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A SUPERONVERGENCE ANALYISIS FOR FINITE ELEMENT SOLUTION BY THE INTERPOLANT POSTPROCESSING ON IRREGULAR MESHES FOR SMOOTH PROBLEM*1)

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Abstract

The post-processing procedure is given by a interpolant postprocessing of the finit element solution by appropriately-defined finite dimensional subspaces. The corresponding superconvergence are established on general quasi-regular finite element partitions.

Key words: Finite element, Superconvergence interpolant postprocessing.

1. Introduction

The results in this paper are based on the idea of interpolation postprocessing in [1] and the techniques of L^2 projection processing in [2]

For simlicity, we consider the model problem: Find$u \in H_0^1(\Omega)$,such that

$$\begin{cases} -\nabla \cdot (a\nabla u) = f, & \text{in}\Omega, \\ \qquad u = 0, & \text{on}\partial\Omega \end{cases} \tag{1.1}$$

Suppose that J^h and J^H are irregular triangulations (or quadrilateral partitions). Their sizes satisfy $h << H$, $(H \to 0)$. Construct piecewise k-order and r-order finite element space S^h and S^H respectively . Let $u^h \in S^h$ be the Galerkin approximation of $u \in H_0^1(\Omega)$, and

$$I_H : C(\bar{\Omega}) \to S^H \tag{1.2}$$

be the interpolation operator, which satisfies the following there conditions:

1)$\|I_H w\|_{1,\infty} \le CH^{-1}\|I_H w\|_{0,\infty}$,

2)$\|u - I_H u\|_{0,\infty} \le C\|u\|_{0,\infty}, \forall u \in C(\Omega)$

3)$\|u - I_H u\|_{m,\infty} \le CH^{r+1-m}\|u\|_{r+1,\infty}, m = 0, 1$

Obriously the standard Lagrange interpolation operator and the projection interpolation operator proposed in [1] satisfy the above three conditions.

It has been shown in [1] the, if S^h and S^H are 1 or 2 order finite elements of uniform triangulation, or Q^k type elements defined on rectangular partition, then the following nonlocal superconvergence estimation holds when the parameters H and r properly macth.

$$\|\nabla(u - I_H u^h)\|_{0,\infty} \le Ch^{l-\epsilon} \tag{1.3}$$

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where $l = k+1$ Here we must impose strict assumptions on the partition, so its applicability is limited.

It has been stated that, if we substitue I_H by the L^2 interpolation operater from $L^2(\Omega)$ onto S^H, then for every irregular partitin J^h and J^H we have the following nonlocal L^2 superconvergence estimation, provided that the two parameters H and r properly match:

$$\|\nabla(u - Q_H u^h\|_0 \le Ch^{l-\epsilon}[\|u\|_l + \|u\|_{r+1}] \tag{1.4}$$

where $l = k+1$

The two processing techniques have their own advantages respectivwly. The former is very easy to perfom interpolation processing, and need not to solve algebraic equations, but strict assumptions must by imposed on the partition. The later is quite contrary.

In this paper, we conclute that ,for every irregular partitions J^H and J^h, if we properly choose the parameters H and r, (1.3) holds. This combines the advantages of the two techniques.

2. Main Results and Proof

Let

$$\bar{k} = \begin{cases} 1, & \text{when} k = 1 \\ 0, & \text{when} k > 1 \end{cases} \tag{2.1}$$

We have the following important Theorem.

Theorem 1. *Suppose that $u \in W^{r+1,\infty}(\Omega) \cap H_0^1(\Omega), (r > k)$,$u^h \in S^h$ is k-order Galerkin appronimation of u, the interpolation operator I_H satisfies the conditions 1),2),3), then we have basic estimation*

$$\|\nabla(u - I_H u^h\|_{0,\infty} \le C(H^r\|u\|_{r+1,\infty} + H^{-1}\|u - u^h\|_{0,\infty}) \tag{2.2}$$

or

$$\|\nabla(u - I_H u^h)\|_{0,\infty} \le C(H^r + H^{-1}h^{k+1}|\log h|^{\bar{k}})\|u\|_{r+1,\infty} \tag{2.3}$$

Proof. Using the triangular inequality, the condition 3), inverse property 1) and the condition 2) we have

$$\begin{aligned} \|\nabla(u - I_H u^h)\|_{0,\infty} &\le \|\nabla(u - I_H u)\|_{0,\infty} + \|\nabla(I_H u - I_H u^h\|_{0,\infty} \\ &\le CH^r\|u\|_{r+1,\infty} + CH^{-1}\|I_H(u - u^h)\|_{0,\infty} \\ &\le CH^r\|u\|_{r+1,\infty} + CH^{-1}\|u - u^h\|_{0,\infty} \end{aligned}$$

Then by the well-known L^∞estimation

$$\|u - u^h\|_{0,\infty} \le Ch^{k+1}|\log h|^{\bar{k}}\|u\|_{k+1,\infty}$$

we obtain (2.3)

Corollary 1. *Under the conditions of Theorem 1, for every $\epsilon > 0$, there exist positive integer number r and $H >> h$, such that*

$$\|\nabla(u - I_H u^h)\|_{0,\infty} \le Ch^{k+1-\epsilon}(\|u\|_{r+1,\infty} + \|u\|_{k+1,\infty}) \tag{2.4}$$

Proof. Select r properly Large so that $\frac{k+1}{r+1} < \epsilon$. Let

$$H = h^{\frac{k+1}{r+1}} \tag{2.5}$$

Replace H in (2.3) by the right side of (2.5) we get

$$\begin{aligned}&H^r + H^{-1}h^{k+1}|\log h|^{\bar{k}}\\&= h^{\frac{(k+1)r}{r+1}} + h^{k+1}|\log h|^{\bar{k}} \cdot h^{-\frac{k+1}{r+1}}\\&= O(h^{k+1-\epsilon})\end{aligned}$$

Thus (2.4) is proved.

Let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ mean that $\Omega_0 \subset \Omega_1 \subset \Omega$ and

$$\mathrm{dist}\{\partial\Omega_0 \setminus \partial\Omega, \partial\Omega_1 \setminus \partial\Omega\} = d > 0$$

where d is independent of h.

Then by local superconvergence estimation theory (See [5] Chapter 5) we have

Theorem 2. *Under the conditions of Theorem 1, if* $\Omega_0 \subset\subset \Omega_1$, *then when* $u \in W^{r+1,\infty}(\Omega_1) \cap H_0^1(\Omega)$ *The following estimation holds*

$$\|\nabla(u - I_H u^h)\|_{0,\infty,\Omega_0} \leq h^{k+1-\epsilon}\|u\|_{r+1,\infty,\Omega_1} + Ch^{-\epsilon}\|u - u^h\|_{-s,\Omega}$$

Where $s \geq 0$ *is any fixed positive integer.*

3. The Superconvergence of Placement

Generally speaking, from the priori estimation

$$\|u\|_{k+1} \leq C\|f\|_{k-1}$$

we can derive that

$$|\textstyle\int_\Omega (u - u^h)dx| \leq C\|u - u^h\|_{-k+1} \leq Ch^{2k}\|u\|_{k+1} \tag{3.1}$$

But

$$\int_\Omega |u - u^h|dx = O(h^{k+1})$$

is the optimal order extimation which can not by improved, so, the error is a function that takes positive and negative values alternatively, therefor it has a lot of zero points. Obriously, these zero opoints are superconvergence points of placement. It has been pointed out that, if Z_0 is a local symmetric point of partition J^h, then it is a superconvergence point. In [5], [6], a large number of superconvergence points are also discovered. We denote by E the set of all these superconvergence points, Then we have the following Theorem which is applicable for Lagrange interpolation I_H.

Theorem 3. $z_0 \in E$ *is a interpolation point, then on the element* e *to which* z_0 *belongs we have superconvergence estimation*

$$\|u - I_H u^h\|_{0,\infty,e} \leq Ch^{k+2-\epsilon}$$

where C depends on u but is independent of h.

Proof. Because Z_0 is a interpolation point of I_H and is also a superconvergence point, we have

$$(u - I_H u^h)(z_0) = (u - u^h)(z_0) = O(h^{k+2-\epsilon})$$

thus for any $x \in e$ by Theorem2 we obtain

$$|(u - I_H u^h)(x)| \leq Ch|\nabla(u - I_H u^h)|_{0,\infty} + |(u - I_H u^h)(z_0)| = O(h^{k+2-\epsilon})$$

The proof is completed.

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