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THE INTERPOLATED COEFFICIENT FEM AND ITS APPLICATION IN COMPUTING THE MULTIPLE SOLUTIONS OF SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. Convergence and superconvergence of the interpolated coefficient finite element method (ICFEM) are discussed as the ICFEM reduces the computation cost greatly. Further, the ICFEM is implemented to compute the multiple solutions of some semilinear elliptic problems.

Key Words. Superconvergence, convergence, the interpolated coefficient finite element method, semilinear elliptic problem, multiple solutions.

1. Introduction

As semilinear partial differential equations arise in physics, biology, energy, and engineering, their study has attracted the attention of many pure and applied mathematicians and physicists. It is well known that the standard finite element method plays a very important role in solving these problems. Unfortunately, the computation cost for implementing the finite element method is usually very expensive.

To overcome this difficulty, a simple and graceful idea called the interpolated coefficient finite element method (ICFEM), which was originally inspired by solving semilinear parabolic problems, was proposed by M. Zlámal [12] et al. Further, he obtained the error estimate $\|(u_h - u)(t)\| = O(h^2)$ for the linear element solution $u_h(t)$ with an unproven assumption that $||u_h(t)||_\infty$ is bounded. Later, Larsson, Thomée, and Zhang [9] studied the linear triangular finite element solution $u_h(t)$ and obtained the error estimate $\|(u_h - u)(t)\| = O(h)$. In [3], implementing some superconvergence techniques, Chen, Larsson, and Zhang derived an almost optimal convergence order $\|(u_h - u)(t)\| = O(h^2 \ln h)$ on piecewise uniform triangular meshes.

In this paper, we show that the interpolated coefficient finite element method for solving the semilinear elliptic equations has the same convergence order or even superconvergence properties as those of the standard finite element method. Moreover, combined with the Improved Search-extension Method [4, 11], the ICFEM is used to compute the multiple solutions of some typical semilinear elliptic problems.

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2. Convergence and superconvergence of the ICFEM

For completeness, below the interpolated coefficient finite element method for solving semilinear elliptic problems is introduced first.

Consider a semilinear elliptic problem with zero Dirichlet boundary condition, i.e.,

$$-D_i(a_{ij}D_ju)+au+f(u)=0 \text{ in } \Omega,\ \ u=0 \text{ on } \partial\Omega, \tag{1}$$

with its weak form

$$Q(u,v)=A(u,v)+(f(u),v)=0,\ \forall v\in S_0, \tag{2}$$

where Ω is a 2-dimensional bounded domain with Lipschitz boundary $\partial\Omega$, $S_0=\{u\in H^1(\Omega), u=0 \text{ on } \partial\Omega\}$, and the bilinear form

$$A(u,v)=\int_\Omega (a_{ij}(x)D_iuD_jv+a(x)uv)dx$$

is assumed to be bounded and S_0-coercive.

We assume that the domain Ω is subdivided into a finite number of elements τ with the subdivision J^h and let $Z_h=\{x_j\}_1^M$ be the set of all interior nodes. Denote by $S^h\subset S_0$ the n-degree finite element subspace and $\{N_j(x)\}_1^M$ the bases of S^h. It is well known that the standard finite element solution $u_h\in S^h$ of (1) can be expressed as $u_h(x)=\sum_{j=1}^M U_jN_j(x)\in S^h$, $U_j=u_h(x_j)$, and satisfies

$$A(u_h,v)+(f(u_h),v)=0, \forall v\in S^h. \tag{3}$$

By taking $v=N_i$, $i=1,2,...,M$, (3) leads to a nonlinear system of equations

$$\sum_{j=1}^M A(N_j,N_i)U_j-(f(\sum_{j=1}^M N_j(x)U_j),N_i)=0,\ \ i=1,2,...M, \tag{4}$$

which is often solved by the Newton method. It is known that the Jacobi matrix is the main concern in the implementation of the Newton method. A direct computation shows that the Jacobi matrix of (4) is

$$J=\{A(N_j,N_i)-(f'(\sum_{k=1}^M N_kU_k)N_j,N_i)\}_{M\times M}, \tag{5}$$

which has to be updated repeatedly as the iterations proceed. Obviously, the integrations for the second term in (5) are quite large and will result in the very time-consuming and expensive computation of the Newton method.

Now we introduce the interpolated coefficient finite element method for solving (1). Substitute the interpolation $I_hf(u_h)=\sum_{j=1}^M N_j(x)f(U_j)$ with $U_j=u_h(x_j)$ rather than $f(u_h)$ into (3) and still denote the interpolated coefficient finite element solution $u_h=\sum_{j=1}^N U_jN_j(x)$. Then we obtain a new finite element equation

$$A(u_h,v)+(I_hf(u_h),v)=0,\ \ \forall v\in S^h. \tag{6}$$

As a result, we obtain a nonlinear algebraic system of equations

$$\sum_{j=1}^M(k_{ij}U_j+m_{ij}f(U_j))=0,\ \ i=1,2,...,M, \tag{7}$$

where the elements of the stiffness matrix $k_{ij} = A(N_j, N_i)$ and the elements of the mass matrix $m_{ij} = (N_j, N_i)$ can be computed once. The Jacobi matrix of (7) is

$$J = \{k_{ij} + m_{ij} f'(U_j)\}_{M\times M}. \tag{8}$$

As k_{ij} and m_{ij} are given, the Jacobi matrix can be obtained simply by multiplying m_{ij} by $f'(U_j)$. Therefore, the computation is reduced greatly compared with that for solving (4).

For simplicity, we assume that Ω is a planar convex polygonal domain. It is well known that the solution $u \in S_0$ of problem $A(u, v) = (h, v), v \in S_0$, satisfies $u \in H^2(\Omega)$ and $||u||_2 \le C||h||$ if $h \in L^2(\Omega)$ is independent of u. We assume that $f(u)$ is suitably smooth, $|f(u)| \le C(1 + |u|)^p$. Moreover, $f'(u) \le 0$ somewhere is admissible. Further, the following basic assumption is essential.

Assumption A. Assume that $u \in H^{n+1}(\Omega) \bigcap S_0$ is a solitary solution of (1), i.e.,

$$Q(u, v) = 0$$

and there is a small neighborhood $N_\epsilon(u) = \{w \in S_0, ||w - u||_1 < \epsilon\}$ such that no solution of (1) exists in $N_\epsilon(u)$. More precisely, for arbitrary w satisfying $w \in S_0, w \ne 0, ||w||_1 \le \epsilon$, there exists a $c = c(u) > 0$, independent of w, such that

$$|Q(u + w, v)| \ge c||w||_1||v||_1,\ v \in S_0.$$

First, we introduce an auxiliary elliptic operator

$$B(u; w, v) = A(w, v) + (f'(u)w, v).$$

Then we have the following theorem to guarantee the uniqueness of the solution of the auxiliary elliptic problem

$$B(u; w, v) = A(w, v) + (f'(u)w, v) = (g, v),\ \forall v \in S_0.$$

Theorem 2.1. *Under the assumption* **(A)***, the problem*

$$B(u; w, v) = A(w, v) + (f'(u)w, v) = 0,\ \forall v \in S_0, \tag{9}$$

has a unique solution $w = 0$.

Proof: We shall prove the conclusion by contradiction. Suppose that $w \ne 0$ and $w \in S_0 \cap H^2(\Omega)$ is a solution of (9). For t sufficiently small, the Taylor expansion implies

$$Q(u+tw, v) = Q(u, v)+t[A(w, v)+(f'(u)w, v)]+t^2(\phi(u, w)w^2, v) = t^2(\phi(u, w)w^2, v),$$

where $\phi(u, w) = \int_0^1 f''(u + tsw)(1 - s)ds$. For any $p, p' \gg 1$ large enough, by the Hölder inequality, we have

$$|Q(u + tw, v)| \le C||tw||^2_{0,2p}||v||_{0,p'} \le C||tw||^2_1||v||_1,\ v \in S_0.$$

This contradicts the assumption **(A)**. ∎

Theorem 2.2. *Under the assumption* **(A)***, suppose that the solution* $u \in H^{n+1}(\Omega)$ *and the subdivision* J^h *is quasi-uniform* [1]*. Then, for* $h > 0$ *sufficiently small, the* $n-$*degree interpolated coefficient finite element solution* u_h *has an optimal order convergence estimate*

$$\|u_h - u\| \le C(u)h^{n+1},$$

where the constant $C(u)$ *depends on the norm* $\|u\|_{n+1,\Omega}$.

Proof: Set $e = u - u_h$. According to (2) and (6), e satisfies

$$A(e, v) + (f(u) - I_h f(u_h), v) = 0,\ v \in S^h. \tag{10}$$

Then we introduce an auxiliary elliptic projection $R_h u \in S^h$ such that

$$B(u; u - R_h u, v) = 0,\ v \in S^h. \tag{11}$$

It is known that

$$||u - R_h u|| \leq C(u)h^{n+1}.$$

The auxiliary elliptic projection $R_h u$ will serve as a comparing function in the discussion later. Further, e can be rewritten as $e = u - u_h = u - R_h u + \theta$ with $\theta = R_h u - u_h$. The combination of (10) and (11) implies

$$B(u;\theta,v) + (r_h, v) = 0,\ v \in S^h, \tag{12}$$

where

$$r_h = (f(u) - I_h f(u)) + I_h(f(u) - f(u_h)) - f'(u)(u - u_h) = r_1 + r_2 + r_3. \tag{13}$$

Denote by $\{\phi_j(x)\}_1^k$ the shape functions on the element τ. Then the interpolation of u on τ can be expressed as $I_h u = \sum_{j=1}^k u_j\phi_j(x)$ which satisfies

$$||u - I_h u||_{0,p,\tau} \leq Ch^{n+1}||u||_{n+1,p,\tau},\ 1 \leq p \leq \infty. \tag{14}$$

Moreover, a similar estimate for r_1 also holds. On the other hand, implementing the Taylor expansion, we have

$$\begin{aligned} f(u(x_j)) - f(u_h(x_j)) &= \{f'(u) + (f'(u(x_j)) - f'(u(x)))\}(u(x_j) - u_h(x_j)) \\ &- \frac{1}{2}f''(\xi_j)(u(x_j) - u_h(x_j))^2. \end{aligned}$$

Thus, on the element τ,

$$\begin{aligned} (r_2 + r_3)|_\tau &= f'(u)(I_h u - u) + \sum_{j=1}^k \{(f'(u(x_j)) - f'(u(x)))(u(x_j) - u_h(x_j)) \\ &- \frac{1}{2}f''(\xi_j)(u(x_j) - u_h(x_j))^2\}\phi_j(x). \end{aligned}$$

Therefore,

$$\begin{aligned} r_h|_\tau &= F + \sum_{j=1}^k \{(f'(u(x_j)) - f'(u(x)))(u(x_j) - u_h(x_j)) \\ &\qquad -\frac{1}{2}f''(\xi_j)(u(x_j) - u_h(x_j))^2\}\phi_j(x) \end{aligned} \tag{15}$$

with $F = (f(u) - I_h f(u)) - f'(u)(u - I_h u)$.

By the Sobolev embedding theorem, $d = 2,\ p = 2$ is a limit case,

$$H^{n+1}(\Omega) \hookrightarrow C^\beta(\bar{\Omega}),\ \beta = n + 1 - d/p - \epsilon = n - \epsilon,\ \epsilon > 0,$$

we have

$$\max_{x,y\in\tau} |f'(u(x)) - f'(u(y))| \leq C \max_{x,y\in\tau} |u(x) - u(y)| \leq Ch^\alpha,\ \alpha = \min(1, \beta). \tag{16}$$

Thus,

$$|r_h|_\tau| \leq |F| + Ch^\alpha \max_\tau |I_h u - u_h| + C_2 \max_\tau |I_h u - u_h|^2. \tag{17}$$

Combining (17) and the inverse inequalities

$$||I_h u - u_h||_{0,\infty,\tau} \leq Ch^{-1}||I_h u - u_h||_{0,2,\tau},$$
$$||I_h u - u_h||_{0,\infty,\tau} \leq Ch^{-1/2}||I_h u - u_h||_{0,4,\tau},$$

we obtain

$$|r_h|_\tau| \leq |F| + Ch^{\alpha-1}||I_h u - u_h||_{0,2,\tau} + Ch^{-1}||I_h u - u_h||_{0,4,\tau}^2.$$

Consequently,

$$||r_h||_{0,2,\tau}^2 \leq C||F||_{0,2,\tau}^2 + Ch^{2\alpha}||I_h u - u_h||_{0,2,\tau} + C||I_h u - u_h||_{0,4,\tau}^4,$$

as the area of τ is $O(h^2)$. Hence

$$||r_h||_{0,2} \le C||F||_{0,2} + Ch^\alpha||I_h u - u_h||_{0,2} + C||I_h u - u_h||_{0,4}^2,$$

in which all the norms are defined on Ω. Combined with the inverse inequality on Ω, $||I_h u - u_h||_{0,4} \le Ch^{-1/2}||I_h u - u_h||_{0,2}$ and (14), the inequality above implies

$$||r_h|| \le Ch^{n+1} + Ch^\alpha||e|| + Ch^{-1}||e||^2. \tag{18}$$

To employ the duality argument, we construct an auxiliary function $w_h \in S^h$ satisfying

$$B(u; v, w_h) = (\theta, v), \;\; v \in S^h. \tag{19}$$

Note that, no matter whether the bilinear form $B(u; v, w)$ for fixed u is coercive or not (under assumption **(A)**), a priori estimate $||w_h||_1 \le C||\theta||$ always holds. By (19) and (12)

$$||\theta||^2 = B(u; \theta, w_h) = (-r_h, w_h) \le ||r_h||||w_h|| \le C||r_h||||\theta||.$$

Then we get $||\theta|| \le C||r_h||$ and thereby,

$$||e|| \le C_1 h^{n+1} + C_2 h^{-1}||e||^2, \tag{20}$$

in which the error estimate $||u - R_h u|| \le C(u)h^{n+1}$ is used.

We shall adopt a simplified homotopy argument [7] to prove the conclusion from (20). Let $h_0 > 0$ be such that

$$||e|| < 2C_1 h^{n+1} \tag{21}$$

holds for any $h < h_0$. Substituting (21) into the right term of (20), we get

$$||e|| \le C_1(1 + 4C_1C_2h^n)h^{n+1}.$$

Taking h_1 small such that $4C_1C_2h_1^n < 1$, we still have estimate (21) for all $h < \min\{h_0, h_1\}$. Consequently, (21) is valid. ∎

Actually, the most exciting fact about the ICFEM is that it has the same superconvergent property as that of the standard finite element method. In Theorem 2.3, we shall concentrate on the discussion for the superconvergence of the ICFEM.

Below we assume that the rectangular or triangular subdivision of domain is uniform, and denote T_h a set of symmetric points, i.e., angular nodes and midpoints on sides (and the center for the rectangular elements also). Introducing the discrete norm

$$||g||_{T_h} = (\sum_{z \in T_h} |g(z)|^2 h^2)^{1/2},$$

we know that for linear elliptic problem the quadratic finite element solution $R_h u$ and interpolation $I_h u$ have the superconvergence estimate

$$||u - I_h u||_{T_h} + ||u - R_h u||_{T_h} = O(h^4)||u||_4. \tag{22}$$

Now we shall prove the same conclusion for the semilinear elliptic problem (1).

Theorem 2.3. *Suppose assumption* **(A)** *holds and the solution* $u \in H^4(\Omega)$. *Let* u_h *be the quadratic rectangular or triangular interpolated coefficient finite element solution of (1) on uniform meshes. Then* u_h *superconverges on* T_h, *i.e.,*

$$||u - u_h||_{T_h} = O(h^4).$$

Proof: The elliptic projection $R_h u$ satisfying (11) will still serve as a comparing function. According to (12), $\theta = u_h - R_h u$ satisfies

$$B(u; \theta, v) = -(r_h, v), v \in S^h.$$

Noting that $\alpha = \min(1, \beta) = \min(1, 2 - \epsilon) = 1$ in (16) and using the approximate orthogonality of interpolation

$$|(u - I_h u, v)| \leq Ch^4 ||u||_4 ||v||_1,\ \forall v \in S^h, \tag{23}$$

by (15), we obtain

$$|(r_h, v)| \leq C(u)(h^4 + h||e|| + h^{-1}||e||^2)||v||_1 \leq C(u)h^4||v||_1,$$

where the results in Theorem 2.2 are used. Due to properties of the auxiliary function w_h defined in (19), we get

$$||\theta||^2 = B(u; \theta, w_h) = -(r_h, w_h) \leq C(u)h^4||w_h||_1 \leq C(u)h^4||\theta||,$$

and the superconvergence estimate

$$||\theta||_{T_h} \leq C||\theta|| \leq C(u)h^4. \tag{24}$$

Finally, combining (22) and using the equality

$$e = u - u_h = (u - R_h u) + (R_h u - u_h) = u - R_h u + \theta,$$

the conclusion directly follows. ∎

Remark 1. From the proof above, the coerciveness of $B(u; w, v)$ is not required for both the optimal and superconvergent estimates. This means that the condition $f'(u) \geq 0$ is not necessary. Consequently, the ICFEM can be used conveniently in the computation of the multiple solutions of the semilinear elliptic equations in which $f'(u) \leq 0$ or $f'(u)$ is sign-changing, as will be seen later.

Remark 2. In 2-dimensional polygonal domains, solutions of the problem $A(u, v) = (f, v)$, $v \in S_0$, in general, have only a lower regularity [2, 8]. But if $f = 0$ on $\partial\Omega$, then the regularity of the solution is higher. For example, the higher regularity of the solution on a rectangular domain can be seen in [2]. Actually, [10] defined a space $\dot{H}^s(\Omega)$ and its norm. Especially, when s is an integer,

$$\dot{H}^s = \{u :\ u \in H^s,\ \Delta^j u = 0 \ on\ \Gamma,\ j < s/2\},$$

whose norm is equivalent to Sobolev norm. As a result, the higher regularity of the solution depends on whether $u = 0, \Delta u = 0, \Delta^2 u = 0, \ldots.$ For nonlinear problem (25) with $f(u)$ expressed in (28) and (29) below, a direct computation shows that we have $u = 0, \Delta u = 0, \Delta^2 u = 0$. For example,

$$\Delta u = -f(u) = -f(0) = 0,\ \Delta^2 u = -\Delta f(u),$$

$$\Delta u^3 = (u^3)_{xx} + (u^3)_{yy} = (3u^2 u_x)_x + (3u^2 u_y)_y = 3u^2 \Delta u + 6u(u_x u_x + u_y u_y) = 0.$$

Therefore the $\dot{H}^4(\Omega)$-regularity can be guaranteed in these examples.

3. The application of the ICFEM

It is well known that the following problem

$$\Delta u + f(u) = 0 \text{ in } \ \Omega,\ u = 0 \ \text{ on } \ \ \partial\Omega, \tag{25}$$

has multiple and possibly even infinitely many solutions (in the case that f is an odd function in the variable u) under some growth conditions. In this section, we shall show how the ICFEM can be implemented to compute the multiple solutions of semilinear elliptic problems combined with the so-called improved search-extension method(ISEM)[4, 11].

For completeness, we shall list the improved search-extension method below. Details can be seen in [11].

Improved Search-extension Algorithm

Step 1. Compute the eigenpairs $\{\lambda_j, \phi_j\}$ of the eigenvalue problem:

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Step 2. Search for the initial values in some subspace $S_N \subset S$.

Assume that $0 < \lambda_1 \leq \lambda_2 \leq .. \leq \lambda_N$ and $\{\phi_j\}_1^N$ form a normalized orthogonal system. Then the solution of (25) can be approximated by the following series

$$u(x) = \sum_{j=1}^{N} a_j \phi_j(x) \in S_N,$$

where the coefficients $\mathbf{a} = \mathbf{a}(N) = [a_1, a_2, ..., a_N]^T$ can be determined by solving

$$\begin{aligned} \frac{\partial J(u)}{\partial a_i} &= F_i(\mathbf{a}) = \lambda_i a_i - g_i(\mathbf{a}) = 0, \\ g_i(\mathbf{a}) &= (f(\sum_{j=1}^{N} a_j \phi_j), \phi_i), \ i = 1, 2, 3, ..., N, \end{aligned} \tag{26}$$

where $J(u)$ is the corresponding functional of (25). Suppose λ_l is a k-fold eigenvalue and its eigenfunctions span the subspace $S_k^* \subset S_N$ with N appropriately large. The solutions of (26) can be searched out and will serve as the initial guesses of the solutions u_l of (25). Actually, we can take $S_N = S_k^*$ in many simple cases.

Step 3. Discretize (25) by the interpolated coefficient finite element method.

By the interpolated coefficient finite element method, the discrete form of (25) is

$$F(U) = K_1 U - K_2 F_1(U) = 0, \tag{27}$$

where K_1 and K_2 are $M \times M$ matrices defined by $K_1(i,j) = (\nabla N_i, \nabla N_j)$, $K_2(i,j) = (N_i, N_j)$, $i, j = 1, 2, ...M$, with N_i, $i = 1, 2, ...M$, the bases of the FEM at the interior nodes, and $F_1(U) = [f(U_1), f(U_2), ...f(U_M)]^T$.

Step 4. Solve (27) by the numerical extension method based on the Newton approach.

Define a homotopy mapping

$$H(U, t) = tF(U) + (1 - t)G(U), \ \ 0 \leq t \leq 1,$$

where $G(U) = DF(U^0)(U - U^0)^T$ with $U^0 = u_l^0(i), i = 1, 2, ..., M$ where u_l^0 is the initial guess obtained in step 2. Then follow the standard numerical extension method to solve (27).

Now the ISEM described above is used to compute the solutions of semilinear Dirichlet problem (25) with

$$f(u) = f_1(u) = u^3 \tag{28}$$

or

$$f(u) = f_2(u) = \begin{cases} u^3, & \text{if } \ u \geq 0, \\ u^5, & \text{if } \ u \leq 0, \end{cases} \tag{29}$$

where $\Omega = \Omega_L = [-1, 1] \times [0, 1] \cup [-1, 0] \times [-1, 0]$, is a non-convex L-shaped domain.

For these two cases, Ding Z.H., et al. [6] could only get at most four solutions numerically in general domains. Actually $f_1(u)$ in (28) is a typical example of odd nonlinearity. It is known that there exist an infinite number of solutions. Indeed,

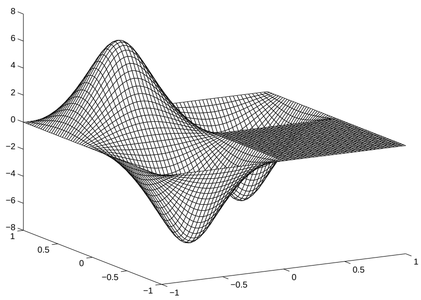

FIGURE 1. A solution of (25) with $f_1(u)$ w.r.t λ_3.

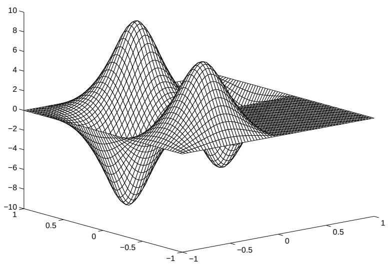

FIGURE 2. A solution of (25) with $f_1(u)$ w.r.t λ_4.

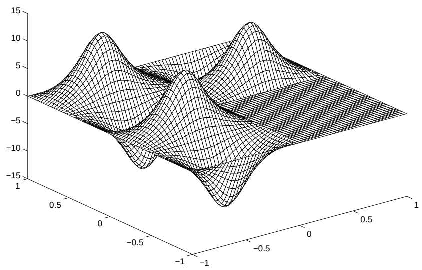

FIGURE 3. The 1st solution of (25) with $f_1(u)$ w.r.t $\lambda_8 = \lambda_9$.

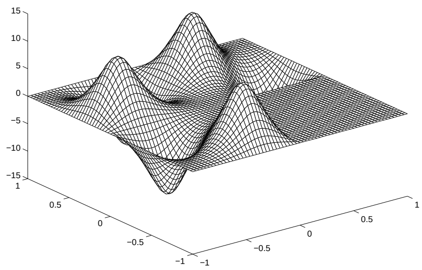

FIGURE 4. The 2nd solution of (25) with $f_1(u)$ w.r.t $\lambda_8 = \lambda_9$.

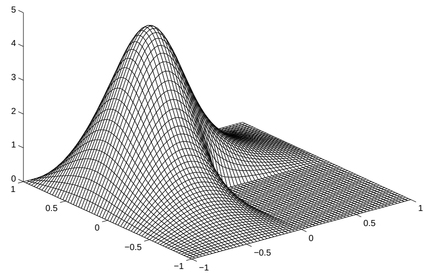

FIGURE 5. A positive mountain pass solution of (25) with $f_2(u)$ w.r.t. λ_1.

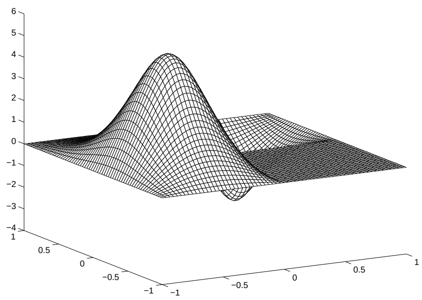

FIGURE 6. A solution of (25) with $f_2(u)$ w.r.t. λ_2.

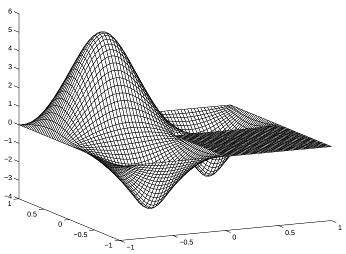

FIGURE 7. A solution of (25) with $f_2(u)$ w.r.t. λ_3.

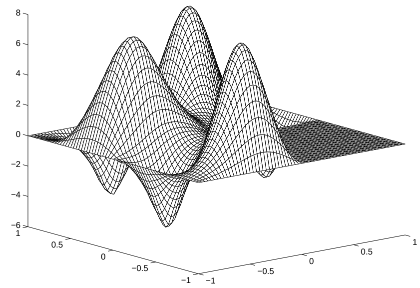

FIGURE 8. A solution of (25) with $f_2(u)$ w.r.t. $\lambda_8 = \lambda_9$.

the ISEM can obtain solutions of (25) as many as one wants when $f(u)$ is odd, no matter whether the considered domains are symmetric. For the non-odd case with $f_2(u)$ in (29), three solutions have been verified to exist theoretically. However, the ISEM can also compute more than 20 solutions in Ω_L. Actually, the ISEM has been used to compute the solutions of other problems of type (25) with non-odd nonlinearity and obtained much more than three solutions. Based on the numerical experiments, we have a conjecture as follows:

Conjecture: If $f(x,u)$ satisfies some growth conditions (see [4, 5, 11]) and

$$\lim_{|u|\to\infty} \frac{f(x,u)}{u} = +\infty,$$

then (25) has an infinite number of solutions.

For simplicity, we only show 4 solutions of (25) with odd nonlinearity $f_1(u)$ in Figures 1-4 and 4 solutions of (25) with non-odd nonlinearity $f_2(u)$ in Figures 5-8, in which $\lambda_8 = \lambda_9$ are double eigenvalues.

Remark 3. As the ISEM applies the numerical extension method based on the Newton approach in which a lot of iterations are needed, the ICFEM reduces the computation cost dramatically.

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