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EXPONENTIALLY FITTED LOCAL DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION PROBLEMS*

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Abstract

In this paper, we study the local discontinuous Galerkin (LDG) method for one-dimensional singularly perturbed convection-diffusion problems by an exponentially fitted technique. We prove that the method is uniformly first-order convergent in the energy norm with respect to the small diffusion parameter.

Mathematics subject classification: 65N30.
Key words: Exponentially fitted, Local discontinuous Galerkin method, Convection-diffusion problem.

1. Introduction

In this paper we consider the one-dimensional convection-diffusion problem:

$$\begin{cases} L_\epsilon u := -\epsilon u'' + (au)' = f(x), \quad x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where ϵ is a small positive diffusion coefficient and the convection velocity a is positive.

This is a fundamental model problem in computational fluid dynamics. In general, the solution of the problem has a boundary layer at $x = 1$ and the width of the layer is $\mathcal{O}(\epsilon \ln(1/\epsilon))$. When ϵ is big enough, the problem can be solved well by standard finite element methods. But when ϵ is too small, that is to say the problem is convection dominated, the standard finite element methods do not work well, except for the partition step $h < \epsilon$. But it maybe is impossible, since the computing cost is too expensive.

In order to avoid the difficulties, many investigators have resorted to methods based on exponentially-fitted techniques. In [8–10, 18], the authors explored the so-called L spline to solve the problem and gave some uniform error estimates with respect to the small parameter ϵ. However, there are quite a few other techniques developed to treat this problem. We refer to two books focusing on this topic [15, 17]. In [11], the author proposed the interesting tailored finite point method to solve the singular perturbation problem. There are some other papers (see, e.g., [12, 16]) about this method. An upwind finite difference scheme with the grid formed by equidistributing a monitor function is proposed in [14].

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Recently, some authors (see, e.g., [1, 4–7, 19]) applied DG or LDG methods to solve the problem. In [6], the authors analyzed the minimal dissipation LDG method (MD-LDG) for convection-diffusion or diffusion problems. They took the stabilization parameter α to zero on all the inter-element faces except on some part of the Dirichlet boundary to guarantee that the method is well defined. In [4], the authors used LDG method to solve one dimensional time-dependent convection-diffusion problem and obtained some optimal priori error estimates. The numerical result shows that, on a uniform mesh, it can not get the accuracy when the mesh-size h is bigger than the small diffusion parameter ϵ.

In fact, in all the above mentioned works, only piecewise polynomials are used in the approximate finite element spaces and all the error estimates had the form like $||u - u_h|| \leq Ch^\alpha ||u||_\beta$, where u is the exact solution and u_h is the numerical solution. In general, for singularly perturbed problems, the constant C and the Sobolev norm $||u||_\beta$ depend on the negative power of the small diffusion parameter ϵ. Therefore, when the mesh-size $h > \epsilon$, this kind of error estimates does not make sense.

As known, one of the advantages of the DG methods is the flexibility with the finite element approximation space. So in [20], the authors used the approximate spaces including non-polynomial functions such as exponentials. With properly selected spaces, they got much more accurate numerical results than only using piecewise polynomial spaces. However there is no theoretical result given on the uniformly convergence of such methods.

In this paper, we will consider a minimal dissipation exponential-fitted LDG method with no penalty involved, i.e. the stabilization parameter α is identically zero everywhere. A first order uniform convergence is obtained in the energy norm as: $\|q - q_h\|_{L^2(0,1)} \leq ch$, with $q = \sqrt{\epsilon}u'$ and q_h the approximation for q. Here 'uniformly' means that the constant $c > 0$ in the above estimate is independent of either the small parameter ϵ and h or the exact solution u. To do so, the ingredient is that only $\|u'\|_{L^1(0,1)}$ is involved in the error estimate. Throughout this paper, the constant c is independent of the parameter ϵ and the exact solution u.

The paper is structured as follows. In Section 2, we review the minimal dissipation LDG method and then present the numerical scheme and the main result on the uniform error estimate, which is proved in Section 3. In Section 4, we show some numerical results.

2. Exponentially Fitted LDG Method

2.1. Review of LDG method

In this subsection, we introduce the minimal dissipation LDG method discussed in [6]. At first, by introducing a new variable, $q = \sqrt{\epsilon}u'$, the problem (1.1) can be rewrite as follows:

$$\begin{cases} (au - \sqrt{\epsilon}q)' = f(x), & x \in (0,1), \\ q + (-\sqrt{\epsilon}u)' = 0, & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Let $\{x_{j+\frac{1}{2}}\}_{j=0}^N$, $j = 1, \cdots, N$ be a uniform partition of the interval $[0,1]$. Denote by $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, and $h = 1/N$. Multiplying (2.1) by smooth functions v, w, and integrating over

I_j, after a simple formal integration by parts, the weak formulation of the exact solution reads

$$\begin{cases} -\int_{I_j}(au-\sqrt{\epsilon}q)v' + (au-\sqrt{\epsilon}q)^-_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}} - (au-\sqrt{\epsilon}q)^+_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}} = \int_{I_j} fv, \\ \int_{I_j} qw + \int_{I_j}\sqrt{\epsilon}uw' - \sqrt{\epsilon}u^-_{j+\frac{1}{2}}w^-_{j+\frac{1}{2}} + \sqrt{\epsilon}u^+_{j-\frac{1}{2}}w^+_{j-\frac{1}{2}} = 0, \end{cases} \tag{2.2}$$

where the notation $v^{\pm}_{j\pm\frac{1}{2}}$ stands for $v(x^{\pm}_{j\pm\frac{1}{2}})$.

In order to define the LDG method, we need the finite element space

$$V_h = \left\{ v \in L^2(0,1) : v|_{I_j} \in P^k(I_j), j = 1, \cdots, N \right\},$$

where $P^k(I)$ denotes the space of polynomials in I of degree at most k. Then, the LDG method is defined by a discrete version of the formulation (2.2): find $u_h \in V_h$ and $q_h \in V_h$, such that

$$\begin{cases} -\int_{I_j}(au_h-\sqrt{\epsilon}q_h)v' + (a\hat{u}^c_h-\sqrt{\epsilon}\hat{q}_h)_{j+\frac{1}{2}}v^-_{j+\frac{1}{2}} - (a\hat{u}^c_h-\sqrt{\epsilon}\hat{q}_h)_{j-\frac{1}{2}}v^+_{j-\frac{1}{2}} = \int_{I_j} fv, \\ \int_{I_j} q_h w + \int_{I_j}\sqrt{\epsilon}u_h w' - \sqrt{\epsilon}(\hat{u}^d_h)_{j+\frac{1}{2}}w^-_{j+\frac{1}{2}} + \sqrt{\epsilon}(\hat{u}^d_h)_{j-\frac{1}{2}}w^+_{j-\frac{1}{2}} = 0, \end{cases} \tag{2.3}$$

for all $v, w \in V_h$. Here $\hat{u}^d_h$ and $\hat{q}_h$ are the numerical fluxes associated with diffusion and $\hat{u}^c_h$ is the numerical flux associated with convection. In the MD-LDG method, the numerical flux $\hat{u}^c_h$ is defined as

$$(\hat{u}^c_h)_{j+\frac{1}{2}} = \begin{cases} u_h(x^-_{j+\frac{1}{2}}), & j = 1, \cdots, N, \\ 0, & j = 0. \end{cases} \tag{2.4}$$

The numerical flux associated with diffusion, $\hat{u}^d_h$, has a similar definition

$$(\hat{u}^d_h)_{j+\frac{1}{2}} = \begin{cases} u_h(x^-_{j+\frac{1}{2}}), & j = 1, \cdots, N-1, \\ 0, & j = 0, N, \end{cases} \tag{2.5}$$

and the numerical flux for q_h is given by

$$(\hat{q}_h)_{j+\frac{1}{2}} = \begin{cases} q_h(x^+_{j+\frac{1}{2}}), & j = 0, \cdots, N-1, \\ q_h(x^-_{j+\frac{1}{2}}) - \alpha u_h(x^-_{j+\frac{1}{2}}), & j = N, \end{cases} \tag{2.6}$$

where α is the stabilization parameter. In our scheme, we can set $\alpha = 0$ and the stability still holds.

For convenience we list some commonly used notations: jumps and averages on the node points

$$[p]_{j+\frac{1}{2}} = \begin{cases} p^+_{j+\frac{1}{2}}, & j = 0, \\ p^+_{j+\frac{1}{2}} - p^-_{j+\frac{1}{2}}, & j = 1, \cdots, N-1, \\ -p^-_{j+\frac{1}{2}}, & j = N, \end{cases}$$

$$\bar{p}_{j+1/2} = \begin{cases} p^+_{j+\frac{1}{2}}, & j = 0, \\ \frac{1}{2}(p^+_{j+\frac{1}{2}} + p^-_{j+\frac{1}{2}}), & j = 1, \cdots, N-1, \\ p^-_{j+\frac{1}{2}}, & j = N. \end{cases}$$

2.2. Exponentially fitted LDG method

We now introduce the numerical scheme. Assume the convection velocity a is a positive constant and $f \in W^{1,\infty}(0,1)$. In order to fit the property of boundary layer, we add exponential functions to the basis space. Then the scheme reads: find $u_h \in V_1 := \{v \in L^2(0,1) : v|_{I_j} \in \text{span}\{1, x, \exp(\frac{a}{\epsilon}(x - x_{j+\frac{1}{2}}))\}\}$, $q_h \in V_2 := \{v \in L^2(0,1) : v|_{I_j} \in \text{span}\{1, \exp(\frac{a}{\epsilon}(x - x_{j+\frac{1}{2}}))\}\}$ such that

$$\begin{cases} -\displaystyle\int_{I_j} (au_h - \sqrt{\epsilon}q_h)v' + (a\hat{u}_h^c - \sqrt{\epsilon}\hat{q}_h)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (a\hat{u}_h^c - \sqrt{\epsilon}\hat{q}_h)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}} = \displaystyle\int_{I_j} fv, \\ \displaystyle\int_{I_j} q_h w + \int_{I_j} \sqrt{\epsilon}u_h w' - \sqrt{\epsilon}(\hat{u}_h^d)_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \sqrt{\epsilon}(\hat{u}_h^d)_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}} = 0, \end{cases} \tag{2.7}$$

for all $v \in V_1$, $\quad w \in V_2$. The numerical flux $\hat{u}_h^c$, $\hat{u}_h^d$, $\hat{q}_h$ are the same as (2.4), (2.5) and (2.6) with the penalty parameter $\alpha = 0$.

Summing over j from 1 to N in (2.7), we obtain

$$B_h(u_h, q_h; v, w) = \sum_{j=1}^{N} \int_{I_j} fv, \tag{2.8}$$

where

$$\begin{aligned} B_h(u_h, q_h; v, w) = &\sum_{j=1}^{N} \int_{I_j} q_h w - \sum_{j=1}^{N} \int_{I_j} \left((au_h - \sqrt{\epsilon}q_h)v' - \sqrt{\epsilon}u_h w'\right) \\ &+ \sum_{j=1}^{N} \left((a\hat{u}_h^c - \sqrt{\epsilon}\hat{q}_h)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (a\hat{u}_h^c - \sqrt{\epsilon}\hat{q}_h)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\right) \\ &+ \sum_{j=1}^{N} \left(-\sqrt{\epsilon}(\hat{u}_h^d)_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \sqrt{\epsilon}(\hat{u}_h^d)_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}}\right) \end{aligned} \tag{2.9}$$

For simplicity, we denote

$$\begin{aligned} \sum_{j=1}^{N} \hat{h}(u_h, q_h; v, w) = &\sum_{j=1}^{N} \left((a\hat{u}_h^c - \sqrt{\epsilon}\hat{q}_h)_{j+\frac{1}{2}} v^-_{j+\frac{1}{2}} - (a\hat{u}_h^c - \sqrt{\epsilon}\hat{q}_h)_{j-\frac{1}{2}} v^+_{j-\frac{1}{2}}\right) \\ &+ \sum_{j=1}^{N} \left(-\sqrt{\epsilon}(\hat{u}_h^d)_{j+\frac{1}{2}} w^-_{j+\frac{1}{2}} + \sqrt{\epsilon}(\hat{u}_h^d)_{j-\frac{1}{2}} w^+_{j-\frac{1}{2}}\right) \end{aligned}$$

Let $v = u_h, w = q_h$ in (2.9), then after a simple calculate, we get

$$B_h(u_h, q_h; u_h, q_h) = \sum_{j=1}^{N} \int_{I_j} (q_h)^2 + \sum_{j=1}^{N+1} \frac{a}{2}[u_h]^2_{j-\frac{1}{2}} = \sum_{j=1}^{N} \int_{I_j} f u_h. \tag{2.10}$$

Next, we will prove the existence and uniqueness of the numerical scheme (2.7).

Theorem 2.1. (*Existence and uniqueness*) *The exponentially fitted LDG methods* (2.7) *has a unique solution* $(u_h, q_h) \in V_1 \times V_2$.

Proof. We only need to prove the uniqueness of the solution. The existence is equivalent to the uniqueness since the discrete problem (2.7) is finite dimensional.

When $f=0$, from (2.10), we have

$$\sum_{j=1}^{N}\int_{I_j}(q_h)^2+\sum_{j=1}^{N+1}\frac{a}{2}[u_h]_{j-\frac{1}{2}}^2=0. \tag{2.11}$$

That is to say $q_h=0$ and $[u_h]_{j-\frac{1}{2}}=0, \forall j=1,\cdots,N+1$. Then from the first equation of (2.7), let $f=0, q_h=0$ and after a simple formal integration by parts, we get

$$\int_{I_j} au_h'v+a[u_h]_{j-\frac{1}{2}}v_{j-\frac{1}{2}}^{+}=0. \tag{2.12}$$

Using the fact $[u_h]_{j-\frac{1}{2}}=0$, and letting $v=u_h'$, we have $\int_{I_j}a(u_h')^2=0$. Then we have the result $u_h'|_{I_j}=0$. As u_h is in the space V_1, we can easily say $u_h=0$ by using the fact $[u_h]_{j-\frac{1}{2}}=0$, $j=1,\cdots,N+1$.

The following is the main result on the error estimate.

Theorem 2.2. *Suppose that u,q be the exact solution of* (2.1). *Let $u_h\in V_1, q_h\in V_2$ be the approximate solution given by* (2.7), *then we have, for $\epsilon>0$ sufficiently small, there exists a positive constant c independent of ϵ, h, u and q, such that*

$$\sum_{j=1}^{N}\int_{I_j}(q-q_h)^2\le c\,h^2. \tag{2.13}$$

3. Error Estimate

This section is devoted to the proof of Theorem 2.2. In order to derive the error estimate for $(e_u,e_q)=(u-u_h,q-q_h)$, we first split it into two parts $(e_u,e_q)=(u-\tilde{u}+\tilde{u}-u_h,q-\tilde{q}+\tilde{q}-q_h):=(\rho_u+\eta_u,\rho_q+\eta_q)$. To define $\tilde{u}\in V_1$ and $\tilde{q}\in V_2$, we first introduce the bubble functions which were used in [21].

3.1. Bubble functions

Let

$$B_j(x)=\frac{x-x_{j-\frac{1}{2}}}{a}-\frac{h}{a}\frac{1-\exp(\frac{a}{\epsilon}(x-x_{j-\frac{1}{2}}))}{1-\exp(\frac{a}{\epsilon}h)}.$$

Then $B_j\in H_0^1(I_j)$ is the solution of the local boundary value problem:

$$\begin{cases} L_0B_j:=-\epsilon B_j''+aB_j'=1, \quad \text{in } I_j=(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}),\\ B_j(x_{j-\frac{1}{2}})=B_j(x_{j+\frac{1}{2}})=0. \end{cases} \tag{3.1}$$

Moreover, the following properties hold:

$$\begin{cases} (1)\ 0\le \dfrac{c_0}{2h}(x-x_{j-\frac{1}{2}})(x_{j+\frac{1}{2}}-x)\le B_j(x)\le \dfrac{h}{a}, \quad c_0=\min\{\dfrac{h}{2\epsilon},\dfrac{1}{a}\},\\ (2)\ \displaystyle\int_{I_j}B_j(x)=\epsilon\int_{I_j}|B_j'(x)|^2,\\ (3)\ \dfrac{c_0h}{12}\le \tilde{h}_j:=\dfrac{1}{h}\displaystyle\int_{I_j}B_j(x)dx\le\dfrac{h}{a},\\ (4)\ 0\le \displaystyle\int_{I_j}\left|B_j'(x)\right|\le\dfrac{2h}{a}. \end{cases} \tag{3.2}$$

It is easily to see that $V_1 \equiv \{v \in L^1(0,1) : v|_{I_j} \in \text{span}\{1, x, B_j\}\}$ and $V_2 \equiv \{v \in L^1(0,1) : v|_{I_j} \in \text{span}\{1, B_j'\}\}$. Then we can split u_h and q_h into two parts respectively as $u_h = u_L + u_B$ and $q_h = q_L + q_B$, where $u_L|_{I_j}$ is linear, $q_L|_{I_j}$ a constant and $u_B|_{I_j} = cB_j, q_B|_{I_j} = cB_j'$. Taking $v = B_j$ in (2.7), we will see that

$$q_B|_{I_j} = \left(-a\sqrt{\epsilon} u_L'|_{I_j} + \frac{\sqrt{\epsilon}\int_{I_j} fB_j}{\int_{I_j} B_j} \right) B_j'. \tag{3.3}$$

3.2. Error estimate in energy norm

Now, we define $\tilde{u} = \tilde{u}_L + \tilde{u}_B$ and $\tilde{q} = \tilde{q}_L + \tilde{q}_B := \sqrt{\epsilon}\tilde{u}_L' + \sqrt{\epsilon}\tilde{u}_B'$ by

$$\tilde{u}_L = \Pi_h u, \tag{3.4}$$

and

$$\epsilon \int_{I_j} (u - \tilde{u})' B_j' + \int_{I_j} a(u - \tilde{u})' B_j = 0, \quad j = 1, \cdots, N, \tag{3.5}$$

where $\Pi_h u$ is the continuous piecewise linear nodal interpolation of u. Using the properties (3.2) of $B_j(x)$ and (3.5), we have

$$\tilde{u}_B|_{I_j} = \left(-a\tilde{u}_L'|_{I_j} + \frac{\int_{I_j} fB_j}{\int_{I_j} B_j} \right) B_j. \tag{3.6}$$

The following Lemmas consider the uniform estimates for ρ_u and ρ_q.

Lemma 3.1. ([8]) *There exists positive constants ϵ_0 and c such that, for all $v \in H^1(s,t)$*

$$\epsilon|v'(x)| + |v(x)| \le c\bigg(||L_\epsilon v||_{L^1(s,t)} + |v(s)| + |v(t)| \bigg),\ s \le x \le t,\ 0 < \epsilon \le \epsilon_0.$$

Lemma 3.2. *For $\rho_u = u - \tilde{u}$ and $\rho_q = q - \tilde{q}$, we have*

$$\epsilon|\rho_u'| + |\rho_u| \le ch^2, \quad \sum_{j=1}^{N} \int_{I_j} (\rho_q)^2 \le ch^2. \tag{3.7}$$

Proof. By the definition of ρ, we can easily see

$$\begin{cases} L_0\rho_u := -\epsilon\rho_u'' + a\rho_u' = f - \dfrac{\int_{I_j} fB_j}{\int_{I_j} B_j} := F_j, \quad x \in I_j \\ \rho_u(x_{j-\frac{1}{2}}) = \rho_u(x_{j+\frac{1}{2}}) = 0. \end{cases} \tag{3.8}$$

Using Lemma 3.1, we obtain

$$\epsilon||\rho_u'||_{L^\infty(I_j)} + ||\rho_u||_{L^\infty(I_j)} \le c||L_0\rho_u||_{L^1(I_j)} = c||F_j||_{L^1(I_j)} \le ch^2.$$

Noting that

$$(\rho_q)^2 = (\sqrt{\epsilon}\rho_u')^2 \le \epsilon|\rho_u'|(|u'| + |\tilde{u}'|),$$

we have by using the fact (see [15]) $||u'||_{L^1(0,1)} \le c$ that

$$\sum_{j=1}^{N}\int_{I_j}(\rho_q)^2 \le \sum_{j=1}^{N}\int_{I_j}\epsilon|\rho_u'|\Big(|u'|+|\tilde{u}'|\Big) \le ch^2\sum_{j=1}^{N}\int_{I_j}\Big(|u'|+|\tilde{u}'|\Big)$$
$$\le ch^2 + ch^2\sum_{j=1}^{N}\int_{I_j}|\tilde{u}'| \le ch^2 + ch^2\sum_{j=1}^{N}\int_{I_j}\Big(|\tilde{u}_L'|+|\tilde{u}_B'|\Big). \tag{3.9}$$

This completes the first part of (3.7).

From the definition (3.4) of $\tilde{u}_L$, we have $\left|\tilde{u}_L'|_{I_j}\right| \le \frac{1}{h}\int_{I_j}|u'|$. Thus,

$$\sum_{j=1}^{N}\int_{I_j}|\tilde{u}_L'| \le \sum_{j=1}^{N}\int_{I_j}|u'| \le c.$$

Using the expression (3.6) of $\tilde{u}_B$ and the properties (3.2), we similarly have

$$\sum_{j=1}^{N}\int_{I_j}|\tilde{u}_B'| \le c.$$

Therefore, the second part of (3.7) is also established. □

Now we give the estimate for (η_u,η_q).

Lemma 3.3. *For $0<\epsilon\le\epsilon_0$, we have*

$$\sum_{j=1}^{N}\int_{I_j}(\eta_q)^2 + \frac{1}{2}\sum_{j=1}^{N+1}a[\eta_u]^2_{j-\frac{1}{2}} \le ch^3. \tag{3.10}$$

Proof. Denote by $\rho=(\rho_u,\rho_q)$ and $\eta=(\eta_u,\eta_q)$. From (2.10) and (2.8), we have

$$\begin{aligned}
&\sum_{j=1}^{N}\int_{I_j}(\eta_q)^2 + \frac{1}{2}\sum_{j=1}^{N+1}a[\eta_u]^2_{j-\frac{1}{2}}\\
&= B_h(\eta;\eta) = B_h(u-u_h,q-q_h;\eta) - B_h(\rho;\eta) = -B_h(\rho,\eta)\\
&= -\sum_{j=1}^{N}\int_{I_j}\rho_q\eta_q + \sum_{j=1}^{N}\hat{h}(\rho_u,\rho_q;\eta_u,\eta_q) + \sum_{j=1}^{N}\int_{I_j}(a\rho_u-\sqrt{\epsilon}\rho_q)\eta_u' - \sum_{j=1}^{N}\int_{I_j}\sqrt{\epsilon}\rho_u\eta_q'\\
&=: T_1+T_2+T_3+T_4.
\end{aligned} \tag{3.11}$$

The first term can be eliminated by the fourth term by using integration by parts and (3.4),

$$\begin{aligned}
T_4 &= -\sum_{j=1}^{N}\int_{I_j}\sqrt{\epsilon}\rho_u\eta_q' = \sum_{j=1}^{N}\int_{I_j}(\sqrt{\epsilon}\rho_u)'\eta_q - \sum_{j=1}^{N}\sqrt{\epsilon}\rho_u\eta_q|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\\
&= \sum_{j=1}^{N}\int_{I_j}\rho_q\eta_q = -T_1.
\end{aligned} \tag{3.12}$$

As to the second term T_2, by Lemma 3.2, we obtain

$$|T_2| = \left|\sum_{j=1}^{N}\hat{h}(\rho_u,\rho_q;\eta_u,\eta_q)\right|$$

$$
\begin{aligned}
&= \left| \sum_{j=1}^{N} \sqrt{\epsilon}\hat{\rho}_q(x_{j-\frac{1}{2}})\eta_u(x^+_{j-\frac{1}{2}}) - \sqrt{\epsilon}\hat{\rho}_q(x_{j+\frac{1}{2}})\eta_u(x^-_{j+\frac{1}{2}}) \right| \\
&\le \sum_{j=1}^{N+1} \left| \sqrt{\epsilon}(\hat{\rho_q})_{j-\frac{1}{2}}[\eta_u]_{j-\frac{1}{2}} \right| \le \sum_{j=1}^{N+1} ch^2 \left| [\eta_u]_{j-\frac{1}{2}} \right| \\
&\le c\sum_{j=1}^{N+1}(h^2)^2 + \frac{1}{4}\sum_{j=1}^{N+1} a[\eta_u]^2_{j-\frac{1}{2}} = ch^3 + \frac{1}{4}\sum_{j=1}^{N+1} a[\eta_u]^2_{j-\frac{1}{2}}
\end{aligned} \tag{3.13}
$$

by using the fact $\rho_u(x^{\pm}_{j-\frac{1}{2}}) = 0,\ j = 1, \cdots, N+1$.

It remains to estimate the term T_3. Denote by $\eta_u = \eta_L + \eta_B$, where η_L is a piecewise linear function and $\eta_B|_{I_j} = cB_j$. We can rewrite the third term as follows:

$$
\begin{aligned}
T_3 &= \sum_{j=1}^{N}\int_{I_j}(a\rho_u - \sqrt{\epsilon}\rho_q)\eta'_u \\
&= \sum_{j=1}^{N}\int_{I_j}(a\rho_u - \sqrt{\epsilon}\rho_q)\eta'_L + \sum_{j=1}^{N}\int_{I_j}(a\rho_u - \sqrt{\epsilon}\rho_q)\eta'_B \\
&= \sum_{j=1}^{N}\int_{I_j}(a\rho_u - \epsilon\rho'_u)\eta'_L + \sum_{j=1}^{N}\int_{I_j}(a\rho_u - \epsilon\rho'_u)\eta'_B.
\end{aligned} \tag{3.14}
$$

By the definition of $\tilde{u}$ (3.5) and (3.4), we derive

$$
\sum_{j=1}^{N}\int_{I_j}(a\rho_u - \epsilon\rho'_u)\eta'_B = -\sum_{j=1}^{N}\int_{I_j}(a\rho'_u\eta_B + \epsilon\rho'_u\eta'_B) + \sum_{j=1}^{N} a\rho_u\eta_B|^{x+\frac{1}{2}}_{x-\frac{1}{2}} = 0. \tag{3.15}
$$

As η'_L is a piecewise constant, from (3.4), we see

$$
\sum_{j=1}^{N}\int_{I_j}(-\epsilon\rho'_u\eta'_L) = -\epsilon\sum_{j=1}^{N}\eta'_L|_{I_j}\int_{I_j}\rho'_u = 0. \tag{3.16}
$$

Thus, thanks to (3.2),

$$
\begin{aligned}
|T_3| &= |\sum_{j=1}^{N}\int_{I_j} a\rho_u\eta'_L| \le \sum_{j=1}^{N}(||\rho_u||_{L^2(I_j)}\tilde{h}_j^{-1/2})(||a\eta'_L||_{L^2(I_j)}\tilde{h}_j^{1/2}) \\
&\le ch^{-1}\sum_{j=1}^{N}||\rho_u||^2_{L^2(I_j)} + \frac{1}{2}\sum_{j=1}^{N}||a\eta'_L||_{L^2(I_j)}\tilde{h}_j.
\end{aligned} \tag{3.17}
$$

Using Lemma 3.2, we obtain

$$
ch^{-1}\sum_{j=1}^{N}||\rho_u||^2_{L^2(I_j)} = ch^{-1}\sum_{j=1}^{N}\int_{I_j}|\rho_u|^2 \le ch^3. \tag{3.18}
$$

From (3.2), (3.3) and (3.6), we see that

$$
\begin{aligned}
\sum_{j=1}^{N} ||a\eta_L'||_{L^2(I_j)}\tilde{h}_j &= \sum_{j=1}^{N}(a\eta_L'|_{I_j})^2 \int_{I_j} B_j = \sum_{j=1}^{N}(a\sqrt{\epsilon}\eta_L'|_{I_j})^2 \int_{I_j}(B_j')^2 \\
&= \sum_{j=1}^{N}\int_{I_j}(a\sqrt{\epsilon}\eta_L' B_j')^2 = \sum_{j=1}^{N}\int_{I_j}(\tilde{q}_B - q_B)^2 \le \sum_{j=1}^{N}\int_{I_j}(\eta_q)^2,
\end{aligned}
$$

where we have used the facts

$$
\int_{I_j}(\eta_q)^2 = \int_{I_j}(\tilde{q}_L - q_L + \tilde{q}_B - q_B)^2 = \int_{I_j}(\tilde{q}_L - q_L)^2 + \int_{I_j}(\tilde{q}_B - q_B)^2.
$$

Therefore, we get

$$
|T_3| \le ch^3 + \frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(\eta_q)^2. \tag{3.19}
$$

Consisting (3.11), (3.13) and (3.19) yields

$$
\sum_{j=1}^{N}\int_{I_j}(\eta_q)^2 + \frac{1}{2}\sum_{j=1}^{N+1} a[\eta_u]^2_{j-\frac{1}{2}} \le ch^3 + \frac{1}{2}\sum_{j=1}^{N}\int_{I_j}(\eta_q)^2 + \frac{1}{4}\sum_{j=1}^{N+1} a[\eta_u]^2_{j-\frac{1}{2}},
$$

which gives the desired estimate (3.10). □

Form Lemmas 3.2 and 3.3, we obtain the main result of Theorem 2.2, i.e., (2.13).

4. Numerical Results

In this section, we present some numerical examples.

Example 1. Consider the problem:

$$
\begin{cases} -\epsilon u'' + 2\,u' = f(x), & x \in (0,1), \\ u(0) = u_0,\ u(1) = u_1, \end{cases}
$$

with the exact solution

$$
u(x) = \exp\left(\frac{2(x-1)}{\epsilon}\right) + 2\,x^3 + x.
$$

Table 4.1: $\epsilon = 1.0,\ h = 1/n,\ \alpha = 0,$ piecewise linear space

n	L^2 error	rate	energy error	rate
4	2.6929e-002	-	5.5963e-002	-
8	7.4939e-003	1.8454	1.4389e-002	1.9595
16	1.9879e-003	1.9145	3.6495e-003	1.9792
32	5.1248e-004	1.9557	9.1908e-004	1.9894
64	1.3014e-004	1.9774	2.3062e-004	1.9947
128	3.2791e-005	1.9887	5.7762e-005	1.9973
256	8.2302e-006	1.9943	1.4454e-005	1.9987
512	2.0616e-006	1.9972	3.6151e-006	1.9994
1024	5.1592e-007	1.9985	9.0395e-007	1.9997

Table 4.2: $\epsilon = 1.0,\ h = 1/n,\ \alpha = 1/h,$ piecewise linear space

n	L^2 error	rate	energy error	rate
4	2.8905e-002	-	5.5963e-002	-
8	7.9843e-003	1.8561	1.4389e-002	1.9595
16	2.0744e-003	1.9445	3.6495e-003	1.9792
32	5.2544e-004	1.9811	9.1908e-004	1.9894
64	1.3192e-004	1.9939	2.3062e-004	1.9947
128	3.3024e-005	1.9981	5.7762e-005	1.9973
256	8.2601e-006	1.9993	1.4454e-005	1.9987
512	2.0654e-006	1.9997	3.6151e-006	1.9994
1024	5.1639e-007	1.9999	9.0396e-007	1.9997

Table 4.3: $\epsilon = 10^{-2},\ h = 1/n,\ \alpha = 0,$ piecewise linear space

n	L^2 error	rate	energy error	rate
4	5.6421e-002	-	9.2245e-001	-
8	4.5013e-002	-	8.4816e-001	-
16	3.7733e-002	-	7.1516e-001	-
32	2.6491e-002	-	5.0798e-001	-
64	5.0798e-001	-	2.7117e-001	-
128	5.1888e-003	-	1.0416e-001	-
256	1.6164e-003	1.6826	3.1723e-002	1.7152
512	4.4252e-004	1.8690	8.6373e-003	1.8769
1024	1.1419e-004	1.9543	2.2430e-003	1.9452

The numerical results are displayed in Tables 4.1– 4.6. The L^2 error is $||u - u_h||_{L^2(0,1)}$ and the error in energy norm is $||q - q_h||_{L^2(0,1)}$. In Tables 4.1 and 4.3, only piecewise linear function space are used for u_h and q_h in the approximation finite element spaces. In Tables 4.4 – 4.5, we use the exponential functions space in (2.7). In Table 4.6, we use the same exponential functions space V_1 for both u_h and q_h.

In Tables 4.1 and 4.3, when only piecewise linear functions are used, we can see that when $h \le \epsilon$, i.e., the grid is fine enough to resolve the boundary layer, the scheme is of order h^2 in L^2 norm and energy norm. But when $h > \epsilon$, the accuracy is destroyed.

From Tables 4.1 and 4.2, whether we take $\alpha = 0$ or $\alpha = 1/h$, the schemes are both of order h^2 in the L^2 norm and energy norm. That means there is no need to add the penalty term.

In Table 4.4, in the case $h \le \epsilon$, the energy error and the L^2 error are of order h^2. But when the problem is singularly perturbed, the error in energy norm is of first order in Table 4.5, for

Table 4.4: $\epsilon = 1.0,\ h = 1/n,\ \alpha = 0,$ exponential functions space V_1 for u_h and V_2 for q_h

n	L^2 error	rate	energy error	rate
4	1.2122e-002	-	2.4279e-002	-
8	3.1554e-003	1.9417	6.3719e-003	1.9299
16	8.0520e-004	1.9704	1.6228e-003	1.9732
32	2.0351e-004	1.9842	4.0890e-004	1.9887
64	5.1167e-005	1.9918	1.0259e-004	1.9949
128	1.2829e-005	1.9958	2.5692e-005	1.9975
256	3.2079e-006	1.9997	6.4284e-006	1.9988
512	7.9139e-007	2.0192	1.6078e-006	1.9994

Table 4.5: $\epsilon = 10^{-6}$, $h = 1/n$, $\alpha = 0$, exponential functions space V_1 for u_h and V_2 for q_h

n	L^2 error	rate	energy error	rate
4	2.6782e-002	-	4.9686e-004	-
8	6.6533e-003	2.0091	2.4961e-004	0.9932
16	1.6559e-003	2.0065	1.2495e-004	0.9983
32	4.1288e-004	2.0038	6.2492e-005	0.9996
64	1.0307e-004	2.0021	3.1247e-005	1.0000
128	2.5745e-005	2.0013	1.5623e-005	1.0000
256	6.4319e-006	2.0010	7.8102e-006	1.0002
512	1.6066e-006	2.0012	3.9040e-006	1.0004
1024	4.0111e-007	2.0019	1.9509e-006	1.0008

Table 4.6: $\epsilon = 10^{-6}$, $h = 1/n$, $\alpha = 0$, exponential functions space V_1 both for u_h and q_h

n	L^2 error	rate	energy error	rate
4	2.6782e-002	-	2.7951e-005	-
8	6.6533e-003	2.0091	6.9876e-006	2.0000
16	1.6559e-003	2.0065	1.7469e-006	2.0000
32	4.1288e-004	2.0038	4.3671e-007	2.0000
64	1.0307e-004	2.0021	1.0917e-007	2.0001
128	2.5745e-005	2.0013	2.7289e-008	2.0002
256	6.4319e-006	2.0010	6.8207e-009	2.0003
512	1.6067e-006	2.0012	1.7043e-009	2.0007
1024	4.0111e-007	2.0019	4.2569e-010	2.0013

Table 4.7: $\epsilon = 10^{-6}$, $h = 1/n$, $\alpha = 0$, exponential functions space V_1 for u_h and V_2 for q_h

n	L^2 error	rate	energy error	rate
4	3.5635e-005	-	6.2361e-002	-
8	1.6658e-005	1.0971	3.1234e-002	0.9975
16	8.0636e-006	1.0467	1.5623e-002	0.9994
32	3.9682e-006	1.0229	7.8123e-003	0.9999
64	1.9685e-006	1.0114	3.9063e-003	0.9999
128	9.8044e-007	1.0056	1.9532e-003	1.0000
256	4.8929e-007	1.0027	9.7662e-004	1.0000
512	2.4443e-007	1.0013	4.8834e-004	0.9999
1024	1.2218e-007	1.0004	2.4420e-004	0.9998

$h > \epsilon$, which confirms our error estimate.

In Table 4.6, the L^2 error and the energy error are of order h^2, when the space for q_h is the same as u_h. The accuracy in energy norm is better than that in Table 4.5. However, this numerical observation can not be verified theoretically.

Example 2. Consider the problem:

$$\begin{cases} -\epsilon u'' + ((1+x)u)' = f(x), & x \in (0,1), \\ u(0) = u_0, \ u(1) = u_1, \end{cases}$$

with the exact solution

$$u(x) = \exp\left(\frac{(x+3)(x-1)}{2\epsilon}\right) + x.$$

The numerical result is displayed in Table 4.7. The result indicate that the scheme is of the first-order accuracy both in L^2 norm and in energy norm.

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