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HIGH RESOLUTION SCHEMES AND DISCRETE ENTROPY CONDITIONS FOR 2-D LINEAR CONSERVATION LAWS*[,1)]

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Abstract

In this paper, fully discrete entropy conditions of a class of high resolution schemes with the MmB property are discussed by using the theory of proper discrete entropy flux for the linear scalar conservation laws in two dimensions. The theoretical resluts show that the high resolution schemes satisfying fully discrete entropy conditions with proper discrete entropy flux cannot preserve second order accuracy in the case of two dimensions.

1. Introduction

Consider 2-D hyperbolic conservation laws:

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}+\frac{\partial g(u)}{\partial y}=0\ ,\\&u(x,y,0)=u_0(x,y)\ .\end{aligned}\tag{1.1}$$

The research of numerical methods for the equations has been developed rapidly in this decade. Since appearance of the concept of TVD(total variation diminishing) schemes, various high resolution schemes (TVD,TVB (total variation bounded[6]), ENO (essentially non-oscillatory[2]), MmB (Maxima minima Bounded preserving[10]) schemes etc.) have been applied successfully to computational fluid dynamics. Recently, the convergence of difference schemes by using every ways are discussed. The convergence of numerical methods for hyperbolic conservation laws depends on the entropy condition and some kinds of stability of difference solutions such as the total variation stability. However, there exists some relationship between the entropy condition and nonlinear stability of numerical solutions. Previously constructing difference schemes always based on some kinds of total variation stability (TVD, TVB, ENO, and MmB etc.). Then these schemes are modified so that the entropy condition can be satisfied. Some quantities depending on the grid width are often introduced when these modifications are made. Generally, the difference schemes only depend on the grid ratio

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but independ of the grid width. So, it is meaningful to construct schemes satisfying the entropy condition without intruducing the quantities depending on the grid width. M. Merriam[3] and T. Sonar[8] put out the concept of the proper discrete entropy flux. That is discretizating the entropy flux by using the proper way so that the entropy condition can be satisfied and simultaneously the difference solution satisfies some kind of total variation stability. In [11], N. Zhao and H. Wu discussed the relationship between entropy conditions and nonlinear stability for 1-D scalar linear conservation laws, and obtained second order accurate TVD schemes using limiters. Based on the similar procedure, in this paper, we discuss the relationship between the discrete entropy conditions and the MmB property in the case of two dimensions. Unfortunately, the theoretical results show that a class of high resolution schemes satisfying the discrete entropy condition with the proper discrete entropy flux cannot preserve second order accuracy for linear scalar hyperbolic conservation laws in two dimensions.

2. MmB Schemes in Two Dimensions

In this section, let us review the MmB schemes in two dimensions introduced by H. Wu and S. Yang in [10].

Consider the difference schemes for 2-D scalar equations

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial y}=0\ ,\ a>0\ ,\ b>0\ , \tag{2.1}$$

where a and b are constants. Let $\lambda = a\Delta t/\Delta x,\ \mu = b\Delta t/\Delta y \geq 0$, be the Courant numbers, and $u^n_{j,k}$ the approximating function value of the solution at the mesh point (x_j, y_k, t^n).

In general, we have the following partially 'upwind' second order accurate scheme to approximate the equation (2.1) (the notations are conventional, $u_{j,k} = u^n_{j,k}$)

$$\begin{aligned}u^{n+1}_{j,k} =& u_{j,k}-\lambda(u_{j,k}-u_{j-1,k})-\frac{\lambda(1-\lambda)}{2}(u_{j+1,k}-2u_{j,k}+u_{j-1,k})\\ &-\mu(u_{j,k}-u_{j,k-1})-\frac{\mu(1-\mu)}{2}(u_{j,k+1}-2u_{j,k}+u_{j,k-1})\\ &+\lambda\mu\left((u_{j,k}-u_{j-1,k})-(u_{j,k-1}-u_{j-1,k-1})\right).\end{aligned} \tag{2.2}$$

The scheme (2.2) is not MmB, it may cause oscillations for non-smooth solutions. So, H. Wu and S. Yang constructed the following flux limited version of the modification of (2.2) in [10]

$$\begin{aligned}u^{n+1}_{j,k} =& u_{j,k}-\lambda\Delta_{j-\frac12,k}u-\frac{\lambda(1-\lambda)}{2}\left(\varphi_{j,k}\Delta_{j+\frac12,k}u-\varphi_{j-1,k}\Delta_{j-\frac12,k}u\right)\\ &+\frac{\lambda\mu}{2}\left[\Theta_{j,k-\frac12}\Delta_{j,k-\frac12}u-\Theta_{j-1,k-\frac12}\Delta_{j-1,k-\frac12}u\right]\\ &-\mu\Delta_{j,k-\frac12}u-\frac{\mu(1-\mu)}{2}\left(\psi_{j,k}\Delta_{j,k+\frac12}u-\psi_{j,k-1}\Delta_{j,k-\frac12}u\right)\end{aligned}$$

$$+ \frac{\lambda\mu}{2}\left[\chi_{j-\frac{1}{2},k}\Delta_{j-\frac{1}{2},k}u - \chi_{j-\frac{1}{2},k-1}\Delta_{j-\frac{1}{2},k-1}u\right] \tag{2.3}$$

or

$$\begin{aligned} u_{j,k}^{n+1} =& u_{j,k} - \lambda\left[1 + \frac{1-\lambda}{2}\left(\left(\frac{\varphi}{r}\right)_{j,k} - \varphi_{j-1,k}\right)\right. \\ & \left. - \frac{\mu}{2}\left(\chi_{j-\frac{1}{2},k} - \chi_{j-\frac{1}{2},k-1}/q_{j-\frac{1}{2},k-1}\right)\right]\Delta_{j-\frac{1}{2},k}u \\ & - \mu\left[1 + \frac{1-\mu}{2}\left(\left(\frac{\psi}{s}\right)_{j,k} - \psi_{j,k-1}\right)\right. \\ & \left. - \frac{\lambda}{2}\left(\Theta_{j,k-\frac{1}{2}} - \Theta_{j-1,k-\frac{1}{2}}/p_{j-1,k-\frac{1}{2}}\right)\right]\Delta_{j,k-\frac{1}{2}}u \ , \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} r_{j,k} &= \frac{\Delta_{j-\frac{1}{2},k}u}{\Delta_{j+\frac{1}{2},k}u} s_{j,k} = \frac{\Delta_{j,k-\frac{1}{2}}u}{\Delta_{j,k+\frac{1}{2}}u} \ , \\ p_{j,k-\frac{1}{2}} &= \frac{\Delta_{j,k-\frac{1}{2}}u}{\Delta_{j-1,k-\frac{1}{2}}u} q_{j-\frac{1}{2},k} = \frac{\Delta_{j-\frac{1}{2},k}u}{\Delta_{j-\frac{1}{2},k-1}u} \ , \end{aligned} \tag{2.5}$$

and $\varphi(r)$, $\psi(s)$, $\Theta(p)$, and $\chi(q)$ are flux limiters, nonnegative, and equal to zero for negative arguments r, s, p, and q defined in (2.5).

It is natural to assume that $\frac{\varphi}{r}$, $\frac{\psi}{s}$, $\frac{\Theta}{p}$, $\frac{\chi}{q}$ and φ, ψ, Θ, χ obey the same limitations, $\mathbf{X}(\theta)$. In [10], the scheme (2.4) is MmB provided

$$\mathbf{X}(\theta) \le \mathbf{Min}\{\frac{2}{1-\lambda+\mu} \ , \ \frac{2}{1-\mu+\lambda}\} \ . \tag{2.6}$$

If let λ, $\mu \le \alpha \le 1$, then

$$\mathbf{X}(\theta) \le \frac{2}{1+\alpha} \ . \tag{2.7}$$

As well known, the scheme (2.3) is second order accurate in the smooth region of the solution of the equation (2.1) under the limiters (2.7) because the critical point $\mathbf{X}(1) = 1$ is located in the limiting area of (2.7).

In the next section we will discuss the discrete entropy condition of the scheme (2.4).

3. The Entropy Condition of MmB Schemes

Consider 2-D conservative schemes to approximate the equations (1.1)

$$u_{j,k}^{n+1} = u_{j,k} - \lambda(f_{j+\frac{1}{2},k} - f_{j-\frac{1}{2},k}) - \mu(g_{j,k+\frac{1}{2}} - g_{j,k-\frac{1}{2}}) \tag{3.1}$$

where

$$\begin{aligned} f_{j+\frac{1}{2},k} &= h^x(u_{j-s+1,k-p}, ..., u_{j+s,k+p}) \ , \quad h^x(u, ..., u) = f(u) \ , \\ g_{j,k+\frac{1}{2}} &= h^y(u_{j-s,k-p+1}, ..., u_{j+s,k+p}) \ , \quad h^y(u, ..., u) = g(u) \ , \end{aligned} \tag{3.2}$$

and $\lambda = \Delta t/\Delta x$, $\mu = \Delta t/\Delta y$.

Similar to the case in [3] and [11], the 2-D conservative schemes based on variables can be written as:

$$u_{j,k}^{n+1} = u_{j,k} - \lambda(f(u_{j+\frac{1}{2},k}) - f(u_{j-\frac{1}{2},k})) - \mu(g(u_{j,k+\frac{1}{2}}) - g(u_{j,k-\frac{1}{2}})) \ , \tag{3.3}$$

where $u_{j+\frac{1}{2},k}$ and $u_{j,k+\frac{1}{2}}$ are some kinds of averages of values $u_{j-s,k-p}$, ..., $u_{j+s,k+p}$.

As well known, the weak solution of (1.1) is not unique. Let the function $U(u)$ be any convex function, so-called the entropy function, and corresponded to the functions $F(u)$ and $G(u)$ (the entropy flux) satisfy $F'(u) = U'(u)f'(u)$ and $G'(u) = U'(u)g'(u)$. (U, F, G) is called an entropy pair. If the weak solution u of (1.1) satisfies the inequality:

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} + \frac{\partial G(u)}{\partial y} \leq 0 \tag{3.4}$$

in distribution to every entropy pair (U, F, G), then the weak solution is the unique physical solution of (1.1), the inequality (3.4) is called the entropy inequality (or the entropy condition).

Corresponding to the conservative schemes (3.1), the discrete entropy inequality is defined as

$$U(u_{j,k}^{n+1}) - U(u_{j,k}) + \lambda(F_{j+\frac{1}{2},k} - F_{j-\frac{1}{2},k}) + \mu(G_{j,k+\frac{1}{2}} - G_{j,k-\frac{1}{2}}) \leq 0 \ , \tag{3.5}$$

where the discrete entropy flux

$$\begin{aligned} F_{j+\frac{1}{2},k} &= H^x(u_{j-s+1,k-p}, ..., u_{j+s,k+p}) \ , \quad H^x(u, ..., u) = F(u) \ , \\ G_{j,k+\frac{1}{2}} &= H^y(u_{j-s,k-p+1}, ..., u_{j+s,k+p}) \ , \quad H^y(u, ..., u) = G(u) \ . \end{aligned} \tag{3.6}$$

Similar to the case in [3] and [11], we define the 2-D proper discrete entropy flux corresponding to the schemes (3.3) as following:

$$F_{j+\frac{1}{2},k} = F(u_{j+\frac{1}{2},k}) \ , \qquad G_{j,k+\frac{1}{2}} = G(u_{j,k+\frac{1}{2}}) \ . \tag{3.7}$$

Now, let us consider the entropy condition of MmB schemes constructed in the section 2.

For the equation (2.1), the schemes (2.3) can be written as

$$u_{j,k}^{n+1} = u_{j,k} - \lambda(u_{j+\frac{1}{2},k} - u_{j-\frac{1}{2},k}) - \mu(u_{j,k+\frac{1}{2}} - u_{j,k-\frac{1}{2}}) \ , \tag{3.8}$$

where

$$\begin{aligned} u_{j+\frac{1}{2},k} =& u_{j,k} + \frac{1-\lambda}{2}\varphi_{j,k}\Delta_{j+\frac{1}{2},k}u - \frac{\mu}{2}\Theta_{j,k-\frac{1}{2}}\Delta_{j,k-\frac{1}{2}}u \ , \\ u_{j,k+\frac{1}{2}} =& u_{j,k} + \frac{1-\mu}{2}\psi_{j,k}\Delta_{j,k+\frac{1}{2}}u - \frac{\lambda}{2}\chi_{j-\frac{1}{2},k}\Delta_{j-\frac{1}{2},k}u \ , \end{aligned} \tag{3.9}$$

and $\lambda = a\Delta t/\Delta x$, $\mu = b\Delta t/\Delta y$.

For conventional, denote

$$\begin{aligned} \varphi_{j,k} &= \varphi, & \varphi_{j-1,k} &= \overline{\varphi}, \\ \Theta_{j,k-\frac{1}{2}} &= \Theta, & \Theta_{j-1,k-\frac{1}{2}} &= \overline{\Theta}, \\ \psi_{j,k} &= \psi, & \psi_{j,k-1} &= \overline{\psi}, \\ \chi_{j-\frac{1}{2},k} &= \chi, & \chi_{j-\frac{1}{2},k-1} &= \overline{\chi}, \end{aligned} \tag{3.10}$$

and

$$r = r_{j,k} \qquad s = s_{j,k}, \qquad p = p_{j,k-\frac{1}{2}}, \qquad q = q_{j-\frac{1}{2},k}. \tag{3.11}$$

Here, we only consider the square entropy function $U(u) = u^2/2$. So, the entropy flux $F(u) = au^2/2$ and $G(u) = bu^2/2$. Thus, the inequality (3.5) changes into

$$\frac{1}{2}(u_{j,k}^{n+1})^2 - \frac{1}{2}(u_{j,k})^2 + \frac{\lambda}{2}\left((u_{j+\frac{1}{2},k})^2 - (u_{j-\frac{1}{2},k})^2\right) + \frac{\mu}{2}\left((u_{j,k+\frac{1}{2}})^2 - (u_{j,k-\frac{1}{2}})^2\right) \le 0. \tag{3.12}$$

Now, we discuss which kinds of the conditions should be satisfied by the limiters in (3.8) when the schemes (3.8) satisfies the inequality (3.12).

Multiplying 2 and the left hand side of (3.12), (LHT), we have

$$\begin{aligned} LHS = & \left[\lambda(u_{j+\frac{1}{2},k} - u_{j-\frac{1}{2},k}) + \mu(u_{j,k+\frac{1}{2}} - u_{j,k-\frac{1}{2}})\right]^2 \\ & + \lambda(u_{j+\frac{1}{2},k} - u_{j-\frac{1}{2},k})(u_{j+\frac{1}{2},k} - 2u_{j,k} + u_{j-\frac{1}{2},k}) \\ & + \mu(u_{j,k+\frac{1}{2}} - u_{j,k-\frac{1}{2}})(u_{j,k+\frac{1}{2}} - 2u_{j,k} + u_{j,k-\frac{1}{2}}) \ . \end{aligned}$$

Substitute (3.9) into LHS, we have

$$\begin{aligned} LHS = & -\lambda\left[(1 + \frac{1-\lambda}{2}(\frac{\varphi}{r} - \overline{\varphi}))\Delta_{j-\frac{1}{2},k}u - \frac{\mu}{2}(\Theta - \frac{\overline{\Theta}}{p})\Delta_{j,k-\frac{1}{2}}u\right] \\ & \cdot \left[(1 - \frac{1-\lambda}{2}(\frac{\varphi}{r} + \overline{\varphi})\Delta_{j-\frac{1}{2},k}u + \frac{\mu}{2}(\Theta + \frac{\overline{\Theta}}{p})\Delta_{j,k-\frac{1}{2}}u\right] \\ & - \mu\left[(1 + \frac{1-\mu}{2}(\frac{\psi}{s} - \overline{\psi}))\Delta_{j,k-\frac{1}{2}}u - \frac{\lambda}{2}(\chi - \frac{\overline{\chi}}{q})\Delta_{j-\frac{1}{2},k}u\right] \\ & \cdot \left[(1 - \frac{1-\mu}{2}(\frac{\psi}{s} + \overline{\psi})\Delta_{j,k-\frac{1}{2}}u + \frac{\lambda}{2}(\chi + \frac{\overline{\chi}}{q})\Delta_{j-\frac{1}{2},k}u\right] \\ & + \left\{\lambda\left[(1 + \frac{1-\lambda}{2}(\frac{\varphi}{r} - \overline{\varphi}))\Delta_{j-\frac{1}{2},k}u - \frac{\mu}{2}(\Theta - \frac{\overline{\Theta}}{p})\Delta_{j,k-\frac{1}{2}}u\right]\right. \\ & \left. + \mu\left[(1 + \frac{1-\mu}{2}(\frac{\psi}{s} - \overline{\psi}))\Delta_{j,k-\frac{1}{2}}u - \frac{\lambda}{2}(\chi - \frac{\overline{\chi}}{q})\Delta_{j-\frac{1}{2},k}u\right]\right\}^2 \ . \end{aligned}$$

Let

$$\begin{aligned} \alpha &= 1 + \frac{1-\lambda}{2}(\frac{\varphi}{r} - \overline{\varphi})\overline{\alpha} = 1 - \frac{1-\lambda}{2}(\frac{\varphi}{r} + \overline{\varphi}), \\ \beta &= 1 + \frac{1-\mu}{2}(\frac{\psi}{s} - \overline{\psi})\overline{\beta} = 1 - \frac{1-\mu}{2}(\frac{\psi}{s} + \overline{\psi}), \end{aligned}$$

$$
\begin{aligned}
\gamma &= \Theta - \frac{\overline{\Theta}}{p}\overline{\gamma} = \Theta + \frac{\overline{\Theta}}{p}, \\
\xi &= \chi - \frac{\overline{\chi}}{q}\overline{\xi} = \chi + \frac{\overline{\chi}}{q}, \\
\mathbf{a} &= \Delta_{j-\frac{1}{2},k}u\mathbf{b} = \Delta_{j,k-\frac{1}{2}}u.
\end{aligned} \tag{3.13}
$$

Then

$$
\begin{aligned}
LHS =& -\lambda(\alpha\mathbf{a} - \frac{\mu}{2}\gamma\mathbf{b})(\overline{\alpha}\mathbf{a} + \frac{\mu}{2}\overline{\gamma}\mathbf{b}) - \mu(\beta\mathbf{b} - \frac{\lambda}{2}\xi\mathbf{a})(\overline{\beta}\mathbf{b} + \frac{\lambda}{2}\overline{\xi}\mathbf{a}) \\
&+ \left(\lambda\alpha\mathbf{a} - \frac{\lambda\mu}{2}\gamma\mathbf{b} + \mu\beta\mathbf{b} - \frac{\lambda\mu}{2}\xi\mathbf{a}\right)^2 \\
=& \mathbf{A}\ \mathbf{a}^2 + \mathbf{B}\ \mathbf{b}^2 + \mathbf{C}\ \mathbf{ab},
\end{aligned} \tag{3.14}
$$

where

$$
\begin{aligned}
\mathbf{A} =& \lambda^2(\alpha - \frac{\mu}{2}\xi)^2 - \lambda\alpha\overline{\alpha} + \mu\frac{\lambda^2}{4}\xi\overline{\xi}, \\
\mathbf{B} =& \mu^2(\beta - \frac{\lambda}{2}\gamma)^2 - \mu\beta\overline{\beta} + \lambda\frac{\mu^2}{4}\gamma\overline{\gamma}, \\
\mathbf{C} =& -\frac{\lambda\mu}{2}(\alpha\overline{\gamma} - \overline{\alpha}\gamma + \beta\overline{\xi} - \overline{\beta}\xi) + 2\lambda\mu(\alpha - \frac{\mu}{2}\xi)(\beta - \frac{\lambda}{2}\gamma).
\end{aligned}
$$

It can be found that the inequality (3.12) holds if and only if the inequality

$$
\mathbf{A}\ \mathbf{a}^2 + \mathbf{B}\ \mathbf{b}^2 + \mathbf{C}\ \mathbf{ab} \le 0 \tag{3.15}
$$

holds to every parameters $\mathbf{a}$, $\mathbf{b} \in (-\infty, \infty)$.

That is

$$
\mathbf{A} \le 0,\ \mathbf{B} \le 0,\ \text{and } \mathbf{C}^2 - 4\mathbf{AB} \le 0 \tag{3.16}
$$

Now, let us discuss the three inequalities in (3.16) respectively.

$$
\begin{aligned}
\mathbf{I} = -\frac{\mathbf{A}}{\lambda} =& \left[1 + \frac{1-\lambda}{2}(\frac{\varphi}{r} - \overline{\varphi})\right]\left[1 - \frac{1-\lambda}{2}(\frac{\varphi}{r} + \overline{\varphi})\right] - \frac{\lambda\mu}{4}\left[\chi - \frac{\overline{\chi}}{q}\right]\left[\chi + \frac{\overline{\chi}}{q}\right] \\
&- \lambda\left[1 + \frac{1-\lambda}{2}(\frac{\varphi}{r} - \overline{\varphi}) - \frac{\mu}{2}(\chi - \frac{\overline{\chi}}{q})\right]^2 \\
=& (1-\lambda) - (1-\lambda)^2\left[\overline{\varphi} - \frac{1-\lambda}{4}\overline{\varphi}^2\right] + \lambda\mu\left[\chi - \frac{1+\mu}{4}\chi^2\right] \\
&- \lambda(1-\lambda)\frac{\varphi}{r} - \frac{(1-\lambda)^2(1+\lambda)}{4}\frac{\varphi^2}{r^2} - \lambda\mu\left[\frac{\overline{\chi}}{q} - \frac{(1-\mu)}{4}\frac{\overline{\chi}^2}{q^2}\right] \\
&- \frac{\lambda\mu(1-\lambda)}{2}\overline{\varphi}\chi - \frac{\lambda\mu(1-\lambda)}{2}\frac{\varphi}{r}\frac{\overline{\chi}}{q} + \frac{\lambda(1-\lambda)}{2}\frac{\varphi}{r}\overline{\varphi} \\
&+ \frac{\lambda\mu(1-\lambda)}{2}\frac{\varphi}{r}\overline{\chi} + \frac{\lambda\mu(1-\lambda)}{2}\overline{\varphi}\frac{\overline{\chi}}{q} + \frac{\lambda\mu^2}{2}\chi\frac{\overline{\chi}}{q}\ .
\end{aligned}
$$

For all of the limiters $\mathbf{X}(\theta)$, take

$$
0 \le \left[\frac{\mathbf{X}(\theta)}{\theta}\ ,\ \mathbf{X}(\theta)\right] \le \mathbf{X} \le 2\ . \tag{3.17}
$$

Let $0 \le \lambda$, $\mu \le \delta < 1$, so $1-\delta \le 1-\lambda$, $1-\mu \le 1$ and

$$0 \le \lambda \le \frac{\delta}{1-\delta}(1-\lambda), \qquad 0 \le \mu \le \frac{\delta}{1-\delta}(1-\mu). \tag{3.18}$$

Then

$$\begin{aligned}\mathbf{I} &\ge (1-\lambda) - (1-\lambda+\lambda\mu)\mathbf{X} - \frac{\lambda\mu(3-4\lambda+\mu)+2\lambda(1-\lambda)^2}{4}\mathbf{X}^2\\ &\ge (1-\lambda)\left[1 - \frac{1-\delta+\delta^2}{1-\delta}\mathbf{X} - \frac{\delta(2+\delta+\delta^2)}{4(1-\delta)}\mathbf{X}^2\right],\end{aligned}$$

where $\delta \le \frac{3}{4}$.

Therefore, $\mathbf{I}\ge 0$ if

$$\mathbf{X} \in \left[0\ ,\ \frac{2(2-\delta-\delta^3)}{(2+\delta+\delta^2)(\sqrt{1+2\delta^2-2\delta^3}+1-\delta+\delta^2)}\ .\right] \tag{3.19}$$

For example, take $\delta = \dfrac{1}{3}$, then $\mathbf{X}\in [0, 0.72099]$. The condition (3.19) should be satisfied by the limiters when $\mathbf{A}\le 0$.

Similar to the case of $\mathbf{I}$, we can get

$$\mathbf{II} = -\frac{\mathbf{B}}{\mu} \ge 0,$$

provided

$$\mathbf{X} \in \left[0\ ,\ \frac{2(2-\delta-\delta^3)}{(2+\delta+\delta^2)(\sqrt{1+2\delta^2-2\delta^3}+1-\delta+\delta^2)}\right]. \tag{3.20}$$

Now, let us discuss the inequality $\mathbf{C}^2 \le 4\mathbf{AB}$. That is

$$\lambda\mu\left[4(\alpha-\frac{\mu}{2}\xi)(\beta-\frac{\lambda}{2}\gamma) - (\alpha\overline{\gamma}-\overline{\alpha}\gamma) - (\beta\overline{\xi}-\overline{\beta}\xi)\right]^2 \le 16\mathbf{I}\cdot\mathbf{II}. \tag{3.21}$$

Denote $\mathbf{III}$ be the left hand side of the above inequality. Then

$$\begin{aligned}\mathbf{III} =& \lambda\mu\left[4\alpha\beta - 2\lambda\alpha\gamma - 2\mu\xi\beta + \lambda\mu\beta\gamma - \alpha\overline{\gamma} + \overline{\alpha}\gamma - \beta\overline{\xi} + \overline{\beta}\xi\right]^2\\ =& \lambda\mu\left[\left(2+(1-\lambda)(\frac{\varphi}{r}-\overline{\varphi})\right)\left(2+(1-\mu)(\frac{\psi}{s}-\overline{\psi})\right) - 2\lambda\left(1+\frac{1-\lambda}{2}(\frac{\varphi}{r}-\overline{\varphi})\right)\right.\\ &\cdot\left(\Theta-\frac{\overline{\Theta}}{p}\right) - 2\mu\left(1+\frac{1-\mu}{2}(\frac{\psi}{s}-\overline{\psi})\right)\left(\chi-\frac{\overline{\chi}}{q}\right) + \lambda\mu\left(\Theta-\frac{\overline{\Theta}}{p}\right)\left(\chi-\frac{\overline{\chi}}{q}\right)\\ &-\left(1+\frac{1-\lambda}{2}(\frac{\varphi}{r}-\overline{\varphi})\right)\left(\Theta+\frac{\overline{\Theta}}{p}\right) + \left(1-\frac{1-\lambda}{2}(\frac{\varphi}{r}+\overline{\varphi})\right)\left(\Theta-\frac{\overline{\Theta}}{p}\right)\\ &\left.-\left(1+\frac{1-\mu}{2}(\frac{\psi}{s}-\overline{\psi})\right)\left(\chi+\frac{\overline{\chi}}{q}\right) + \left(1-\frac{1-\mu}{2}(\frac{\psi}{s}+\overline{\psi})\right)\left(\chi-\frac{\overline{\chi}}{q}\right)\right]^2\\ =& \lambda\mu\left[4 + 2(1-\mu)\frac{\psi}{s} + 2(1-\lambda)\frac{\varphi}{r} - 2(1-\mu)\overline{\psi}\right.\end{aligned}$$

$$- 2(1-\lambda)\overline{\varphi} - 2\lambda\Theta - 2(1-\lambda)\frac{\overline{\Theta}}{p} - 2(1-\mu)\frac{\overline{\chi}}{q} - 2\mu\chi$$
$$- (1-\lambda^2)\frac{\varphi}{r}\Theta + \lambda(1-\lambda)\frac{\varphi}{r}\frac{\overline{\Theta}}{p} + \mu(1-\mu)\frac{\psi}{s}\frac{\overline{\chi}}{q} - (1-\mu^2)\frac{\psi}{s}\chi$$
$$+ \lambda(1-\lambda)\overline{\varphi}\Theta + \mu(1-\mu)\overline{\psi}\chi + (1-\mu)^2\overline{\psi}\frac{\overline{\chi}}{q} + (1-\lambda)^2\overline{\varphi}\frac{\overline{\Theta}}{p}$$
$$+ (1-\lambda)(1-\mu)(\frac{\varphi}{r} - \overline{\varphi})(\frac{\psi}{s} - \overline{\psi}) + \lambda\mu(\Theta - \frac{\overline{\Theta}}{p})(\chi - \frac{\overline{\chi}}{q})\Big]^2 .$$

Under the conditions (3.17) and (3.18), we have

$$\begin{aligned}\mathbf{III} \le & \lambda\mu\left[4 + (4-2\mu-2\lambda)\mathbf{X} + (3-\lambda-\mu-(\lambda-\mu)^2)\mathbf{X}^2\right]^2 \\ \le & \lambda\mu(4+4\mathbf{X}+3\mathbf{X}^2)^2 \\ \le & (1-\lambda)(1-\mu)(4+4\mathbf{X}+3\mathbf{X}^2)^2\left(\frac{\delta}{1-\delta}\right)^2 .\end{aligned}$$

Thus, the inequality $\mathbf{C}^2 \le 4\mathbf{AB}$ holds, if

$$\begin{aligned}&(1-\lambda)(1-\mu)(4+4\mathbf{X}+3\mathbf{X}^2)^2\left(\frac{\delta}{1-\delta}\right)^2 \\ &\le (1-\lambda)(1-\mu)\left[4 - \frac{4(1-\delta+\delta^2)}{1-\delta}\mathbf{X} - \frac{\delta(2+\delta+\delta^2)}{(1-\delta)}\mathbf{X}^2\right]^2 .\end{aligned}$$

When $\delta \le \frac{1}{2}$,

$$4(1-2\delta) - 4(1+\delta^2)\mathbf{X} - \delta(5+\delta+\delta^2)\mathbf{X}^2 \ge 0 . \tag{3.22}$$

Therefore, the condition which should be satisfied by the limiters for $\mathbf{C}^2 \le 4\mathbf{AB}$ is

$$\mathbf{X} \in \left[0\ ,\ \frac{2(5-9\delta-\delta^2-2\delta^3)}{(5+\delta+\delta^2)\left[\sqrt{1+5\delta-7\delta^2-\delta^3-\delta^4}+1+\delta^2\right]}\right] . \tag{3.23}$$

For example, take $\delta = \frac{1}{3}$, then $\mathbf{X} \in [0, 0.27019]$.

Remark. From the conditions (3.19), (3.20), and (3.23), we can find that the region of limiters $\mathbf{X}$ can tend to 1 when δ tends to 0. But, the critical point $\mathbf{X}(1) = 1$ is not located in the region of $\mathbf{X}$. Thus, the MmB schemes can not preserve the second order accuracy if they satisfy the entropy condition with the proper discrete entropy flux.

4. Conclusions

In the last sections, we have discussed the discrete entropy conditions of the high resolution MmB schemes by using the theory of proper discrete entropy flux for linear hyperbolic equations in two dimensions. Unfortunately, the theoretical results show that the 2-D high resolution MmB schemes, which satisfy the entropy condition with the proper discrete entropy flux, can not preserve the second order accuracy that is the case in one dimension. So, it should be further researched how to discretizate the entropy flux properly such that the discrete entropy conditions can be better consistent with the nonlinear stability of the difference schemes for hyperbolic conservation laws.

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