+\hat{v}^3)-(\hat{v}^2+\hat{v}^4)}{4}+\frac{1}{2}(\hat{v}^2-\hat{v}^4)\xi+\frac{1}{2}(\hat{v}^3-\hat{v}^1)\eta+\frac{3}{4}(-\hat{v}^1+\hat{v}^2-\hat{v}^3+\hat{v}^4)\xi^2,$$

(2.7)
$$\hat{I}^4\hat{v} = \frac{3(\hat{v}^2+\hat{v}^4)-(\hat{v}^1+\hat{v}^3)}{4}+\frac{1}{2}(\hat{v}^2-\hat{v}^4)\xi+\frac{1}{2}(\hat{v}^3-\hat{v}^1)\eta+\frac{3}{4}(\hat{v}^1-\hat{v}^2+\hat{v}^3-\hat{v}^4)\eta^2,$$

respectively.

Lemma 2.1. *The interpolation operators $\hat{I}^i(i = 1,2)$ defined by (2.4) and (2.5) have the anisotropic interpolation property, i.e., $\forall\hat{v}\in H^2(\hat{K}), \alpha = (\alpha_1,\alpha_2)$ with $|\alpha| = 1$, there hold*

(2.8) $$\|\hat{D}^\alpha(\hat{v}-\hat{I}^i\hat{v})\|_{0,\hat{K}} \le C|\hat{D}^\alpha\hat{v}|_{1,\hat{K}}, \quad i = 1,2.$$

Here and later, the positive constant C will be used as a generic constant, which is independent of h_K and $\frac{h_K}{\rho_K}$.

Proof. When $\alpha = (1,0)$

$$\hat{D}^\alpha\hat{I}^1\hat{v} = \frac{\partial\hat{I}^1\hat{v}}{\partial\xi} = \frac{1}{2}(\hat{v}^2-\hat{v}^4)+\frac{1}{2}(\hat{v}^2+\hat{v}^4-2\hat{v}^5)\varphi'(\xi),$$

$$\hat{D}^\alpha\hat{I}^2\hat{v} = \frac{\partial\hat{I}^2\hat{v}}{\partial\xi} = \frac{1}{4}(\hat{v}^7+\hat{v}^8-\hat{v}^6-\hat{v}^9).$$

Note that $dim\hat{D}^\alpha\hat{P}^1 = 2$, $\hat{D}^\alpha\hat{P}^1 = span\{1,\varphi'(\xi)\}$ and $dim\hat{D}^\alpha\hat{P}^2 = 1$. Let

$$\hat{D}^\alpha\hat{I}^1\hat{v} = \beta_1+\beta_2\varphi'(\xi),\ \hat{D}^\alpha\hat{I}^2\hat{v} = \beta_3,$$

where

$$\beta_1 = \frac{1}{2}(\hat{v}^2-\hat{v}^4) = \frac{1}{4}(\int_{\hat{l}_2}\hat{v}(1,\eta)d\eta-\int_{\hat{l}_4}\hat{v}(-1,\eta)d\eta) = \frac{1}{|\hat{K}|}\int_{\hat{K}}\frac{\partial\hat{v}}{\partial\xi}d\xi d\eta,$$

$$\begin{aligned}\beta_2 &= \frac{1}{2}(\hat{v}^2+\hat{v}^4-2\hat{v}^5) = \frac{1}{4}(\int_{\hat{l}_2}\hat{v}(1,\eta)d\eta+\int_{\hat{l}_4}\hat{v}(-1,\eta)d\eta-\int_{\hat{K}}\hat{v}(\xi,\eta)d\xi d\eta)\\ &= \frac{1}{|\hat{K}|}\int_{\hat{K}}\xi\frac{\partial\hat{v}}{\partial\xi}d\xi d\eta,\end{aligned}$$

$$\beta_3 = \frac{1}{4}(\hat{v}^7+\hat{v}^8-\hat{v}^6-\hat{v}^9) = \frac{1}{4}(\int_{\hat{l}_1}\frac{\partial\hat{v}(\xi,-1)}{\partial\xi}d\xi+\int_{\hat{l}_3}\frac{\partial\hat{v}(\xi,1)}{\partial\xi}d\xi).$$

$\forall\hat{w}\in H^1(\hat{K})$, let

$$F_1(\hat{w}) = \frac{1}{|\hat{K}|}\int_{\hat{K}}\hat{w}d\xi d\eta,\ F_2(\hat{w}) = \frac{1}{|\hat{K}|}\int_{\hat{K}}\xi\hat{w}d\xi d\eta,$$

$$F_3(\hat{w}) = \frac{1}{4}(\int_{\hat{l}_1} \hat{w} d\xi + \int_{\hat{l}_3} \hat{w} d\xi).$$

Apparently $F_j \in (H^1(\hat{K}))^{'}, j = 1, 2, 3$. Employing the basic anisotropic interpolation theorem [14], there yields

$$\|\hat{D}^\alpha(\hat{v} - \hat{I}^1\hat{v})\|_{0,\hat{K}} \leq C|\hat{D}^\alpha\hat{v}|_{1,\hat{K}}.$$

Similarly, we can prove that (2.8) is valid for $\alpha = (0, 1)$ and $i = 2$. This completes the proof.

Let $\Omega \subset R^2$ be a polygon with boundaries parallel to the axes, $T_i^h(i = 1, 2, 3)$ be an axis parallel rectangular meshes of Ω, where T_1^h and T_2^h don't need to satisfy the regularity assumption or quasi-uniform assumption, but T_3^h is required to satisfy the above regular assumption.
$\forall K \in T_i^h(i = 1, 2, 3)$, let

$$K = [x_K - h_x, x_K + h_x] \times [y_K - h_y, y_K + h_y], h_K = \max\{h_x, h_y\}, h_i = \max_{K\in T_i^h} h_K.$$

Define the affine mapping $F_K : \hat{K} \longrightarrow K$ as follows:

$$\left\{ \begin{array}{l} x = x_K + h_x\xi, \\ y = y_K + h_y\eta. \end{array} \right. \tag{2.9}$$

Then the associated finite element spaces $V_i^h(i = 1, 2, 3)$ and $W_j^h(j = 2, 3)$ are defined by

$$V_1^h = \{\vec{v} = (v_1, v_2), \hat{v}_m|_{\hat{K}} = v_m|_K \circ F_K \in \hat{P}^1, \forall K \in T_1^h, \int_F [\vec{v}]ds = 0, F \subset \partial K\}, \tag{2.10}$$

$$\begin{aligned} V_2^h &= \{\vec{v} = (v_1, v_2), \hat{v}_m|_{\hat{K}} = v_m|_K \circ F_K \in \hat{P}^2, \forall K \in T_2^h, \int_F [\vec{v}]ds = 0, F \subset \partial K\}, \\ W_2^h &= \{w \in L^2(\Omega), w|_K \in Q_{0,0}(K), \forall K \in T_2^h\}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} V_3^h = \quad & \{\vec{v} = (v_1, v_2), \hat{v}_1|_{\hat{K}} = v_1|_K \circ F_K \in \hat{P}^3, \hat{v}_2|_{\hat{K}} = v_2|_K \circ F_K \in \hat{P}^4, \forall K \in T_3^h, \\ & \int_F [\vec{v}]ds = 0, F \subset \partial K\}, \\ W_3^h = & \left\{w \in L^2(\Omega), w|_K \in Q_{0,0}(K), \forall K \in T_3^h\right\}, \end{aligned} \tag{2.12}$$

respectively, where $m = 1, 2$, $[\vec{v}]$ stands for the jump of $\vec{v}$ across the edge F if F is an internal edge, and it is equal to $\vec{v}$ itself if F belongs to $\partial\Omega$, $Q_{0,0}(K)$ is a space of polynomials whose degrees for x, y are equal to 0, respectively.

We define the operators $\Pi^i : \vec{v} \in H_0(curl; \Omega) \cap (H^2(\Omega))^2 \longmapsto \Pi^i\vec{v} \in V_i^h(i = 1, 2, 3)$ as follows:

$$\Pi_K^i\vec{v} = (\hat{I}^i\hat{v}_1\circ F_K^{-1}, \hat{I}^i\hat{v}_2\circ F_K^{-1})(i = 1, 2), \Pi_K^3\vec{v} = (\hat{I}^3\hat{v}_1\circ F_K^{-1}, \hat{I}^4\hat{v}_2\circ F_K^{-1}), \Pi^i\,|_K = \Pi_K^i,$$

respectively. Obviously, for any $\vec{v} \in H_0(curl; \Omega) \cap (H^2(\Omega))^2$, the interpolations $\Pi^i\vec{v} \in V_i^h(i = 1, 2, 3)$ satisfy

$$\left\{ \begin{array}{l} \displaystyle\int_{l_k} (\vec{v} - \Pi^1\vec{v})ds = 0, \ k = 1, 2, 3, 4, \\ \displaystyle\int_K (\vec{v} - \Pi^1\vec{v})dxdy = 0, \end{array} \right. \tag{2.13}$$

$$
\begin{cases} \dfrac{1}{|l_k|}\displaystyle\int_{l_k}\Pi^2\vec{v}ds = \dfrac{1}{2}(\vec{v}(d_k)+\vec{v}(d_{k+1})),\ k=1,2,3,4, d_5=d_1, \\ \dfrac{1}{|l_1|}\displaystyle\int_{l_1}\Pi^2\vec{v}ds + \dfrac{1}{|l_3|}\displaystyle\int_{l_3}\Pi^2\vec{v}ds = \dfrac{1}{|l_2|}\displaystyle\int_{l_2}\Pi^2\vec{v}ds + \dfrac{1}{|l_4|}\displaystyle\int_{l_4}\Pi^2\vec{v}ds \end{cases} \tag{2.14}
$$

and

$$
\int_{l_k}(\vec{v}-\Pi^3\vec{v})ds = 0,\ k=1,2,3,4, \tag{2.15}
$$

where $l_k = \hat{l}_k\circ F_K^{-1},\ d_k = \hat{d}_k\circ F_K^{-1}(k=1,2,3,4)$ are the four edges and the four vertices of K, respectively.

At the same time, for any $w\in L^2(\Omega)$, we define the interpolations $R^jw\in W_j^h(j=2,3)$, on element K, as follows

$$
\int_K(w-R^jw)dxdy = 0,\ j=2,3. \tag{2.16}
$$

Consider the following two-dimensional Maxwell's equations [22] :

$$
\begin{cases} \epsilon\vec{E}_t+\sigma\vec{E}-rotH=-\vec{J}, & in\ \Omega\times(0,T), \\ \mu H_t + curl\vec{E} = 0, & in\ \Omega\times(0,T), \\ \vec{n}\times\vec{E}=0, & on\ \partial\Omega\times(0,T), \\ \vec{E}(0)=\vec{E}_0, H(0)=H_0, \end{cases} \tag{2.17}
$$

where $\epsilon=\epsilon(\mathbf{x})$ and $\mu=\mu(\mathbf{x})$ denote the dielectric constant and the magnetic permeability of the material in Ω, respectively; $\sigma=\sigma(\mathbf{x})$ denotes the conductivity of the medium; $\vec{E}(\mathbf{x},t)$ and $H(\mathbf{x},t)$ denote, respectively, the electric and magnetic fields; $\vec{J}=\vec{J}(\mathbf{x},t)$ is a known function specifying the applied current, $\mathbf{x}=(x,y)$; $\vec{E}_0=\vec{E}_0(\mathbf{x},t)$ and $H_0=H_0(\mathbf{x},t)$ are given functions. The coefficients $\epsilon(\mathbf{x}),\mu(\mathbf{x})$ and $\sigma(\mathbf{x})$ are bounded, and there exist constants ϵ_{min} and μ_{min} such that

$$
0<\epsilon_{min}\le\epsilon(\mathbf{x}),\ 0<\mu_{min}\le\mu(\mathbf{x}),\ \forall\mathbf{x}\in\Omega.
$$

Furthermore, the conductivity σ is a nonnegative function on $\overline{\Omega}$. $\vec{E}=(E_1,E_2)$, $rotH=(\frac{\partial H}{\partial y},-\frac{\partial H}{\partial x})$, $curl\vec{E}=\frac{\partial E_2}{\partial x}-\frac{\partial E_1}{\partial y}$, $\vec{n}\times\vec{E}=E_1n_2-E_2n_1$, $\vec{n}=(n_1,n_2)$ is the unit outward normal vector on $\partial\Omega$.

In the following, we will use the notations:

$$
\|\cdot\|_l,\ \|\cdot\|_{l,K}\ \text{ for } H^l(\Omega) \text{ or } (H^l(\Omega))^2, H^l(K) \text{ or } (H^l(K))^2\text{-norm},
$$

$$
|\cdot|_m,\ |\cdot|_{m,K} \text{ for } H^m(\Omega) \text{ or } (H^m(\Omega))^2, H^m(K) \text{ or } (H^m(K))^2\text{-seminorm},
$$

where $l\ge0, m>0$ are integer numbers, $H^0(\Omega)=L^2(\Omega)$ and $H^0(K)=L^2(K)$.

Let

$$
H(curl;\Omega)=\{\vec{v}=(v_1,v_2)\in(L^2(\Omega))^2; curl\vec{v}\in L^2(\Omega)\},
$$
$$
H_0(curl;\Omega)=\{\vec{v}\in H(curl;\Omega),\vec{n}\times\vec{v}\mid_{\partial\Omega}=0\}
$$

with norm

$$
\|\vec{v}\|_{H(curl;\Omega)}=(\|\vec{v}\|_0^2+\|curl\vec{v}\|_0^2)^{\frac{1}{2}}.
$$

We denote

$$
(p,q)=\int_\Omega pqdxdy,\ (p,q)_h=\sum_{K\in T_i^h}\int_K pqdxdy(i=1,2,3).
$$

Then two discrete schemes presented in [1] and [3] are described as follows:

(1) A mixed finite element scheme

The variational formulation to (2.17) reads as:
find $(\vec{E},H)\in H_0(curl;\Omega)\times L^2(\Omega)$, such that

$$\left\{\begin{array}{l}(\epsilon\vec{E}_t,\vec{\Phi})+(\sigma\vec{E},\vec{\Phi})-(H,curl\vec{\Phi})=-(\vec{J},\vec{\Phi}),\ \forall\vec{\Phi}\in H_0(curl;\Omega),\\ (\mu H_t,\Psi)+(curl\vec{E},\Psi)=0,\ \forall\Psi\in L^2(\Omega),\\ \vec{E}(0)=\vec{E}_0,\ H(0)=H_0.\end{array}\right. \tag{2.18}$$

Then the mixed finite element approximations based on (2.18) is:
find $(\vec{E}^h,H^h)\in V_i^h\times W_i^h(i=2,3)$, such that

$$\left\{\begin{array}{l}(\epsilon\vec{E}_t^h,\vec{\Phi})_h+(\sigma\vec{E}^h,\vec{\Phi})_h-(H^h,curl\vec{\Phi})_h=-(\vec{J},\vec{\Phi})_h,\ \forall\vec{\Phi}\in V_i^h,\\ (\mu(H^h)_t,\Psi)_h+(curl\vec{E}^h,\Psi)_h=0,\ \forall\Psi\in W_i^h,\\ \vec{E}^h(0)=\Pi^i\vec{E}_0,\ H^h(0)=R^iH_0,\end{array}\right. \tag{2.19}$$

where $\Pi^i\vec{E}_0$ and R^iH_0 are the finite element interpolations of $\vec{E}(0)$ and $H(0)$, respectively. Since the discrete scheme is a system of ordinary differential equations with respect to t, it has a unique solution.

(2) A finite element scheme

By taking the time differentiation of the first equation of (2.17) and using the second equation of (2.17), we obtain the following electric field equation

$$\left\{\begin{array}{l}\epsilon\vec{E}_{tt}+\sigma\vec{E}_t+rot(\frac{1}{\mu}curl\vec{E})=\vec{G},\ in\ \Omega\times(0,T),\\ \vec{E}(0)=\vec{E}_0,\ \vec{E}_t(0)=\vec{E}_{t0},\end{array}\right. \tag{2.20}$$

where $\vec{G}(\mathbf{x},t)=-\vec{J}_t(\mathbf{x},t),\vec{E}_{t0}=\frac{1}{\epsilon}[-\vec{J}(\mathbf{x},0)+rotH_0-\sigma\vec{E}_0]$.
Then the weak form of (2.20) is:
find $\vec{E}\in H_0(curl;\Omega)$, such that

$$\left\{\begin{array}{l}(\epsilon\vec{E}_{tt},\vec{\Phi})+(\sigma\vec{E}_t,\vec{\Phi})+(\frac{1}{\mu}curl\vec{E},curl\vec{\Phi})=(\vec{G},\vec{\Phi}),\ \forall\vec{\Phi}\in H_0(curl;\Omega),\\ \vec{E}(0)=\vec{E}_0,\ \vec{E}_t(0)=\vec{E}_{t0}.\end{array}\right. \tag{2.21}$$

The finite element scheme of (2.21) is:
find $\vec{E}^h\in V_i^h(i=1,2,3)$, such that

$$\left\{\begin{array}{l}(\epsilon\vec{E}_{tt}^h,\vec{\Phi})_h+(\sigma\vec{E}_t^h,\vec{\Phi})_h+(\frac{1}{\mu}curl\vec{E}^h,curl\vec{\Phi})_h=(\vec{G},\vec{\Phi})_h,\ \forall\vec{\Phi}\in V_i^h,\\ \vec{E}^h(0)=\Pi^i\vec{E}_0,\ \vec{E}_t^h(0)=\Pi^i\vec{E}_{t0},\end{array}\right. \tag{2.22}$$

where $\Pi^i\vec{E}_0$ and $\Pi^i\vec{E}_{t0}$ are interpolations of $\vec{E}_0$ and $\vec{E}_{t0}$, respectively.

We define mesh dependent norms:

$$\|\vec{v}\|_{0h}^2=(\vec{v},\vec{v})_h=\sum_{K\in T_i^h}\sum_{j=1}^2\|v_j\|_{0,K}^2,\ \|\vec{v}\|_{1h}^2=\sum_{K\in T_i^h}\sum_{j=1}^2|v_j|_{1,K}^2.$$

Then it is easy to see that $\|\cdot\|_{0h}$ and $\|\cdot\|_{1h}$ are the norms over $V_i^h(i=1,2,3)$. We have the following important lemmas.

Lemma 2.2. *$\forall\vec{v}\in(H^2(\Omega))^2\cap H_0(curl;\Omega)$, we have*

$$\|\vec{v}-\Pi^i\vec{v}\|_{0h}\leq Ch_i|\vec{v}|_1,\ i=1,2,3, \tag{2.23}$$

$$\|curl(\vec{v}-\Pi^i\vec{v})\|_{0h}\leq Ch_i|\vec{v}|_2,\ i=1,2,3. \tag{2.24}$$

Proof. By Lemma 2.1 and the interpolation theorem [9,14], it is obvious that

$$\|\vec{v}-\Pi^i\vec{v}\|_{0h}\leq Ch_i|\vec{v}|_1,\ i=1,2,3,$$

$$\| \vec{v} - \Pi^3 \vec{v} \|_{1h} \leq C h_3 |\vec{v}|_2.$$

When $i = 1, 2$, by Lemma 2.1, we have

$$\begin{aligned}
\| \vec{v} - \Pi^i \vec{v} \|_{1h} &= \Big(\sum_{K \in T_i^h} |\vec{v} - \Pi_K^i \vec{v}|_{1,K}^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{K \in T_i^h} \sum_{|\alpha|=1} \| D^\alpha (\vec{v} - \Pi_K^i \vec{v}) \|_{0,K}^2\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{K \in T_i^h} \sum_{|\alpha|=1} h_K^{-2\alpha} (h_x h_y) \| \hat{D}^\alpha (\hat{\vec{v}} - \hat{\Pi}_K^i \hat{\vec{v}}) \|_{0,\hat{K}}^2\Big)^{\frac{1}{2}} \\
&\leq C\Big(\sum_{K \in T_i^h} \sum_{|\alpha|=1} h_K^{-2\alpha} (h_x h_y) |\hat{D}^\alpha \hat{\vec{v}}|_{1,\hat{K}}^2\Big)^{\frac{1}{2}} \\
&\leq C\Big(\sum_{K \in T_i^h} \sum_{|\alpha|=1} \sum_{|\beta|=1} h_K^{2\beta} \| D^{\alpha+\beta} \vec{v} \|_{0,K}^2\Big)^{\frac{1}{2}} \\
&\leq C h_i |\vec{v}|_2,
\end{aligned}$$

so (2.24) follows by $\| curl(\vec{v} - \Pi^i \vec{v}) \|_{0h} \leq \| \vec{v} - \Pi^i \vec{v} \|_{1h}$.

Lemma 2.3. *$\forall \vec{v} \in (H^1(\Omega))^2 \cap H_0(curl;\Omega))$, we have*

$$(curl(\vec{v} - \Pi^3 \vec{v}), \Psi)_h = 0, \ \forall \Psi \in W_3^h, \tag{2.25}$$

$$(curl(\vec{v} - \Pi^3 \vec{v}), curl \vec{\Phi})_h = 0, \ \forall \vec{\Phi} \in V_3^h, \tag{2.26}$$

$$(w - R^i w, curl \vec{\Phi})_h = 0, \ \forall \vec{\Phi} \in V_i^h, i = 2, 3, \forall w \in L^2(\Omega). \tag{2.27}$$

Proof. Note that $\forall q |_K \in Q_{0,0}(K), rot q |_K$ vanishes. By Green's formula and the definition of Π^3, we get

$$\int_K curl(\vec{v} - \Pi^3 \vec{v}) q dx dy = \int_K (\vec{v} - \Pi^3 \vec{v}) \cdot rot q dx dy + \int_{\partial K} \vec{n} \times (\vec{v} - \Pi^3 \vec{v}) q ds = 0.$$

Since for any $\vec{\Phi} \in V_3^h$, $curl \vec{\Phi}|_K$ is a constant,

$$(curl(\vec{v} - \Pi^3 \vec{v}), curl \vec{\Phi})_h = 0, \ \forall \vec{\Phi} \in V_3^h.$$

Similarly, by the definitions of $R^i (i = 2, 3)$, we have

$$(w - R^i w, curl \vec{\Phi})_h = 0, \ \forall \vec{\Phi} \in V_i^h, i = 2, 3, \forall w \in L^2(\Omega).$$

Lemma 2.4. *$\forall H \in H^2(\Omega)$, we have*

$$\sum_{K \in T_i^h} \int_{\partial K} H \ \vec{n} \times \vec{\Phi} ds \leq C h_i |H|_2 \| \vec{\Phi} \|_{0h}, \ \forall \vec{\Phi} \in V_i^h, i = 1, 2, 3. \tag{2.28}$$

Proof. $\forall K \in T_3^h, H \in H^2(K)$ and $\forall \vec{\Phi} = (\Phi_1, \Phi_2) \in V_3^h$, let

$$P_{0k} H = \frac{1}{2h_x} \int_{l_k} H dx, k = 1, 3,$$

$$P_{0k} H = \frac{1}{2h_y} \int_{l_k} H dy, k = 2, 4.$$

Then we have

$$\begin{aligned}
&\sum_{K\in T_3^h}\int_{\partial K} H\vec{n}\times\vec{\Phi}ds\\
&=\sum_{K\in T_3^h}\int_{\partial K}(H\Phi_1 n_2-H\Phi_2 n_1)ds\\
&=\sum_{K\in T_3^h}[\int_{l_1}-(\Phi_1-P_{01}\Phi_1)(H-P_{01}H)dx\\
&\quad+\int_{l_3}(\Phi_1-P_{03}\Phi_1)(H-P_{03}H)dx\\
&\quad+\int_{l_2}-(\Phi_2-P_{02}\Phi_2)(H-P_{02}H)dy\\
&\quad+\int_{l_4}(\Phi_2-P_{04}\Phi_2)(H-P_{04}H)dy]\\
&=\sum_{K\in T_3^h}[I_1+I_3+I_2+I_4],
\end{aligned}$$

where

$$\begin{aligned}
I_1&=\int_{l_1}-(\Phi_1-P_{01}\Phi_1)(H-P_{01}H)dx,\\
I_2&=\int_{l_2}-(\Phi_2-P_{02}\Phi_2)(H-P_{02}H)dy,\\
I_3&=\int_{l_3}(\Phi_1-P_{03}\Phi_1)(H-P_{03}H)dx,\\
I_4&=\int_{l_4}(\Phi_2-P_{04}\Phi_2)(H-P_{04}H)dy.
\end{aligned}$$

Since

$$\begin{aligned}
I_1+I_3=&-\int_{x_K-h_x}^{x_K+h_x}[H(x,y_K-h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}H(x,y_K-h_y)dx]\\
&\cdot[\Phi_1(x,y_K-h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\Phi_1(x,y_K-h_y)dx]dx\\
&+\int_{x_K-h_x}^{x_K+h_x}[H(x,y_K+h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}H(x,y_K+h_y)dx]\\
&\cdot[\Phi_1(x,y_K+h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\Phi_1(x,y_K+h_y)dx]dx
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_1(x,y_K-h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\Phi_1(x,y_K-h_y)dx\\
&=\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}[\Phi_1(x,y_K-h_y)-\Phi_1(t,y_K-h_y)]dt\\
&=\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x\frac{\partial\Phi_1}{\partial z}(z,y_K-h_y)dzdt\\
&=\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x\frac{\partial\Phi_1}{\partial z}(z,y_K+h_y)dzdt\\
&=\Phi_1(x,y_K+h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\Phi_1(x,y_K+h_y)dx,
\end{aligned}$$

by the use of the special property: $\frac{\partial \Phi_1}{\partial x} \in span\{1, x\}$ and the same argument of [15], we have

$$|I_1 + I_3| \leq \frac{4h_x^2}{3} \| \frac{\partial^2 H}{\partial x \partial y} \|_{0,K} \| \frac{\partial \Phi_1}{\partial x} \|_{0,K} .$$

Similarly, since $\frac{\partial \Phi_2}{\partial y} \in span\{1, y\}$, we can get

$$|I_2 + I_4| \leq \frac{4h_y^2}{3} \| \frac{\partial^2 H}{\partial x \partial y} \|_{0,K} \| \frac{\partial \Phi_2}{\partial y} \|_{0,K} .$$

Note that $\| \frac{\partial \Phi_1}{\partial x} \|_{0,K} \leq Ch_x^{-1} \| \Phi_1 \|_{0,K}$, $\| \frac{\partial \Phi_2}{\partial y} \|_{0,K} \leq Ch_y^{-1} \| \Phi_2 \|_{0,K}$, we have

$$\sum_{K \in T_3^h} \int_{\partial K} H \ \vec{n} \times \vec{\Phi} ds \leq Ch_3 |H|_2 \| \vec{\Phi} \|_{0h}, \ \forall \vec{\Phi} \in V_3^h.$$

Since for any $\vec{\Phi} \in V_i^h (i = 1, 2)$, $\frac{\partial \Phi_1}{\partial x}$ and $\frac{\partial \Phi_2}{\partial y}$ have nothing to do with the variable y and x, respectively, thus (2.28) holds for these elements, i.e.

$$\sum_{K \in T_i^h} \int_{\partial K} H \ \vec{n} \times \vec{\Phi} ds \leq Ch_i |H|_2 \| \vec{\Phi} \|_{0h}, \ \forall \vec{\Phi} \in V_i^h, i = 1, 2,$$

which completes the proof.

3. The convergence analysis

Now, based on the lemmas in Section 2, we can get the main results of this paper. For the sake of simplicity, we will assume that $\epsilon = \mu = 1$ and $\sigma = 0$.

Theorem 3.1. *Assume that* $(\vec{E}, H) \in H_0(curl; \Omega) \times L^2(\Omega)$, $(\vec{E}^h, H^h) \in V_i^h \times W_i^h (i = 2, 3)$ *are the solutions of (2.17) and (2.19), respectively,* $\vec{E}_t \in (H^1(\Omega))^2, \vec{E} \in (H^2(\Omega))^2$ *and* $H \in H^2(\Omega)$. *Then we have*

$$\| \vec{E}^h - \Pi^2 \vec{E} \|_{0h} + \| H^h - R^2 H \|_{0h} \leq Ch_2 (\int_0^t (|\vec{E}_t|_1^2 + |H|_2^2 + |\vec{E}|_2^2) d\tau)^{\frac{1}{2}}, \tag{3.1}$$

$$\| \vec{E} - \vec{E}^h \|_{0h} + \| H - H^h \|_{0h} \leq Ch_2 [|\vec{E}|_1 + |H|_1 + (\int_0^t (|\vec{E}_t|_1^2 + |H|_2^2 + |\vec{E}|_2^2) d\tau)^{\frac{1}{2}}], \tag{3.2}$$

$$\| \vec{E}^h - \Pi^3 \vec{E} \|_{0h} + \| H^h - R^3 H \|_{0h} \leq Ch_3 (\int_0^t (|\vec{E}_t|_1^2 + |H|_2^2) d\tau)^{\frac{1}{2}}, \tag{3.3}$$

$$\| \vec{E} - \vec{E}^h \|_{0h} + \| H - H^h \|_{0h} \leq Ch_3 [|\vec{E}|_1 + |H|_1 + (\int_0^t (|\vec{E}_t|_1^2 + |H|_2^2) d\tau)^{\frac{1}{2}}]. \tag{3.4}$$

Proof. We define

$$A((\vec{E}, H); (\vec{\Phi}, \Psi)) = (\vec{E}_t, \vec{\Phi}) - (H, curl \vec{\Phi}) + (H_t, \Psi) + (curl \vec{E}, \Psi). \tag{3.5}$$

Then we have

$$A((\vec{E}, H); (\vec{E}, H)) = (\vec{E}_t, \vec{E}) + (H_t, H) = \frac{1}{2} \frac{d}{dt} (\| \vec{E} \|_0^2 + \| H \|_0^2).$$

Due to for any $(\vec{\Phi},\Psi)\in V_i^h\times W_i^h(i=2,3)$,

$$(rotH,\vec{\Phi})_h=\sum_{K\in T_i^h}\int_{\partial K}H\ \vec{n}\times\vec{\Phi}ds+\sum_{K\in T_i^h}\int_K H\ curl\vec{\Phi}dxdy,$$

it follows from (2.17) and (2.19) that

$$A_h((\vec{E}-\vec{E}^h,H-H^h);(\vec{\Phi},\Psi))=\sum_{K\in T_i^h}\int_{\partial K}H\ \vec{n}\times\vec{\Phi}ds,$$

where $A_h((\vec{E},H);(\vec{\Phi},\Psi))=(\vec{E}_t,\vec{\Phi})_h-(H,curl\vec{\Phi})_h+(H_t,\Psi)_h+(curl\vec{E},\Psi)_h$.
Let $(\vec{\zeta},\theta)=(\vec{E}^h-\Pi^i\vec{E},H^h-R^iH)$, we have

$$\begin{aligned}&A_h((\vec{\zeta},\theta);(\vec{\Phi},\Psi))=A_h((\vec{E}^h-\Pi^i\vec{E},H^h-R^iH);(\vec{\Phi},\Psi))\\&=A_h((\vec{E}-\Pi^i\vec{E},H-R^iH);(\vec{\Phi},\Psi))-\sum_{K\in T_i^h}\int_{\partial K}H\ \vec{n}\times\vec{\Phi}ds\\&=((\vec{E}-\Pi^i\vec{E})_t,\vec{\Phi})_h-(H-R^iH,curl\vec{\Phi})_h+(H_t-R^iH_t,\Psi)_h\\&+(curl(\vec{E}-\Pi^i\vec{E}),\Psi)_h-\sum_{K\in T_i^h}\int_{\partial K}H\ \vec{n}\times\vec{\Phi}ds.\end{aligned}\tag{3.6}$$

by (2.23)-(2.25), (2.27), (2.28) and the definitions of $R^i(i=2,3)$

$$A_h((\vec{\zeta},\theta);(\vec{\Phi},\Psi))\le Ch_2(|\vec{E}_t|_1+|H|_2)\parallel\vec{\Phi}\parallel_{0h}+Ch_2|\vec{E}|_2\parallel\Psi\parallel_{0h},\tag{3.7}$$

$$A_h((\vec{\zeta},\theta);(\vec{\Phi},\Psi))\le Ch_3(|\vec{E}_t|_1+|H|_2)\parallel\vec{\Phi}\parallel_{0h}.\tag{3.8}$$

Taking $(\vec{\Phi},\Psi)=(\vec{\zeta},\theta)$ in (3.7),(3.8)and applying Schwarz inequality, there yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\parallel\vec{\zeta}\parallel_{0h}^2+\parallel\theta\parallel_{0h}^2)=\frac{1}{2}\frac{d}{dt}(\parallel\vec{E}^h-\Pi^2\vec{E}\parallel_{0h}^2+\parallel H^h-R^2H\parallel_{0h}^2)\\&\le Ch_2^2(|\vec{E}_t|_1^2+|H|_2^2+|\vec{E}|_2^2)+\frac{1}{2}\parallel\vec{\zeta}\parallel_{0h}^2+\frac{1}{2}\parallel\theta\parallel_{0h}^2,\end{aligned}\tag{3.9}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\parallel\vec{\zeta}\parallel_{0h}^2+\parallel\theta\parallel_{0h}^2)&=\frac{1}{2}\frac{d}{dt}(\parallel\vec{E}^h-\Pi^3\vec{E}\parallel_{0h}^2+\parallel H^h-R^3H\parallel_{0h}^2)\\&\le Ch_3^2(|\vec{E}_t|_1^2+|H|_2^2)+\frac{1}{2}\parallel\vec{\zeta}\parallel_{0h}^2.\end{aligned}\tag{3.10}$$

Because $\vec{\zeta}(0)=(0,0),\theta(0)=0$, integrating (3.9),(3.10) for t, by Gronwall inequality, there yield

$$\parallel\vec{E}^h-\Pi^2\vec{E}\parallel_{0h}+\parallel H^h-R^2H\parallel_{0h}\le Ch_2(\int_0^t(|\vec{E}_t|_1^2+|H|_2^2+|\vec{E}|_2^2)d\tau)^{\frac{1}{2}},$$

$$\parallel\vec{E}^h-\Pi^3\vec{E}\parallel_{0h}+\parallel H^h-R^3H\parallel_{0h}\le Ch_3(\int_0^t(|\vec{E}_t|_1^2+|H|_2^2)d\tau)^{\frac{1}{2}}.$$

By the triangle inequality, we can prove (3.2) and (3.4). The proof is completed.
Theorem 3.2. *Assume that $\vec{E}\in H_0(curl;\Omega)$, $\vec{E}^h\in V_i^h(i=1,2,3)$ are the solutions of (2.20) and (2.22) respectively, and $\Pi^i\vec{E}\in V_i^h(i=1,2,3)$ are the interpolations of $\vec{E}\in(H^3(\Omega))^2,\vec{E}_t,\vec{E}_{tt}\in(H^2(\Omega))^2$. Then*

$$\begin{aligned}&\parallel(\vec{E}^h-\Pi^i\vec{E})_t\parallel_{0h}+\parallel curl(\vec{E}^h-\Pi^i\vec{E})\parallel_{0h}\\&\le Ch_i[|\vec{E}|_2^2+\int_0^t(|\vec{E}_{tt}|_1^2+|\vec{E}_t|_2^2+|\vec{E}|_3^2)d\tau]^{\frac{1}{2}},\ i=1,2,\end{aligned}\tag{3.11}$$

$$\begin{aligned}&\| (\vec{E}-\vec{E}^h)_t \|_{0h} + \| curl(\vec{E}-\vec{E}^h) \|_{0h}\\ &\le Ch_i[|\vec{E}_t|_1 + |\vec{E}|_2 + (|\vec{E}|_2^2 + \int_0^t (|\vec{E}_{tt}|_1^2 + |\vec{E}_t|_2^2 + |\vec{E}|_3^2)d\tau)^{\frac{1}{2}}],\ i=1,2,\end{aligned} \tag{3.12}$$

$$\| (\vec{E}^h - \Pi^3\vec{E})_t \|_{0h} + \| curl(\vec{E}^h - \Pi^3\vec{E}) \|_{0h} \le Ch_3[\int_0^t (|\vec{E}_{tt}|_1^2 + |\vec{E}|_3^2)d\tau]^{\frac{1}{2}}, \tag{3.13}$$

(3.14)

$$\| (\vec{E}-\vec{E}^h)_t \|_{0h} + \| curl(\vec{E}-\vec{E}^h) \|_{0h} \le Ch_3[|\vec{E}_t|_1 + |\vec{E}|_2 + (\int_0^t (|\vec{E}_{tt}|_1^2 + |\vec{E}|_3^2)d\tau)^{\frac{1}{2}}].$$

Proof. $\forall \vec{\Phi} \in V_i^h (i=1,2,3)$, by (2.20) and (2.22), we can obtain

$$\begin{aligned}&((\vec{E}^h - \Pi^i\vec{E})_{tt}, \vec{\Phi})_h + (curl(\vec{E}^h - \Pi^i\vec{E}), curl\vec{\Phi})_h\\ &= ((\vec{E} - \Pi^i\vec{E})_{tt}, \vec{\Phi})_h + (curl(\vec{E} - \Pi^i\vec{E}), curl\vec{\Phi})_h\\ &+ \sum_{K\in T_i^h} \int_{\partial K} curl\vec{E}\ \vec{n} \times \vec{\Phi} ds.\end{aligned} \tag{3.15}$$

Let $\vec{\zeta} = \vec{E}^h - \Pi^i\vec{E}, \vec{\Phi} = \vec{\zeta}_t$, then (3.15) can be rewritten as

(3.16)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\| \vec{\zeta}_t \|_{0h}^2 + \| curl\vec{\zeta} \|_{0h}^2) = (\vec{\zeta}_{tt}, \vec{\zeta}_t)_h + (curl\vec{\zeta}, curl\vec{\zeta}_t)_h\\ &= ((\vec{E} - \Pi^i\vec{E})_{tt}, \vec{\zeta}_t)_h + (curl(\vec{E} - \Pi^i\vec{E}), curl\vec{\zeta}_t)_h + \sum_{K\in T_i^h} \int_{\partial K} curl\vec{E}\ \vec{n} \times \vec{\zeta}_t ds\\ &= ((\vec{E} - \Pi^i\vec{E})_{tt}, \vec{\zeta}_t)_h + \frac{d}{dt}(curl(\vec{E} - \Pi^i\vec{E}), curl\vec{\zeta})_h - (curl(\vec{E} - \Pi^i\vec{E})_t, curl\vec{\zeta})_h\\ &+ \sum_{K\in T_i^h} \int_{\partial K} curl\vec{E}\ \vec{n} \times \vec{\zeta}_t ds.\end{aligned}$$

Because $\vec{\zeta}(0) = \vec{\zeta}_t(0) = (0,0)$, integrating (3.16) for t, we get

$$\begin{aligned}&\| \vec{\zeta}_t \|_{0h}^2 + \| curl\vec{\zeta} \|_{0h}^2\\ &= 2\int_0^t ((\vec{E} - \Pi^i\vec{E})_{tt}, \vec{\zeta}_t)_h d\tau + 2(curl(\vec{E} - \Pi^i\vec{E}), curl\vec{\zeta})_h\\ &-2\int_0^t (curl(\vec{E} - \Pi^i\vec{E})_t, curl\vec{\zeta})_h d\tau\\ &+2\int_0^t \sum_{K\in T_i^h} \int_{\partial K} curl\vec{E}\ \vec{n} \times \vec{\zeta}_t ds d\tau,\end{aligned} \tag{3.17}$$

if $i=1,2$, by (2.23), (2.24), (2.28) and the Schwarz inequality

$$\begin{aligned}&\| \vec{\zeta}_t \|_{0h}^2 + \| curl\vec{\zeta} \|_{0h}^2\\ &\le Ch_i^2[|\vec{E}|_2^2 + \int_0^t (|\vec{E}_{tt}|_1^2 + |\vec{E}_t|_2^2 + |\vec{E}|_3^2)d\tau] + \int_0^t (\| \vec{\zeta}_t \|_{0h}^2\\ &+ \| curl\vec{\zeta} \|_{0h}^2)d\tau + \frac{1}{2} \| curl\vec{\zeta} \|_{0h}^2,\end{aligned}$$

if $i=3$, by (2.23), (2.26), (2.28) and the Schwarz inequality

$$\| \vec{\zeta}_t \|_{0h}^2 + \| curl\vec{\zeta} \|_{0h}^2 \le Ch_3^2[\int_0^t (|\vec{E}_{tt}|_1^2 + |\vec{E}|_3^2)d\tau] + \int_0^t \| \vec{\zeta}_t \|_{0h}^2\, d\tau.$$

Hence (3.11) and (3.13) follow from the Gronwall inequality. Thus, employing the triangle inequality, we can prove (3.12) and (3.14). Then the proof is completed.

Remark 1: It can be checked that some very popular elements, such as the rotated Q_1 *element [23] and low order element [15,24-25] can not be applied to Maxwell's equations on the mixed finite element scheme discussed in this paper because they do not satisfy the important property (2.27), although they also have some advantages in superconvergence analysis. However, we get the convergence results of the element [15] on the finite element scheme under anisotropic meshes. Moreover, the above results can be extended to the three-dimensional problem for the first element and the second element spaces. The convergence analysis of the last element for three-dimensional Maxwell's equations will be investigated in our further study.*

Remark 2: Since the superconvergence analysis of the above nonconforming mixed finite elements to Maxwell's equations under anisotropic meshes is very important in the electric-magnetic fields and can hardly be treated, it will be one of our further studying topics.

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References

[1] P. Monk, A comparison of three mixed methods for the time-dependent Maxwell's equations, SIAM J. Sci. Stat. Comput., 13 (1992), 1097-1122.

[2] P. Monk, A mixed method for approximating Maxwell's equations, SIAM J. Numer. Anal., 28(6) (1991), 1610-1634.

[3] P. Monk, Analysis of a finite element method for Maxwell's equations, SIAM J. Numer. Anal., 29(3) (1992), 714-729.

[4] P. Monk, Superconvergence of finite element approximation to Maxwell's equations, Numer. Meth. for PDEs., 10 (1994), 793-812.

[5] Q. Lin and N. N. Yan, Global superconvergence for Maxwell's equations, Math. Comput., 69 (1999), 159-176.

[6] Q. Lin and N. N. Yan, Superconvergence of mixed finite element methods for Maxwell's equations, J. Engrg. Math., 13(Supp) (1996), 1-10.

[7] J. H. Brandts, Superconvergence of mixed finite element semi-discretizations of two time-dependent problems, Appl. Math., 44(1) (1999), 43-53.

[8] J. F. Lin and Q. Lin, Global superconvergence of mixed finite element methods for 2-D Maxwell's equations, J. Comput. Math., 21(5) (2003), 637-646.

[9] P. G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, New York, 1978.

[10] A. Zenisek and M. Vanmaele, The interpolation theorem for narrow quadrilateral isoparametric finite elements, Numer. Math., 72 (1995), 123-141.

[11] A. Zenisek and M. Vanmaele, Applicability of the Bramble-Hilbert lemma in interpolation problems of narrow quadrilateral isoparametric finite elements, J. Comput. Appl. Math., 65 (1995), 109-122.

[12] T. Apel and M. Dobrowolski, Anisotropic interpolation with applications to the finite element method, Computing, 47(3) (1992), 277-293.

[13] T. Apel, Anisotropic finite elements: Local estimates and Applications, B.G. Teubner Stuttgart Leipzig, 1999.

[14] S. C. Chen, D. Y. Shi and Y. C. Zhao, Anisotropic interpolations and quasi-Wilson element for narrow quadrilateral meshes, IMA J. Numer. Anal., 24 (2004), 77-95

[15] D. Y. Shi, S. P. Mao and S. C. Chen, An anisotropic nonconforming finite element with some superconvergence results, J. Comput. Math., 23(3) (2005), 261-274.

[16] S. C. Chen, Y. C. Zhao and D. Y. Shi, Anisotropic interpolations with application to nonconforming elements, Appl. Numer. Math., 49(2) (2004), 135-152.

[17] D. Y. Shi, S. P. Mao and S. C. Chen, A locking-free anisotropic nonconforming finite element for planar linear elasticity, Acta. Math. Sci., 27B(1) (2007), 193-202.

[18] D. Y. Shi and H. Q. Zhu, The superconvergence analysis of an anisotropic element, J. Sys. Sci. Complex., 18(4) (2005), 478-487.
[19] D. Y. Shi, S. P. Mao and S. C. Chen, On the anisotropic accuracy analysis of ACM's nonconforming finite element, J. Comput. Math., 23(6) (2005), 635-646.
[20] H. D. Han, Nonconforming elements in the mixed finite element method, J. Comput. Math., 2(3) (1984), 223-233.
[21] C. J. Park and D. W. Sheen, P_1-nonconforming quadrilateral finite element methods for second order elliptic problems, SIAM J. Numer. Anal., 41 (2003), 624-640.
[22] Q. Lin and J. F. Lin, Finite Element Methods: Accuracy and Improvement, Science Press, 2006.
[23] R. Rannacher and S. Turek, A simple nonconforming quadrilateral stokes element, Numer. Meth. for PDEs., 8 (1992), 97-111.
[24] Q. Lin, Lutz Tobiska and A. H. Zhou, Superconvergence and extrapolation of nonconforming low order elements applied to the Poisson equation, IMA J. Numer. Anal., 25 (2005), 160-181.
[25] U. Risch, Superconvergence of nonconforming low order finite element, Appl. Numer. Math., 54 (2005), 324-338.

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