Journal of Computational Mathematics, Vol.19, No.1, 2001, 77–86.

A NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR EXTERIOR 3-D PROBLEM*[1)]

De-hao Yu
(*State Key Laboratory of Scientific and Engineering Computing,
Institute of Computational Mathematics and Scientific/Engineering Computing,
Chinese Academy of Sciences, Beijing 100080, China*)

Ji-ming Wu
(*State Key Laboratory of Scientific and Engineering Computing,
Institute of Computational Mathematics and Scientific/Engineering Computing,
Chinese Academy of Sciences, Beijing 100080, China*)
(*Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics,
Beijing 100088, China*)

Dedicated to the 80th birthday of Professor Feng Kang

Abstract

In this paper, a nonoverlapping domain decomposition method, which is based on the natural boundary reduction(cf. [4, 13, 15]), is developed to solve the boundary value problem in exterior three-dimensional domain of general shape. Convergence analyses both for the exterior spherical domain and the general exterior domain are made. Some numerical examples are also provided to illustrate the method.

Key words: Domain decomposition, D-N Algorithm, Exterior 3-D problem.

1. Introduction

In recent years, the elliptic boundary value problems in unbounded domains have drawn more and more attention. To solve an equation in an unbounded domain numerically, a basic idea is to limit the computation to a bounded domain by introducing an artificial boundary. Based on this idea, many numerical methods, such as the coupling of BEM and FEM, the FEM with boundary conditions at artificial boundary, the coupled finite-infinite element method, the DDM(domain decomposition method)(cf.,e.g., [7, 6, 12, 3, 5, 2, 16, 17] and so on), have been put forward. All these methods have their own advantages as well as limitations. It is a practicable way to combine the natural boundary element method with the traditional FEM and DDM to solve problems in unbounded domains. However, the methods given in some published papers are only for two-dimensional cases(cf. [16, 17]) and cannot be directly extended to three-dimensional problems. In this paper, by taking Poisson equation as an example, we shall suggest a nonoverlapping DDM for exterior three-dimensional problems. By choosing a sphere as an interface, we turn the original problem into two subproblems, i.e., one in a bounded domain and the other in a regular unbounded domain(exterior spherical domain). We then solve the two subproblems alternately to acquire an approximate solution of the original problem. The subproblem in bounded domain is treated by the traditional FEM. The unique aspect of our method is to adopt the recent results of the natural boundary element method (cf. [10]) to solve the subproblem in unbounded domain, which makes our method simple in analysing and easy to be implemented.

The rest of this paper is organized as follows. Section 2 develops the D-N alternating algorithm; Section 3 studies the convergence of the D-N method for exterior spherical domain; Section 4 extends the result of Section 3 to general exterior domain; Section 5 discusses the discrete form of the D-N alternating algorithm; Section 6 presents some numerical results.

* Received March 26, 1999.

1) The Project supported by the Special Funds for State Major Basic Research Projects, the Chinese National Key Project for Basic Research and the National Natural Science Foundation of China.

2. Dirichlet-Neumann(D-N) Alternating Algorithm

Consider the following exterior boundary value problem:

$$\begin{cases} -\triangle u = f, & \text{in } \Omega^c, \\ u = g, & \text{on } \Sigma_0, \end{cases} \tag{2.1}$$

where $\Omega \subset R^3$ is a bounded domain and Ω^c denotes $R^3\backslash\overline{\Omega}$. $\Sigma_0 = \partial\Omega$ is a piecewise smooth surface. $g \in H^{\frac{1}{2}}(\Sigma_0)$ and $f \in L^2(\Omega^c)$ are given functions. To guarantee ensure the existence and uniqueness of the solution of (2.1), we must assume that u vanishes at infinity(cf. [18]). In the following discussion we shall also assume that function f has compact support.

Introduce a sphere $\Sigma_1 = \{(r, \theta, \varphi)|r = R_1\}$ for an appropriate R_1 to enclose boundary Σ_0 and the support of f. Make sure that $\text{dist}(\Sigma_1,\ \Sigma_0) > 0$. Then Ω^c is decomposed into two mutually disjoint subdomains, i.e., an interior subdomain denoted by Ω_1 and an exterior subdomain denoted by Ω_2. Ω_1 and Ω_2 are nonoverlapping domains. For exterior boundary value problem (2.1), we suggest the following D-N(Dirichlet-Neumann) alternating algorithm:

$$\begin{cases} -\Delta u_2^n = 0, & \text{in } \Omega_2, \\ u_2^n = \lambda^n, & \text{on } \Sigma_1, \\ \lim\limits_{r\to\infty} u_2^n = 0, \end{cases} \tag{2.2a}$$

$$\begin{cases} -\Delta u_1^n = f, & \text{in } \Omega_1, \\ u_1^n = g, & \text{on } \Sigma_0, \\ \dfrac{\partial u_1^n}{\partial n_1} = -\dfrac{\partial u_2^n}{\partial n_2}, & \text{on } \Sigma_1, \end{cases} \tag{2.2b}$$

$$\lambda^{n+1} = \theta_n u_1^n + (1-\theta_n)\lambda^n, \tag{2.2c}$$

where u_1^n and u_2^n are the n-th approximate solutions in Ω_1 and Ω_2; n_1 and n_2 denote the unit outward normals of Σ_1 with respect to the two neighboring subdomains; θ_n denotes the n-th relaxation factor and λ^0 is an arbitrary function in $H^{\frac{1}{2}}(\Sigma_1)$.

Note that, on interface Σ_1, only the value of the normal derivative of the solution of (2.2a) is needed in solving (2.2b). So it is unnecessary to solve (2.2a). Actually we can obtain $\dfrac{\partial u_2^n}{\partial n_2}$ directly from λ^n by making use of the following natural integral equation(cf. [10]):

$$\frac{\partial u_2^n}{\partial n_2} = -\frac{1}{16\pi R_1}\int_0^{2\pi}\int_0^{\pi} \frac{\lambda^n(\theta',\ \varphi')\sin\theta'}{\sin^3\frac{\gamma}{2}} d\theta' d\varphi'. \tag{2.3}$$

γ in (2.3) satifies $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$ and the hypersingular integral in (2.3) must be understood as the normalization of divergent integral in the sense of generalized function. The details about the computation of this kind of integrals can be found in [8, 13, 14, 10].

For $\lambda(\theta,\ \varphi) \in H^{\frac{1}{2}}(\Sigma_1)$, make harmonic extensions of $\lambda(\theta,\ \varphi)$ to Ω_1 and Ω_2 respectively to acquire functions $H_1\lambda$ and $H_2\lambda$, namely, $H_1\lambda$ satifies

$$\begin{cases} -\Delta H_1\lambda = 0, & \text{in } \Omega_1, \\ H_1\lambda = \lambda, & \text{on } \Sigma_1, \\ H_1\lambda = 0, & \text{on } \Sigma_0, \end{cases} \tag{2.4}$$

while $H_2\lambda$ satifies

$$\begin{cases} -\Delta H_2\lambda = 0, & \text{in } \Omega_2, \\ H_2\lambda = \lambda, & \text{on } \Sigma_1, \\ \lim\limits_{r\to+\infty} H_2\lambda = 0. \end{cases} \tag{2.5}$$

Define

$$S_i = \frac{\partial}{\partial n_i}(H_i \cdot), \quad i = 1,\ 2. \tag{2.6}$$

S_2 is the natural integral operator of the exterior spherical domain Ω_2 and $S = S_1 + S_2$ is none other than the Poincaré-Steklov operator on Σ_1 which might be preconditioned by using the method given in [11]. Follow [9], we can prove that the D-N alternating algorithm (2.2) is equivalent to the following preconditioned Richardson iterative method:

$$S_1(\lambda^{n+1} - \lambda^n) = \theta_n(b - S\lambda^n), \tag{2.7}$$

where b is a function dependent on f and g. So, in order to analyse the convergence of this D-N alternating method, what we need to do is just to estimate the the eigenvalues of operator $S_1^{-1}S$, i.e., to estimate the upper bound and the lower bound of the following ratio:

$$\frac{(S\lambda,\ \lambda)}{(S_1\lambda,\ \lambda)} = 1 + \frac{(S_2\lambda,\ \lambda)}{(S_1\lambda,\ \lambda)}. \tag{2.8}$$

3. Convergence Analysis for Exterior Spherical Domain

In order to estimate (2.8) in general case, we first take into consideration a special case in which $\Sigma_0 = \{(r,\theta,\varphi)|r = R_0\}$ and the interface $\Sigma_1 = \{(r,\theta,\varphi)|r = R_1,\ R_1 > R_0\}$. We claim that

Lemma 1. *Suppose*

$$f_n(x) = \frac{n}{2n+1} + \frac{n+1}{2n+1}x^{2n+1},\ x \in [0,\ 1],\ n = 0,\ 1,\ 2,\ \cdots. \tag{3.1}$$

Set

$$U(x) = \sup_{n\geq 0,\ n\in Z}\{f_n(x)\},\ L(x) = \inf_{n\geq 0,\ n\in Z}\{f_n(x)\}. \tag{3.2}$$

Then we have

$$U(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq \frac{1}{2}, \\ x, & \frac{1}{2} < x \leq 1, \end{cases} \qquad L(x) = \begin{cases} f_0(x), & 0 \leq x \leq x_0, \\ f_{k+1}(x), & x_k < x \leq x_{k+1}, \end{cases} \tag{3.3}$$

where x_k is the only root of equation $f_{k+1}(x) - f_k(x) = 0$ in interval $(0,\ 1)$ and $x_k < x_{k+1}$ (k is a nonnegative integer).

Here we omit the proof of this lemma for it is primary and tedious. For convenience, we write out the approximate values of the first eleven roots:

$$\begin{array}{lll} x_0 = 0.36602540378444, & x_1 = 0.50663226438840, & x_2 = 0.58946986810183, \\ x_3 = 0.64549392091075, & x_4 = 0.68642756803965, & x_5 = 0.71788512503709, \\ x_6 = 0.74294473351832, & x_7 = 0.76345385249658, & x_8 = 0.78059641404708, \\ x_9 = 0.795170011037, & x_{10} = 0.807733615911, & \cdots\ \cdots\ \cdots\ \cdots \end{array}$$

Now, for $\lambda(\theta,\ \varphi) \in H^{\frac{1}{2}}(\Sigma_1)$, at least in the L^2-sense there holds the following expansion(cf. [1]):

$$\lambda(\theta,\ \varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \Lambda_l^m\ Y_l^m(\theta,\ \varphi), \tag{3.4}$$

with

$$Y_l^m(\theta,\ \varphi)=\sqrt{\frac{(l-m)!}{(l+m)!}\frac{2l+1}{4\pi}}P_l^m(\cos\theta)e^{im\varphi},$$

$$\Lambda_l^m=\int_0^{2\pi}\int_0^{\pi}\lambda(\theta,\ \varphi)\ \overline{Y_l^m(\theta,\ \varphi)}\sin\theta d\theta d\varphi,$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1+l^2)^{1/2}|\Lambda_l^m|^2<+\infty,$$

where $P_l^m(x)$ denotes the associated Legendre function of the first kind; $\overline{Y_l^m(\theta,\ \varphi)}$ denotes the complex conjugate of $Y_l^m(\theta,\ \varphi)$. By separation of variables, we obtain the harmonic extension in annular domain Ω_1

$$H_1\lambda=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{R_1^{l+1}\Lambda_l^m}{R_1^{2l+1}-R_0^{2l+1}}(r^l-\frac{R_0^{2l+1}}{r^{l+1}})Y_l^m(\theta,\ \varphi),\quad R_0<r<R_1 \tag{3.5}$$

and the harmonic extension in exterior spherical domain Ω_2

$$H_2\lambda=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{R_1^{l+1}\Lambda_l^m}{r^{l+1}}Y_l^m(\theta,\ \varphi),\quad r>R_1. \tag{3.6}$$

Since

$$S_1\lambda=\frac{\partial}{\partial n_1}(H_1\lambda)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{[lR_1^{2l+1}+(l+1)R_0^{2l+1}]\Lambda_l^m}{R_1(R_1^{2l+1}-R_0^{2l+1})}Y_l^m(\theta,\ \varphi),$$

$$S_2\lambda=\frac{\partial}{\partial n_2}(H_2\lambda)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{(l+1)\Lambda_l^m}{R_1}Y_l^m(\theta,\ \varphi),$$

so there hold

$$(S_1\lambda,\ \lambda)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{lR_1^{2l+1}+(l+1)R_0^{2l+1}}{R_1^{2l+1}-R_0^{2l+1}}R_1|\Lambda_l^m|^2,$$

$$(S_2\lambda,\ \lambda)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(l+1)R_1|\Lambda_l^m|^2.$$

Noting that

$$\begin{aligned}\frac{R_1-R_0}{R_1+R_0}&=\inf_{l\geq 0,l\in Z}\{\frac{R_1^{2l+1}-R_0^{2l+1}}{R_1^{2l+1}+R_0^{2l+1}}\}\leq\inf_{l\geq 0,l\in Z}\{\frac{(l+1)(R_1^{2l+1}-R_0^{2l+1})}{lR_1^{2l+1}+(l+1)R_0^{2l+1}}\}\\&\leq\frac{(l+1)(R_1^{2l+1}-R_0^{2l+1})}{lR_1^{2l+1}+(l+1)R_0^{2l+1}}\leq\sup_{l\geq 0,l\in Z}\{\frac{(l+1)(R_1^{2l+1}-R_0^{2l+1})}{lR_1^{2l+1}+(l+1)R_0^{2l+1}}\}\\&\leq\max\{\frac{R_1}{R_0}-1,2\},\end{aligned}$$

we get

$$\frac{2R_1}{R_1+R_0}\leq\frac{1}{U(R_0/R_1)}\leq\frac{(S\lambda,\ \lambda)}{(S_1\lambda,\ \lambda)}\leq\frac{1}{L(R_0/R_1)}\leq\max\{\frac{R_1}{R_0},\ 3\}. \tag{3.7}$$

Following the usual theory of D-N alternating algorithm(cf., e. g., [9]) we obtain the following convergence result.

Theorem 1. *If relaxation factor θ_n satifies*

$$0<\theta_n<2L(\frac{R_0}{R_1}), \tag{3.8}$$

then the D-N alternating algorithm (2.2) *converges. Particularly, the optimal relaxation factor is*

$$\theta_{opt} = \frac{2U(R_0/R_1)L(R_0/R_1)}{U(R_0/R_1)+L(R_0/R_1)}, \tag{3.9}$$

while the optimal compression ratio of the iteration is

$$\delta_{opt} = \frac{U(R_0/R_1)-L(R_0/R_1)}{U(R_0/R_1)+L(R_0/R_1)}. \tag{3.10}$$

The values of θ_{opt} and δ_{opt} can be obtained by using Lemma 1. Here we give some examples in the following table.

Table 1.

R_0/R_1	1/5	1/4	1/3	1/2	2/3
$L(R_0/R_1)$	1/5	1/4	1/3	5/12	81292/177147
$U(R_0/R_1)$	1/2	1/2	1/2	1/2	2/3
θ_{opt}	2/7	1/3	2/5	5/11	162584/299085
δ_{opt}	3/7	1/3	1/5	1/11	18403/99695

4. Convergence Analysis for General Exterior Domain.

The convergence result about the exterior spherical domain given in Section 3 can be extended to a general exterior domain. We assume that Σ_0 is a piecewise smooth closed surface and sphere Σ_1 is the interface with radius R_1. Let Σ_0' be a sphere with radius R_0' inside Σ_0, and let Σ_0'' be a sphere with radius R_0'' between Σ_0 and Σ_1; see Fig. 1 below. Make sure that $\text{dist}\{\Sigma_0,\ \Sigma_0'\} = \text{dist}\{\Sigma_0,\ \Sigma_0''\} = 0$. By translation of axes to make the origin to be the common center of the aforementioned three spheres and R_0' to be maximal. Obviously, it holds that $0 < R_0' < R_0'' < R_1$. We denote by Ω_1' the annular domain between Σ_0' and Σ_1, and Ω_1'' the annular domain between Σ_0'' and Σ_1. It is easy to see that $\Omega_1'' \subset \Omega_1 \subset \Omega_1'$.

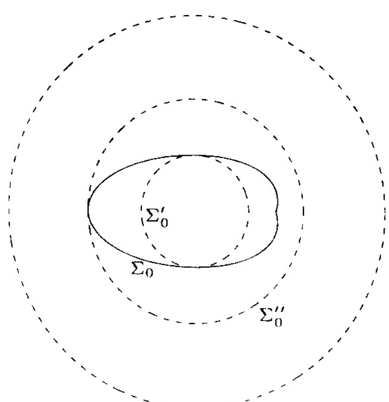

Figure 1.

Let u_1, u_1' and u_1'' be the solutions of the following Dirichlet BVPs in Ω_1, Ω_1' and Ω_1'' respectively:

$$\begin{cases} -\Delta u_1 = 0, & \text{in } \Omega_1, \\ u_1 = \lambda, & \text{on } \Sigma_1, \\ u_1 = 0, & \text{on } \Sigma_0, \end{cases} \tag{4.1a}$$

$$\begin{cases} -\Delta u_1' = 0, & \text{in } \Omega_1', \\ u_1' = \lambda, & \text{on } \Sigma_1, \\ u_1' = 0, & \text{on } \Sigma_0', \end{cases} \tag{4.1b}$$

$$\begin{cases} -\Delta u_1'' = 0, & \text{in } \Omega_1'', \\ u_1'' = \lambda, & \text{on } \Sigma_1, \\ u_1'' = 0, & \text{on } \Sigma_0''. \end{cases} \tag{4.1c}$$

Thus $u_1 = H_1\lambda, u_1' = H_1'\lambda, u_1'' = H_1''\lambda$, where H_1, H_1' and H_1'' are harmonic extension operators of functions defined on Σ_1 to Ω_1, Ω_1' and Ω_1'' respectively. Now let us consider the following bilinear forms:

$$D_{\Omega_i}(u,\ v) = \iiint_{\Omega_i} \nabla u \cdot \nabla v dxdydz,\ i = 1, 2, \tag{4.2}$$

$$D_{\Omega_1'}(u,\ v) = \iiint_{\Omega_1'} \nabla u \cdot \nabla v dxdydz, \tag{4.3}$$

$$D_{\Omega_1''}(u,\ v) = \iiint_{\Omega_1''} \nabla u \cdot \nabla v dxdydz, \tag{4.4}$$

Note that the PDEs in (4.1) are homogeneous and the boundary conditions on Σ_0, Σ_0' and Σ_0'' are also homogeneous. Thus the solution u_1'' of BVPs (4.1c), extended by zero in $\Omega_1 \backslash \Omega_1''$, may be considered as a function in the solution space of BVPs (4.1a). We also note that the solution u_1 of BVPs (4.1a) is the minimizer of the energy functional associated with (4.1a). Therefore we have

$$D_{\Omega_1}(u_1,\ u_1) \le D_{\Omega_1}(u_1'', u_1'') = D_{\Omega_1''}(u_1'', u_1'').$$

Similarly, it holds that

$$D_{\Omega_1'}(u_1', u_1') \le D_{\Omega_1}(u_1,\ u_1).$$

The above discussion concludes that

$$D_{\Omega_1'}(H_1'\lambda,\ H_1'\lambda) \le D_{\Omega_1}(H_1\lambda,\ H_1\lambda) \le D_{\Omega_1''}(H_1''\lambda,\ H_1''\lambda). \tag{4.5}$$

Further we have

$$\frac{D_{\Omega_2}(H_2\lambda, H_2\lambda)}{D_{\Omega_1''}(H_1''\lambda,\ H_1''\lambda)} \le \frac{D_{\Omega_2}(H_2\lambda, H_2\lambda)}{D_{\Omega_1}(H_1\lambda,\ H_1\lambda)} \le \frac{D_{\Omega_2}(H_2\lambda, H_2\lambda)}{D_{\Omega_1'}(H_1'\lambda,\ H_1'\lambda)}. \tag{4.6}$$

$H_2\lambda$ is defined by (2.5). Using Green's formula we get

$$\frac{(S_2\lambda,\ \lambda)}{(\partial H_1''\lambda/\partial n_1,\ \lambda)} \le \frac{(S_2\lambda,\ \lambda)}{(S_1\lambda,\ \lambda)} \le \frac{(S_2\lambda,\ \lambda)}{(\partial H_1'\lambda/\partial n_1,\ \lambda)}. \tag{4.7}$$

Following the practice of Section 3, by expansion in spherical harmonics we come to the result that

$$\frac{2R_1}{R_1 + R_0''} \le \frac{1}{U(R_0''/R_1)} \le \frac{(S\lambda,\ \lambda)}{(S_1\lambda,\ \lambda)} \le \frac{1}{L(R_0'/R_1)} \le \max\{\frac{R_1}{R_0'},\ 3\}. \tag{4.8}$$

Similar to Theorem 1, we claim

Theorem 2. *If relaxation factor θ_n satifies*

$$0 < \theta_n < 2L(\frac{R_0'}{R_1}), \tag{4.9}$$

then for general exterior domain, the D-N alternating algorithm (2.2) *converges.*

Remark. As we use (4.7) to get (4.8), here we have been unable to obtain the expressions of the optimal relaxation factor and the optimal compression ratio of the iteration. However, it is definite that they depend on R_1 and the geometry of Ω^c in some way, which is confirmed by our numerical results(see Section 6).

5. The Weak Form and Discritization

The weak form of problem (2.2b) is: Find $u_1^n \in H_g^1(\Omega_1)$, such that

$$D_{\Omega_1}(u_1^n,\ v) = \iiint_{\Omega_1} fv\ dxdydz - \iint_{\Sigma_1} v\frac{\partial u_2^n}{\partial n_2}ds, \quad \forall\ v\ \in \overset{\circ}{H^1}(\Omega_1), \tag{5.1}$$

where

$$H_g^1(\Omega_1) = \{v \in H^1(\Omega_1) \mid v|_{\Sigma_0} = g\},\ \overset{\circ}{H^1}(\Omega_1) = \{v \in H^1(\Omega_1) \mid v|_{\Sigma_0} = 0\}.$$

Obviously, $v|_{\Sigma_1} \in H^{\frac{1}{2}}(\Sigma_1)$, so it holds that, in the sense of mean convergence,

$$v|_{\Sigma_1} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\ \varphi)\int_0^{2\pi}\int_0^{\pi} v(R_1,\ \theta',\ \varphi')\ \overline{Y_l^m(\theta',\ \varphi')}\sin\theta' d\theta' d\varphi',$$

As (2.3) is equivalent to(cf. [10])

$$\frac{\partial u_2^n}{\partial n_2} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{l+1}{R_1} Y_l^m(\theta,\ \varphi)\int_0^{2\pi}\int_0^{\pi} \lambda^n(\theta',\ \varphi')\ \overline{Y_l^m(\theta',\ \varphi')}\sin\theta' d\theta' d\varphi',$$

we have

$$\begin{aligned}\iint_{\Sigma_1} v\frac{\partial u_2^n}{\partial n_2}ds = &\sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_1(l+1)\int_0^{2\pi}\int_0^{\pi} v(R_1,\ \theta',\ \varphi')\ Y_l^m(\theta',\ \varphi')\sin\theta' d\theta' d\varphi' \\ &\times\int_0^{2\pi}\int_0^{\pi} \lambda^n(\theta',\ \varphi')\ \overline{Y_l^m(\theta',\ \varphi')}\sin\theta' d\theta' d\varphi'.\end{aligned}$$

Define

$$\begin{aligned}\hat{D}^c(\lambda,\ \mu) = &\sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_1(l+1)\int_0^{2\pi}\int_0^{\pi} \mu(\theta',\ \varphi')\ Y_l^m(\theta',\ \varphi')\sin\theta' d\theta' d\varphi' \\ &\times\int_0^{2\pi}\int_0^{\pi} \lambda(\theta',\ \varphi')\ \overline{Y_l^m(\theta',\ \varphi')}\sin\theta' d\theta' d\varphi',\end{aligned} \tag{5.2}$$

then

$$\iint_{\Sigma_1} v\frac{\partial u_2^n}{\partial n_2}ds = \hat{D}^c(\lambda^n,\ v). \tag{5.3}$$

Now the weak form (5.1) can be rewritten as : Find $u_1^n \in H_g^1(\Omega_1)$, such that

$$D_{\Omega_1}(u_1^n,\ v) = \iiint_{\Omega_1} fv\ dxdydz - \hat{D}^c(\lambda^n,\ v), \quad \forall\ v \in \overset{\circ}{H^1}(\Omega_1), \tag{5.4}$$

(5.4), together with (2.2c), is the weak form of D-N method (2.2). Now let us consider the discritization of this weak form. Discretize Ω_1 into a finite number of element domains. Let $\overset{\circ}{S_h}(\Omega_1)$ denote the linear subspace of $\overset{\circ}{H^1}(\Omega_1)$ corresponding to this partition. Define $M_0,\ M_1$ and M_i to be the sets of all nodes belong to $\Sigma_0,\ \Sigma_1$ and Ω_1 respectively. Denote by $N_X(x,\ y,\ z)$ a basis function corresponding to each node X. Obviously

$$\overset{\circ}{S_h}(\Omega_1) = \mathrm{span}\{N_{A^i}(x,\ y,\ z),\ N_{A^1}(x,\ y,\ z);\ A^i \in M_i, A^1 \in M_1\}.$$

The finite element approximation of u_1^n can be expressed as

$$\begin{aligned}u_{1h}^n = &\sum_{A^i \in M_i} U_{A^i}^n N_{A^i}(x,\ y,\ z) + \sum_{A^1 \in M_1} U_{A^1}^n N_{A^1}(x,\ y,\ z) \\ &+ \sum_{A^0 \in M_0} g(x_{A^0},\ y_{A^0},\ z_{A^0}) N_{A^0}(x,\ y,\ z).\end{aligned}$$

By (5.4) and (2.2c), it is not difficult to acquire the following discrete form of the D-N alternating algorithm:

$$\begin{pmatrix} Q^{ii} & Q^{i1} \\ Q^{1i} & Q^{11} \end{pmatrix} \begin{pmatrix} U^n_{A^i} \\ U^n_{A^1} \end{pmatrix} = \begin{pmatrix} F^i \\ F^1 - K_h \Lambda_n \end{pmatrix} \tag{5.5a}$$

$$\Lambda_{n+1} = \theta_n U^n_{A^1} + (1-\theta_n)\Lambda_n, \tag{5.5b}$$

with

$$K_h = [\hat{D}^c(N_{A^1},\ N_{B^1})],\quad Q^{ii} = [D_{\Omega_1}(N_{A^i},\ N_{B^i})],\quad Q^{i1} = [D_{\Omega_1}(N_{A^1},\ N_{B^i})],$$

$$Q^{1i} = [D_{\Omega_1}(N_{A^i},\ N_{B^1})],\quad Q^{11} = [D_{\Omega_1}(N_{A^1},\ N_{B^1})],$$

$$F^i = \iiint_{\Omega_1} f N_{B^i} dxdydz - \sum_{A^0 \in M_0} g(x_{A^0},\ y_{A^0},\ z_{A^0}) D_{\Omega_1}(N_{A^0},\ N_{B^i}),$$

$$F^1 = \iiint_{\Omega_1} f N_{B^1} dxdydz - \sum_{A^0 \in M_0} g(x_{A^0},\ y_{A^0},\ z_{A^0}) D_{\Omega_1}(N_{A^0},\ N_{B^1}),$$

$$A^0,\ B^0 \in M_0,\ A^1,\ B^1 \in M_1,\ A^i,\ B^i \in M_i.$$

It is easy to see that, K_h is none other than the matrix derived by the natural boundary reduction on interface Σ_1, the stiffness matrix in the right-hand side of (5.5a), which is banded and sparse, is derived by finite element method. Obviously, during the whole process of iteration, we only need to form the stiffness matrix and K_h only once. Moreover it can be checked that the discrete D-N alternating algorithm (5.5) is equivalent to the following preconditioned Richardson iteration:

$$S_h^{(1)}(\Lambda_{n+1} - \Lambda_n) = \theta_n(\overline{F^1} - S_h\Lambda_n), \tag{5.6}$$

where $S_h^{(1)} = Q^{11} - Q^{1i}(Q^{ii})^{-1}Q^{i1},\ S_h = S_h^{(1)} + K_h,\ \overline{F^1} = F^1 - Q^{1i}(Q^{ii})^{-1}F^i$.

Following the inference of [17], we can obtain the following results.

Theorem 3. *The condition number of the iterative matrix of the discrete D-N alternating algorithm* (5.5), *i.e., the condition number of* $(S_h^{(1)})^{-1}S_h$, *is independent of mesh parameter* h *of domain* Ω_1.

Theorem 4. *If* $0 < \min\theta_n \le \max\theta_n < 2L(\frac{R_0'}{R_1})$ *(the meanings of* R_0' *and* R_1 *are the same as those in Section 4), then the discrete D-N alternating algorithm* (5.5) *converges and the convergence rate is independent of mesh parameter* h *of domain* Ω_1.

6. Numerical Results

Example. Use the discrete D-N alternating algorithm (5.5) to solve BVPs (2.1). Suppose that $\Omega^c = \{(x,y,z)|\ |x| > 1$ or $|y| > 1$ or $|z| > 1\}$ is the exterior domain of cube $[-1,1] \times [-1,1] \times [-1,1]$. Choose $f = 0$. The exact solution is $u = (x+z)/r^3$ and consequently $g(x,\ y,\ z)$ is determined by u. Choose the interface as $\Sigma_1 = \{(r,\theta,\varphi)|\ r = R_1,\ R_1 > \sqrt{3}\}$. Use a uniform square mesh with mesh size $\frac{2}{N}$ to get a partition of the interior boundary Σ_0. Draw a line from each node on Σ_0 in the radial direction to get the corresponding node on the interface Σ_1 and subdivide each line segment between these two nodes into N equivalent parts to get $N-1$ nodes. Then by using all the above nodes we get an eight-node trilinear isoparametric finite elements. We substitute $\sum_{l=0}^{M}$ for $\sum_{l=0}^{\infty}$ in the computing of the entries of K_h. Denote by m the total number of nodes on $\overline{\Omega_1}$. By computing, the results are as follows:

Table 2. Maximum node-error on $\overline{\Omega_1}$ for $\theta_n = 0.5,\ R_1 = 2.0$

N	m	M	Iterative times					
			0	1	2	3	4	10
2	78	4	0.34087	0.09297	0.05016	0.04715	0.04713	0.04713
4	490	12	0.32986	0.04694	0.01504	0.01367	0.01352	0.01352
8	3474	22	0.32047	0.04376	0.00631	0.00346	0.00336	0.00335
16	26146	36	0.31854	0.05121	0.00216	0.00079	0.00085	0.00084

Table 3. Maximum node-error on $\overline{\Omega_1}$ for $\theta_n = 0.4,\ R_1 = 2.0$

N	m	M	Iterative times					
			0	1	2	3	4	10
2	78	4	0.34087	0.14107	0.07438	0.05198	0.04806	0.04713
4	490	12	0.32986	0.09061	0.03376	0.01775	0.01449	0.01352
8	3474	22	0.32047	0.07966	0.02542	0.00888	0.00439	0.00335
16	26146	36	0.31854	0.08404	0.02361	0.00671	0.00216	0.00084

Table 2 and Table 3 show that $\theta_n = 0.5$ is a better choice than $\theta_n = 0.4$. But things might be quite different if we change the value of R_1(see Table 4 and Table 5 below). These facts show that the optimal relaxation factor is dependent on R_1, which is in accord with our analyses.

Table 4. Maximum node-error on $\overline{\Omega_1}$ for $\theta_n = 0.5,\ R_1 = 4.0$

N	m	M	Iterative times					
			0	1	2	3	4	10
2	78	4	0.22764	0.06505	0.06902	0.03877	0.05165	0.04782
4	490	12	0.17336	0.06576	0.03249	0.00999	0.01159	0.01109
8	3474	22	0.16190	0.06306	0.02539	0.00864	0.00462	0.00303
16	26146	36	0.15760	0.04651	0.00767	0.00079	0.00080	0.00081

Table 5. Maximum node-error on $\overline{\Omega_1}$ for $\theta_n = 0.4,\ R_1 = 4.0$

N	m	M	Iterative times					
			0	1	2	3	4	10
2	78	4	0.22764	0.03133	0.05011	0.04748	0.04785	0.04780
4	490	12	0.17336	0.01794	0.01140	0.01107	0.01110	0.01109
8	3474	22	0.16190	0.01808	0.00321	0.00303	0.00304	0.00303
16	26146	36	0.15760	0.00557	0.00648	0.00081	0.00084	0.00081

The above four tables show that the convergence rate of the discrete D-N alternating algorithm is independent of mesh parameter h. From the last columns of the above four tables we can also see that as we refine the mesh the maximum node-error on $\overline{\Omega_1}$ is roughly of $O(h^2)$ order, which shows that our method is very practicable.

References

[1] R. Courant, D. Hilbert, Methods of Mathematical Physics (in Chinese, translated by M. Qian and D. R. Guo), Science Press, Beijing, 1981.

[2] L. Demkowicz, F. Ihlenburg, Analysis of a coupled finite-infinite element method for exterior helmholtz problems, to appear.

[3] K. Feng, Asymptotic radiation conditions for reduced wave equation, *J. Comput. Math.*, **2** (1984), 130-138.

[4] K. Feng, D.H. Yu, Canonical Integral Equations of Elliptic Boundary Value Problems and Their Numerical Solutions, Proceedings of the China-France Symposium on FEM, Science Press, Beijing 1983, 211-252.

[5] H.D. Han, X.N. Wu, The approximation of the exact boundary conditions at an artificial boundary for linear elastic equations and its application, *Math. Comp.*, **59** (1992), 21-37.

[6] G.C. Hsiao, The coupling of BEM and FEM—a brief review, *Boundary Elements X*, Computational Mechanics Publications, Southampton, 1988, 431-445.

[7] C. Johnson, J.C. Nédélec, On the coupling of boundary integral and finite element methods, *Math. Comp.*, **35** (1980), 1063-1079.
[8] P. Linz, On the approximate computation of certain strongly singular integrals, *Computing*, **35** (1985), 345-353.
[9] T. Lü, J.M. Shi, Z.B. Lin, Domain Decomposition Method—A New Technique for Numereical Solution of Partial Differential Equations, Science Press, Beijing, 1992.
[10] J.M. Wu, D.H. Yu, The natural integral equations of 3-D harmonic problems and their numerical solutions, *Math. Numer. Sinica*, **20** (1998), 419-430.
[11] J.C. Xu, S. Zhang, Preconditioning the Poincaré-Steklov operator by using Green's function, *Math. Comp.*, **66** (1997), 125-138.
[12] D.H. Yu, A direct and natural coupling of BEM and FEM, Boundary Elements XIII, Computational Mechanics Publications, Southampton, 1991, 995-1004.
[13] D.H. Yu, Mathematical Theory of Natural Boundary Element Method, Science Press, Beijing, 1993.
[14] D.H. Yu, The numerical computation of hypersingular integrals and its application in BEM, *Advances in Engineering Software*, **18** (1993), 103-109.
[15] D.H. Yu, Natural boundary element method and adaptive boundary element method—some new developments in boundary element methods in China, *Contemp. Math.*, **163** (1994), 185-204.
[16] D.H. Yu, A domain decomposition method based on the natural boundary reduction over unbounded domain, *Math. Numer. Sinica*, **16** (1994), 448-459; translation in *Chinese J. Numer. Math. & Appl.*, **17**:1 (1995), 95-105.
[17] D.H. Yu, A disretization of a nonoverlapping domain decomposition method for unbounded domains and its convergence, *Math. Numer. Sinica*, **18** (1996), 328-336; translation in *Chinese J. Numer. Math. & Appl.*, **18**:4 (1996), 93-102.
[18] J.L. Zhu, Boundary Element Analysis for Elliptic Boundary Value Problems, Science Press, Beijing, 1991.