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SPURIOUS NUMERICAL SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS *1)

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Abstract

This paper deals with the relationship between asymptotic behavior of the numerical solution and that of the true solution itself for fixed step-sizes. The numerical solution is viewed as a dynamical system in which the step-size acts as a parameter. We present a unified approach to look for bifurcations from the steady solutions into spurious solutions as step-size varies.

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1. Introduction

It is well-known that a numerical method which is convergent in a finite interval does not necessarily yield the same asymptotic behavior as the underlying differential equation. In many circumstances, we are interested in the asymptotic behavior in the differential equations. The asymptotic states of a dynamical system are captured in the $\omega-$ and $\alpha-$ limit sets which may concern equilibria, periodic orbits, attractors, etc. It is desirable to design numerical schemes for which these sets are close to the corresponding limit sets of the underlying differential equation, and to understand and hence to avoid conditions under which spurious members of the limit sets are introduced by the time discretization.

Runge-Kutta and linear multistep methods are commonly used to obtain a numerical solution of ordinary differential equations (ODEs). Dynamics of the numerical solution produced by Runge-Kutta and linear multistep methods solving ODEs has been extensively studied (see, for example, [3, 6, 7, 8, 9, 10, 12, 17]).

In this paper, we are concerned with the nonlinear delay differential equation with a constant lag in the form

$$\begin{array}{rcl} y'(t) & = & f(y(t), y(t-\tau)), \quad t>0, \\ y(t) & = & \phi(t), \quad -\tau \le t \le 0, \end{array} \tag{1}$$

where y, f are real scalar functions and $\tau > 0$ is a constant lag. The solution (if it exists) is determined by a choice of initial function ϕ. The results on existence, uniqueness and continuous dependence of solution of (1) can be found in the books by Hale and Lunel [4] and Driver [2]. We assume throughout that the initial function ϕ is continuous.

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Consider approximating the solution of (1) using a consistent numerical method with a fixed step-size h such that $h = \frac{\tau}{m} > 0$, where $m > \widetilde{k}$ is some positive integer and $\widetilde{k}$ is a positive integer depending upon the specific method. Let y_n denote our approximation to $y(t_n)$, where $t_n = nh$. Typically the sequence y_n is derived from a map of the form

$$\mathcal{F}(y_{n-m}, \cdots, y_{n+\widetilde{k}}; h) = 0, \quad n = 0, 1, \ldots, \tag{2}$$

together with $\widetilde{k}$ initial conditions. Thus (2) must be solved for $y_{n+\widetilde{k}}$ given $y_{n-m}, \cdots, y_{n+\widetilde{k}-1}$. By introducing a new vector $U_n = [y_{n-m}, \cdots, y_{n+\widetilde{k}-1}]^T \in \mathbb{R}^{\widetilde{k}+m}$ we may write (2) as a one-step map of the form

$$\mathcal{H}(U_n, U_{n+1}; h) = 0. \tag{3}$$

Definition 1.1.

1. *The numerical scheme (2) is regular of degree* 1*, denoted* $R^{[1]}$*, if every fixed point* $\widehat{u} \in \mathbb{R}$ *of (2) satisfies* $f(\widehat{u}, \widehat{u}) = 0$ *of (1) for all* $h > 0$ *and all equations (1) with* $f \in C^2$*. Otherwise it is irregular of degree* 1*.*

2. *The numerical scheme (2) is regular of degree* 2*, denoted* $R^{[2]}$*, if (2) does not admit real period two solution in* n *for all* $h > 0$ *and all equations (1) with* $f \in C^2$*. Otherwise it is irregular of degree* 2*.*

The following two lemmas are used in the proofs of our main results. The first one concerns the bifurcation of fixed points from simple eigenvalue, while the second concerns the bifurcation of period 2 solutions in the map (3).

Lemma 1.2 [7] *Let the function* $\mathcal{H}(a, b; h)$ *satisfy* $C^r(\mathbb{R}^{\widetilde{k}+m} \times \mathbb{R}^{\widetilde{k}+m}, \mathbb{R})$ *for some integer* $r \geq 2$*. Assume that the map (3) has a fixed point* $\widehat{U}$ *for all* $h > 0$*. Assume also that* $\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h)$ *is singular at* $h = h_c$ *and there exists a nonzero vector* $\eta \in \mathbb{R}^{\widetilde{k}+m}$ *such that* $Null(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)) = span\{\eta\}$*. If*

$$\frac{d}{dh}\left(\frac{\partial \mathcal{H}}{\partial a}(U, U; h) + \frac{\partial \mathcal{H}}{\partial b}(U, U; h)\right)\bigg|_{U=\widehat{U},\ h=h_c} \eta \notin Range\left(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)\right).$$

Then, for $0 < \epsilon \ll 1$*, there exists a fixed point of (3) with the form*

$$\begin{aligned} h(\epsilon) &= h_c + \mathcal{O}(|\epsilon|), \\ U_n(\epsilon) &= \epsilon\eta + \mathcal{O}(|\epsilon|^2) \end{aligned}$$

which is C^{r-1} *in* ϵ*.*

Lemma 1.3 [7] *Let the function* $\mathcal{H}(a, b; h)$ *satisfy* $C^r(\mathbb{R}^{\widetilde{k}+m} \times \mathbb{R}^{\widetilde{k}+m}, \mathbb{R})$ *for some integer* $r \geq 2$*. Assume that the map (3) has a fixed point* $\widehat{U}$ *for all* $h > 0$*. Assume also that there exists a nonzero vector* $\vartheta \in \mathbb{R}^{\widetilde{k}+m}$ *such that* $Null\left(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) - \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)\right) = span\{\vartheta\}$ *and that* $\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) + \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)$ *is invertible. If*

$$\frac{d}{dh}\left(\frac{\partial \mathcal{H}}{\partial a}(U, U; h) - \frac{\partial \mathcal{H}}{\partial b}(U, U; h)\right)\bigg|_{U=\widehat{U},\ h=h_c} \vartheta \notin Range\left(\frac{\partial \mathcal{H}}{\partial a}(\widehat{U}, \widehat{U}; h_c) - \frac{\partial \mathcal{H}}{\partial b}(\widehat{U}, \widehat{U}; h_c)\right).$$

Then, for $0 < \epsilon \ll 1$*, there exists a period* 2 *solution of (3) with the form*

$$\begin{aligned} h(\epsilon) &= h_c + \mathcal{O}(|\epsilon|), \\ U_n(\epsilon) &= \widehat{U} + \epsilon(-1)^n\vartheta + \mathcal{O}(|\epsilon|^2) \end{aligned}$$

which is C^{r-1} *in* ϵ*.*

Many authors have investigated linear stability and contractivity of Runge-Kutta and linear multistep methods solving DDEs (see, for example, [1, 5, 14, 15, 16, 19]). Regularity properties of Runge-Kutta and linear multistep methods for DDEs (1) have been widely investigated (see, for example, [11, 13, 18]). The aim of this paper is to present a unified approach to look for bifurcations from the steady solutions into spurious solutions as step-size varies.

2. Runge-Kutta methods

Rung-Kutta methods are natural candidates for solving DDEs, because they can be more readily adapted to cope with discontinuities and appear to be well suited to problems where frequently step-size changing is required.

Let (A, b, c) denote a given Runge-Kutta method with an $s \times s$ matrix $A = (a_{ij})$ and vectors $b = (b_1, b_2, \cdots, b_s)^T$, $c = (c_1, c_2, \cdots, c_s)^T$. Let $h > 0$ be a given step-size such that $\tau = mh$ for some positive integer m, and define grid-points $t_n = nh, n = -m, -m+1, \cdots$. Then the approximations y_{n+1} to $y(t_{n+1})$ $(n = 0, 1, 2, \cdots)$ are defined by

$$\begin{array}{rcl} Y_i^n & = & y_n + h\sum_{j=1}^s a_{ij} f(Y_j^n, Y_j^{n-m}), \quad i = 1, 2, \cdots, s, \\ y_{n+1} & = & y_n + h\sum_{i=1}^s b_i f(Y_i^n, Y_i^{n-m}), \end{array} \tag{4}$$

where the arguments Y_i^n are approximations to $y(t_n + c_i h)$ $(i = 1, 2, \cdots, s)$.

We assume that $\widehat{y} \in \mathbb{R}$ is a hyperbolic equilibrium of DDE (1) which satisfies that the characteristic equation $P(z) \equiv z - f_1' - f_2' e^{-z\tau} = 0$ has no zeros on the imaginary axis, where $f_1' = \frac{\partial f}{\partial y}(\widehat{y}, \widehat{y}), f_2' = \frac{\partial f}{\partial z}(\widehat{y}, \widehat{y})$, and $f(y, z)$ is the right-hand function of (1). The following lemma is easy to prove.

Lemma 2.1 [9] *Consider the real $(p+1) \times (p+1)$ bordered matrix*

$$\Xi = \begin{bmatrix} \Lambda & \zeta \\ \chi^T & \kappa \end{bmatrix},$$

where Λ is a $p \times p$ matrix, ζ and χ are vectors of length p, and κ is a scalar. Then

1. *If $rank(\Lambda) = p$, then Ξ is singular if and only if $\kappa - \chi^T \Lambda^{-1} \zeta = 0$.*
2. *If $rank(\Lambda) = p - 1$, then Ξ is singular if and only if $\chi^T \psi = 0$ or $\phi^T \zeta = 0$. Here $\Lambda\psi = 0$ and $\phi^T \Lambda = 0$.*

Let vectors $\xi, \eta \in \mathbb{R}^s$ satisfy $[h(f_1' + f_2')A - I]\xi = 0$ and $[h(f_1' + f_2')A^T - I]\eta = 0$. Denote $\mathbf{e} = [1, 1, \cdots, 1]^T$.

Theorem 2.2. *Let $\widehat{y}$ be a hyperbolic steady solution of DDE (1). Spurious fixed points of (4) in n bifurcate from the fixed points $y_n = \widehat{y}$ at $h_c = \frac{\tau}{m}$ for some positive integer m, where*

1. *$b^T[I - h_c(f_1' + f_2')A]^{-1}\mathbf{e} = 0$, provided that $b^T[I - h_c(f_1' + f_2')A]^{-2}\mathbf{e} \neq 0$ and that $(I - h_c(f_1' + f_2')A)$ is invertible; or where*
2. *$[I - h_c(f_1' + f_2')A]$ is singular with one-dimensional null-space and with left and right eigenvectors η and ξ, respectively, such that $\eta^T\mathbf{e} = 0, b^T\xi \neq 0$, and **Technical Condition A** holds; or where*
3. *$[I - h_c(f_1' + f_2')A]$ is singular with one-dimensional null-space and with left and right eigenvectors η and ξ, respectively, such that $\eta^T\mathbf{e} \neq 0, b^T\xi = 0$, and **Technical Condition B** holds.*

Technical Condition A: *Let $\eta^T\mathbf{e} = 0, b^T\xi \neq 0$, and let α be the unique vector with $\alpha^T b = 0$ satisfying*

$$[I - h_c(f_1' + f_2')A]\alpha - \mathbf{e} = 0.$$

Then

$$\eta^T \alpha \neq 0.$$

Technical Condition B*: Let* $\eta^T \mathbf{e} \neq 0, b^T \xi = 0$*, and let* β *be the unique vector with* $\beta^T \mathbf{e} = 0$ *satisfying*

$$[I - h_c(f_1' + f_2')A^T]\beta - h_c(f_1' + f_2')b = 0.$$

Then

$$\beta^T \xi \neq 0.$$

Proof. Let $Y^n = (Y_1^n, Y_2^n, \cdots, Y_s^n)^T$. We introduce a new vector Z_n as

$$Z_n = ((Y^{n-m})^T, (Y^{n-m+1})^T, \cdots, (Y^{n-1})^T, y_n)^T.$$

Then the Runge-Kutta formula (4) can be written as

$$Z_{n+1} = BZ_n + F(Z_{n+1}, Z_n) \tag{5}$$

where

$$B = \begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & I & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mathbf{e} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

and

$$F(Z_{n+1}, Z_n; h) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ h\sum_{j=1}^{s} a_{1j} f(Y_j^n, Y_j^{n-m}) \\ \vdots \\ h\sum_{j=1}^{s} a_{sj} f(Y_j^n, Y_j^{n-m}) \\ h\sum_{j=1}^{s} b_j f(Y_j^n, Y_j^{n-m}) \end{bmatrix}.$$

Denote $H(a, b; h) \equiv a - Bb - F(a, b; h), H_a = \frac{\partial H}{\partial a}\left(\widehat{Z}, \widehat{Z}; h\right)$, and $H_b = \frac{\partial H}{\partial b}\left(\widehat{Z}, \widehat{Z}; h\right)$, where $\widehat{Z} = (\widehat{y}\mathbf{e}^T, \widehat{y}\mathbf{e}^T, \cdots, \widehat{y}\mathbf{e}^T, \widehat{y})^T$. Then

$$H_a = \begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I - hf_1'A & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -hf_1'b^T & 1 \end{bmatrix}$$

and

$$H_b = \begin{bmatrix} 0 & -I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & -I & 0 \\ -hf_2'A & 0 & 0 & 0 & \cdots & 0 & 0 & -\mathbf{e} \\ -hf_2'b^T & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}.$$

For seeking spurious fixed points, consider $H(Z, Z; h) = 0$ and solve the equation

$$X\Phi = 0, \tag{6}$$

where

$$X = H_a + H_b = \begin{bmatrix} I & -I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & -I & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & -I & 0 \\ -hf_2'A & 0 & 0 & 0 & \cdots & 0 & I - hf_1'A & -\mathbf{e} \\ -hf_2'b^T & 0 & 0 & 0 & \cdots & 0 & -hf_1'b^T & 0 \end{bmatrix}, \tag{7}$$

and $\Phi = [\phi_1^T, \phi_2^T, \cdots, \phi_m^T, \phi_{m+1}]^T \in \mathbb{R}^{ms+1}$, $\phi_i \in \mathbb{R}^s, i = 1, 2 \cdots, m$, and $\phi_{m+1} \in \mathbb{R}$.

Case 1: For $[I - h_c(f_1' + f_2')A]$ is invertible, the null-space of X at $h = h_c$ is spanned by $\Phi = \left[((I - h_c(f_1' + f_2')A)^{-1}\mathbf{e})^T, \cdots, ((I - h_c(f_1' + f_2')A)^{-1}\mathbf{e})^T, 1\right]^T$.

Next, we need to show that

$$\left.\frac{dX}{dh}\right|_{h=h_c} \Phi \notin Range(X|_{h=h_c}). \tag{8}$$

Suppose there is a vector $\Gamma \in \mathbb{R}^{ms+1}$ such that

$$X|_{h=h_c}\Gamma = \left.\frac{dX}{dh}\right|_{h=h_c} \Phi, \tag{9}$$

where $\Gamma = (\gamma_1^T, \cdots, \gamma_m^T, \gamma_{m+1})^T \in \mathbb{R}^{ms+1}$, $\gamma_i \in \mathbb{R}^s, i = 1, 2, \cdots, m$, and $\gamma_{m+1} \in \mathbb{R}$. From the equation (9) we obtain

$$\begin{cases} \gamma_1 = \gamma_2 = \cdots = \gamma_m, \\ -A(f_1' + f_2')\phi_m = [I - h_cA(f_1' + f_2')]\gamma_m - \mathbf{e}\gamma_{m+1}, \\ h_cb^T(f_1' + f_2')\gamma_m = b^T(f_1' + f_2')\phi_m. \end{cases}$$

Then

$h_cb^T\gamma_m = b^T[I - h_cA(f_1' + f_2')]^{-1}\mathbf{e} - b^T[I - h_cA(f_1' + f_2')]^{-2}\mathbf{e} + h_cb^T[I - h_cA(f_1' + f_2')]^{-1}\mathbf{e}\gamma_{m+1}$.

One has $b^T[I - h_cA(f_1' + f_2')]^{-2}\mathbf{e} = 0$. This is a contradiction and Lemma 1.2 yields the desired result.

Case 2: First we show that the matrix $X|_{h=h_c}$ is singular with one-dimensional null-space. From the equation (6), we obtain

$$\begin{cases} \phi_1 = \phi_2 = \cdots = \phi_m, \\ I\phi_m - h_cA(f_1' + f_2')\phi_m = \mathbf{e}\phi_{m+1}, \\ h_cb^T(f_1' + f_2')\phi_m = 0. \end{cases}$$

Suppose $\phi_{m+1} = 0$. Then $b^T\xi = 0$ and this is a contradiction. Without loss of generality, let $\phi_{m+1} = 1$. According to **Technical Condition A**, the null-space of $X|_{h=h_c}$ is spanned by $\Phi = (\alpha^T, \alpha^T, \cdots, \alpha^T, 1)^T$. We need to show (8) holds. From the equation (9), we have

$$\begin{cases} \gamma_1 = \gamma_2 = \cdots = \gamma_m, \\ -A(f_1' + f_2')\alpha = [I - h_c A(f_1' + f_2')]\gamma_m - \mathbf{e}\gamma_{m+1}, \\ h_c b^T (f_1' + f_2')\gamma_m = b^T (f_1' + f_2')\alpha, \end{cases}$$

which implies $0 = -h_c\eta^T[I - h_c A(f_1' + f_2')]\gamma_m + h_c\eta^T\mathbf{e}\gamma_{m+1} = h_c(f_1' + f_2')\eta^T A\alpha = \eta^T\alpha \neq 0$. This is a contradiction.

Case 3: The proof is similar to that of **Case 2**. If $[I - h_c(f_1' + f_2')A]$ is singular of rank $s-1$, then $X|_{h=h_c}$ is singular if $b^T\xi = 0$ or $\eta^T e = 0$, where ξ is a right eigenvector and η^T is a left eigenvector. It is easy to show that the null-space of $X|_{h=h_c}$ is spanned by $\Phi = [(h_c f_2'(A^T\beta + b))^T, \cdots, (h_c f_2'(A^T\beta + b))^T, \beta^T, 1]^T$. This completes the proof.

Theorem 2.3. *Let $\widehat{y}$ be a hyperbolic steady solution of (1). Period 2 solutions of (4) in n bifurcate from the steady solution $\widehat{y}$ at $h_c = \frac{\tau}{m}$ for some even m, where*

$$b^T[I - h_c A(f_1' + f_2')]^{-1}\mathbf{e} + \frac{2}{h_c(f_1' + f_2')} = 0 \tag{10}$$

provided that

$$b^T[I - h_c A(f_1' + f_2')]^{-2}\mathbf{e} \neq 0 \tag{11}$$

and that the matrix $[I - h_c A(f_1' + f_2')]$ is invertible.

Proof. Our aim is to find where period 2 solutions of (4) bifurcate from a genuine steady solution. First we will show that $H_a - H_b$ at $h = h_c$ is singular with one-dimensional null-space. Let $Z = H_a - H_b$. Then the matrix Z is given by

$$Z = H_a - H_b = \begin{bmatrix} I & I & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & I & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & I & 0 \\ hf_2'A & 0 & 0 & 0 & \cdots & 0 & I - hf_1'A & \mathbf{e} \\ hf_2'b^T & 0 & 0 & 0 & \cdots & 0 & -hf_1'b^T & 2 \end{bmatrix}. \tag{12}$$

Solve

$$(H_a - H_b)|_{h=h_c}\Phi = 0, \tag{13}$$

where $\Phi = (\phi_1^T, \cdots, \phi_m^T, \phi_{m+1})^T \in \mathbb{R}^{sm+1}$, $\phi_i \in \mathbb{R}^s, i = 1, 2, \cdots, m$, and $\phi_{m+1} \in \mathbb{R}$. Since m is even, we thus have

$$\begin{cases} \phi_1 = -\phi_2 = \cdots = \phi_{m-1} = -\phi_m, \\ (I - h_c A(f_1' + f_2'))\phi_m = -\mathbf{e}\phi_{m+1}, \\ h_c b^T (f_1' + f_2')\phi_m = 2\phi_{m+1}, \end{cases}$$

which implies $-b^T(f_1' + f_2')h_c[I - h_c A(f_1' + f_2')]^{-1}\mathbf{e}\phi_{m+1} = 2\phi_{m+1}$.
Since

$$b^T[I - h_c A(f_1' + f_2')]^{-1}\mathbf{e} + \frac{2}{h_c(f_1' + f_2')} = 0,$$

$(H_a - H_b)|_{h=h_c}$ is singular with one-dimensional null-space.

Next we show that $H_a + H_b$ is invertible at $h = h_c$. Consider the equation

$$X|_{h=h_c}\Gamma = 0, \tag{14}$$

where $\Gamma = (\gamma_1^T, \gamma_2^T, \cdots, \gamma_m^T, \gamma_{m+1})^T$. This is equivalent to

$$\begin{cases} \gamma_1 = \gamma_2 = \cdots = \gamma_{m-1} = \gamma_m, \\ -h_c A f_2' \gamma_1 + [I - h_c A f_1']\gamma_m - \mathbf{e}\gamma_{m+1} = 0, \\ -h_c b^T f_2' \gamma_1 - h_c b^T f_1' \gamma_m = 0, \end{cases}$$

which implies $-b^T h_c(f_1' + f_2')[I - h_c A(f_1' + f_2')]^{-1}\mathbf{e}\gamma_{m+1} = 0$. Since

$$-b^T(f_1' + f_2')[I - h_c A(f_1' + f_2')]^{-1}\mathbf{e} = \frac{2}{h_c} \neq 0,$$

we deduce that $\gamma_{m+1} = 0, \gamma_m = \gamma_{m-1} = \cdots = \gamma_1 = 0$. Hence $H_a + H_b$ at $h = h_c$ is invertible.

Finally, we show that

$$\left.\frac{dZ}{dh}\right|_{h=h_c} \Phi \notin Range\ (Z|_{h=h_c}). \tag{15}$$

Consider

$$\left.\frac{dZ}{dh}\right|_{h=h_c} \Phi = Z|_{h=h_c}\Gamma. \tag{16}$$

From the equation (16), we obtain

$$\begin{cases} \gamma_1 = -\gamma_2 = \cdots = \gamma_{m-1} = -\gamma_m, \\ -A(f_1' + f_2')\phi_m = [I - h_c A(f_1' + f_2')]\gamma_m + \mathbf{e}\gamma_{m+1}, \\ -b^T(f_1' + f_2')\phi_m = -h_c b^T(f_1' + f_2')\gamma_m + 2\gamma_{m+1}, \end{cases}$$

which implies

$$b^T(f_1' + f_2')[I - h_c A(f_1' + f_2')]^{-1}\phi_m + b^T(f_1' + f_2')h_c[I - h_c A(f_1' + f_2')]^{-1}\mathbf{e}\gamma_{m+1} + 2\gamma_{m+1} = 0.$$

Thus $b^T(f_1' + f_2')[I - h_c A(f_1' + f_2')]^{-2}\mathbf{e}\phi_{m+1} = 0$, a contradiction. It follows from Lemma 1.3 that period 2 solutions in n bifurcate from the steady solution $\widehat{y}$ at $h = h_c = \frac{\tau}{m}$ for some even m. This completes the proof.

3. Linear Multistep Methods

Consider approximating the solution of (1) using a general consistent $k-$step linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f(y_{n+j}, y_{n+j-m}) \tag{17}$$

with fixed step-size h such that $h = \frac{\tau}{m} > 0$ for some positive integer $m > k$. Here y_n is the numerical approximation to $y(t_n)$, and $t_n = nh, n = -m, -m+1, \cdots$. It is assumed the starting values $y_0, y_1, \cdots, y_{k-1}$ are given.

Define the polynomials $\rho(z), \sigma(z)$ by

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j, \quad \sigma(z) = \sum_{j=0}^{k} \beta_j z^j. \tag{18}$$

We will assume throughout that the linear multistep method is consistent and zero-stable. This implies that

$$\rho(1) = 0, \quad \rho'(1) = \sigma(1) \neq 0. \tag{19}$$

The following result shows that a consistent and zero-stable linear multistep method is regular of degree 1.

Lemma 3.1 [13] *For a consistent and zero-stable linear multistep method (17), $\widehat{y}$ is a fixed point of (17) if and only if $f(\widehat{y},\widehat{y})=0$.*

The following lemma is easy to prove and will be used to prove Theorem 3.3.

Lemma 3.2 [9] *Consider the real $l\times l$ companion matrix*

$$E=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{l-1} \end{bmatrix}$$

and the polynomial

$$\varrho(x)\equiv x^l+a_{l-1}x^{l-1}+\cdots+a_1x+a_0.$$

Then $\det(xI-E)=\varrho(x), \det(I-E)=\varrho(1)$, *and* $\det(I+E)=(-1)^l\varrho(-1)$.

Theorem 3.3. *Let $\widehat{y}$ be a hyperbolic steady solution of (1). Let the linear multistep method (17) be consistent and zero-stable with $\rho(-1)\neq 0, \sigma(-1)\neq 0$. Then period 2 solutions in n bifurcate from the steady solution $\widehat{y}$ at $h_c=\frac{\rho(-1)}{(f_1'+f_2')\sigma(-1)}=\frac{\tau}{m}$ for some even m.*

Proof. Without loss of generality we assume $\alpha_k=1$. Let $Y^i=(y_i,\cdots,y_{i+k})^T$ and $Y^i_j=y_{i+j}, i=-m,-m+1,\cdots,j=0,1,\cdots,k$. The linear multistep method (17) can be rewritten for $n\geq 0$ as

$$\begin{pmatrix} Y^{n-m+1} \\ Y^{n-m+2} \\ \vdots \\ Y^{n-1} \\ Y^n \end{pmatrix}=\begin{pmatrix} 0 & I & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & 0 & B \end{pmatrix}\begin{pmatrix} Y^{n-m} \\ Y^{n-m+1} \\ \vdots \\ Y^{n-2} \\ Y^{n-1} \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\sum_{j=0}^k \beta_j f(Y^n_j, Y^{n-m}_j) \end{pmatrix}, \tag{20}$$

where

$$B=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{k-1} \end{pmatrix}.$$

Denote

$$A=\begin{pmatrix} 0 & I & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ 0 & 0 & \cdots & 0 & B \end{pmatrix},\ Z_n=\begin{pmatrix} Y^{n-m+1} \\ Y^{n-m+2} \\ \vdots \\ Y^{n-1} \\ Y^n \end{pmatrix},\ F(Z_n;h)=\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ h\sum_{j=0}^k \beta_j f(Y^n_j, Y^{n-m}_j) \end{pmatrix}.$$

So the equation (20) can be rewritten as $Z_n=AZ_{n-1}+F(Z_n;h)$.

Let $H(a,b;h) \equiv a - Ab - F(a;h)$. Denote $H_a = \frac{\partial H}{\partial a}\left(\widehat{Z},\widehat{Z};h\right), H_b = \frac{\partial H}{\partial b}\left(\widehat{Z},\widehat{Z};h\right)$, where $\widehat{Z} = (\widehat{y}\mathbf{e}^T, \widehat{y}\mathbf{e}^T, \cdots, \widehat{y}\mathbf{e}^T)^T \in \mathbb{R}^{m(k+1)}$. Then

$$H_a = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ 0 & 0 & \cdots & 0 & A_0 \end{pmatrix}, \quad H_b = \begin{pmatrix} 0 & -I & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -I \\ -B_0 & 0 & \cdots & 0 & -B \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ -h\beta_0 f_1' & -h\beta_1 f_1' & \cdots & -h\beta_{k-1} f_1' & 1 - h\beta_k f_1' \end{pmatrix}$$

and

$$B_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ h\beta_0 f_2' & h\beta_1 f_2' & \cdots & h\beta_{k-1} f_2' & h\beta_k f_2' \end{pmatrix}.$$

We have

$$H_a + H_b = \begin{pmatrix} I & -I & \cdots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & -I \\ -B_0 & 0 & \cdots & 0 & A_0 - B \end{pmatrix}, \quad H_a - H_b = \begin{pmatrix} I & I & \cdots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & I \\ B_0 & 0 & \cdots & 0 & A_0 + B \end{pmatrix}.$$

It follows that

$$\det(H_a + H_b) = \det(A_0 - B - B_0) = \det \begin{pmatrix} 1 & -1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ c_0 & c_1 & \cdots & c_{k-1} & c_k \end{pmatrix},$$

where

$$c_0 = -h\beta_0(f_1' + f_2'), \qquad c_k = 1 - h\beta_k(f_1' + f_2') + \alpha_{k-1},$$

and

$$c_j = -h\beta_j(f_1' + f_2') + \alpha_{j-1}, \qquad j = 1, 2, \cdots, k-2, k-1.$$

From Lemma 3.2 we obtain

$$\det(H_a + H_b) = 1 + (c_k - 1) + \cdots + c_1 + c_0 = \sum_{j=0}^{k} \alpha_j - h(f_1' + f_2') \sum_{j=0}^{k} \beta_j.$$

Due to the consistency and zero-stability of the method, we arrive at

$$\det(H_a + H_b) = -h(f_1' + f_2')\sigma(1) \neq 0.$$

Since m is even, then

$$\begin{aligned}
\det(H_a - H_b) &= \det((-1)^{m-1}B_0 + B + A_0) \\
&= \det(A_0 + B - B_0) \\
&= \det \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ d_0 & d_1 & \cdots & d_{k-1} & d_k \end{pmatrix},
\end{aligned}$$

where

$$d_0 = -h\beta_0(f_1' + f_2'), \qquad d_k = 1 - h\beta_k(f_1' + f_2') - \alpha_{k-1}$$

and

$$d_j = -h\beta_j(f_1' + f_2') - \alpha_{j-1}, \qquad j = 1, 2, \cdots, k-2, k-1.$$

It follows from Lemma 3.2 that

$$\begin{aligned}
\det(H_a - H_b) &= (-1)^{k+1}[d_0 + (-1)d_1 + \cdots + (-1)^k(d_k + 1) + (-1)^{k+1}] \\
&= (-1)^{k+1}[-h\beta_0(f_1' + f_2') + h\beta_1(f_1' + f_2') + \alpha_0 + \cdots \\
&\qquad +(-1)^k(-\alpha_{k-1} - h\beta_k(f_1' + f_2')) + (-1)^k\alpha_k] \\
&= (-1)^{k+1}\sum_{j=0}^{k}[h(f_1' + f_2')\beta_j(-1)^{j+1} + \alpha_j(-1)^j] \\
&= (-1)^{k+1}[-h(f_1' + f_2')\sigma(-1) + \rho(-1)].
\end{aligned}$$

Since $h_c = \frac{\rho(-1)}{(f_1'+f_2')\sigma(-1)}$, then $(H_a - H_b)|_{h=h_c}$ is singular. In addition,

$$Null(H_a - H_b)|_{h=h_c} = span\{\Phi\}$$

where $\Phi = [\phi^T, -\phi^T, \cdots, -\phi^T]^T \in \mathbb{R}^{m(k+1)}$ with $\phi = [1, -1, \cdots, (-1)^k]^T$.

Now we are in position to prove $(\frac{d}{dh}H_a - \frac{d}{dh}H_b)|_{h=h_c}\Phi \notin Range\ ((H_a - H_b)|_{h=h_c})$. From the expression of H_a and H_b, we have

$$\frac{d}{dh}H_a - \frac{d}{dh}H_b = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ B_0^* & 0 & \cdots & 0 & A_0^* \end{pmatrix},$$

where

$$B_0^* = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \beta_0 f_2' & \beta_1 f_2' & \cdots & \beta_{k-1} f_2' & \beta_k f_2' \end{pmatrix},$$

$$A_0^* = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -\beta_0 f_1' & -\beta_1 f_1' & \cdots & -\beta_{k-1} f_1' & -\beta_k f_1' \end{pmatrix}.$$

Suppose $(\frac{d}{dh}H_a - \frac{d}{dh}H_b)|_{h=h_c}\Phi \in Range((H_a - H_b)|_{h=h_c})$. Then there exists $\Gamma = [\gamma_1^T, \cdots, \gamma_m^T]^T$ such that

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ B_0^* & 0 & \cdots & 0 & A_0^* \end{pmatrix} \begin{pmatrix} \phi \\ -\phi \\ \vdots \\ \phi \\ -\phi \end{pmatrix} = \begin{pmatrix} I & I & \cdots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & I \\ B_0|_{h=h_c} & 0 & \cdots & 0 & B + A_0|_{h=h_c} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{m-1} \\ \gamma_m \end{pmatrix},$$

where $\gamma_i \in \mathbb{R}^{k+1}, i = 1, 2, \cdots, m$. From the left-hand side of the equation, we have

$$\beta_0(f_1' + f_2') + (-1)\beta_1(f_1' + f_2') + \cdots + (-1)^k\beta_k(f_1' + f_2') = (f_1' + f_2')\sigma(-1) \neq 0.$$

While from the right-hand side of the equation, we have

$$-(d_0 - d_1 + d_2 + \cdots + (-1)^k d_k) = -[h_c(f_1' + f_2')\sigma(-1) - \rho(-1)] = 0.$$

Consequently $\left(\frac{d}{dh}H_a - \frac{d}{dh}H_b\right)\big|_{h=h_c}\Phi \notin Range\,(H_a - H_b)|_{h=h_c}$. Application of Lemma 1.3 yields the desired result and this completes the proof.

Example 3.1. Consider the following delay differential equation

$$y'(t) = 2[y(t-1)^3 - y(t)].$$

From the equation, we have $\widehat{y} = 1, f_1' + f_2' = 4$. Application of the 2-step method with $\rho(z) = z^2 - z$ and $\sigma(z) = 1$ to the delay differential equation gives

$$y_{n+2} - y_{n+1} = 2h[y_{n-m}^3 - y_n].$$

It is easy to see that $\rho(1) = 0, \rho'(1) = \sigma(1) = 1, \rho(-1) = 2, \sigma(-1) = 1$. When $h_c = \frac{1}{2}$, the numerical scheme has a period 2 solution

$$y_n = (-1)^n\sqrt{3}.$$

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