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A-POSTERIORI LOCAL ERROR ESTIMATES OF BOUNDARY ELEMENT METHODS WITH SOME PSEUDO-DIFFERENTIAL EQUATIONS ON CLOSED CURVES*

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Abstract

In this paper we show local error estimates for the Galerkin finite element method applied to strongly elliptic pseudo-differential equations on closed curves. In these local estimates the right hand sides are obtained as the sum of a local norm of the residual, which is computable, and additional terms of higher order with respect to the meshwidth. Hence, asymptotically, here the residual is an error indicator which provides a corresponding self-adaptive boundary element method.

§1. Introduction

Adaptive procedures for finite element methods as well as for boundary element methods play an increasingly decisive role in corresponding algorithms and have been recently analyzed also rigorously (see e.g. [2], [5], [6], [12], [13], [21], [30-37]). The heart of adaptivity is some computable expression defined by the approximate solution which can serve as an error indicator and which, on the other hand, is related to a reasonable a-posteriori error estimate. In [30] we have already shown that for strongly elliptic boundary integral equations and some boundary element approximations the residual is a local error indicator. These results were based on a discrete analogon to the pseudo-locality of pseudo-differential operators in terms of the so called influence index and corresponding restrictions for the family of meshes.

Here we obtain again for the boundary element Galerkin method that the residual can serve as local error indicator, however, we do not need the influence index anymore, we only assume a local property, i.e. K-meshes. These new results are based on optimal order global error estimates as well as on local estimates. The asymptotic global

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error estimates require for two-dimensional problems which are governed by pseudo-differential equations on closed curves, that for the spline Galerkin boundary element methods the boundary integral operators must be strongly elliptic.[25] (For this class of operators see also [29].) Our local error estimates apply in particular to Symm's integral equation of the first kind which is of order -1 (see [15]). For quasiregular families of meshes and this special case, Braun in [8],[9] gave a local error estimate. But there the right hand side contains terms in local norm as well as in global norm which are both of the same order. For more general equations but also for quasiregular mesh families, Saranen proved in [24] local estimates with right hand sides of the same form. Hence, these results are not desirable as a-posteriori error bounds or indicators in adaptive methods.

In this paper we will improve our results in [30] by avoiding the concept of influence indeces and by using the local estimates for K-meshes. These meshes can be used for adaptive refinement and therefore are desirable for a feedback method based on our local a-posteriori estimates. Here, the right hand side consists of the local norm of the residual which is computable by using the approximate solution, and additional terms of higher order which are negligible for the local error indicator.

As in [3] we will restrict our presentation here to two-dimensional boundary value problems with corresponding strongly elliptic boundary integral equations on closed boundary curves. Moreover, for technical reasons we consider only the case $|\alpha| \leq 2$ where α denotes the order of the corresponding pseudo-differential operator. If this restriction could be omitted then our results could be extended to naive boundary element collocation involving smoothest splines of odd degree.[23]

For 3-D boundary element methods on two-dimensional surfaces and product splines we would need additional approximate derivatives of the solution; these generalizations are yet to be done.

§2. The Main Result

Let Γ be a plane Jordan curve given by a regular parametric representation

$$\Gamma : z = (z_1(s), z_2(s)) \cong z_1(s) + iz_2(s),$$

where z is a 1-periodic function of a real variable s and $|dz/ds| > 0$. In boundary element methods, Γ is the boundary of a given domain associated with some boundary value problem. Via the parametrization we have a one-to-one correspondence between functions on Γ and 1-periodic functions. More generally, for a system of mutually disjoint Jordan curves $\Gamma = \cup_{j=1}^{L} \Gamma_j$ we may parametrize each and identify functions on Γ with L-vector valued 1-periodic functions. We thus limit ourselves without loss of generality to equations of the form

$$Au = f \tag{1}$$

where $u = (u_1(s), \cdots, u_p(s))$ denotes the unknown 1-periodic p-vector valued function, $f = (f_1(s), \cdots, f_p(s))$ is the right hand side, and A is a given $p \times p$ matrix of bounded linear pseudo-differential operators of real order $\alpha = 2\gamma$ mapping $H^\gamma \to H^{-\gamma}$ continuously. We further assume that (1) has a unique solution. Here H^ρ denotes p copies of the periodic Sobolev space of order $\rho \in R$, i.e. the closure of all smooth real-valued 1-periodic functions with respect to the norm

$$\|f\|_\rho := \Big\{ \sum_{j=1}^{p} (|\hat{f}_j(0)|^2 + \sum_{0 \neq k \in Z} |\hat{f}_j(k)|^2 |2\pi k|^{2\rho}) \Big\}^{\frac{1}{2}},$$

where

$$\hat{f}_j(k) = \int_0^1 e^{-2\pi i k s} f_j(s) ds, \quad k \in Z,$$

are the Fourier coefficients. By $(\cdot, \cdot)$ we denote the L_2 scalar product on the periodic functions which also can be expressed via Parseval's equality by

$$(f, g) = \sum_{j=1}^{p} \sum_{k \in Z} \hat{f}_j(k) \overline{\hat{g}_j(k)}.$$

Note that the scalar product extends to a duality pairing between H^ρ and $H^{-\rho}$ for arbitrary $\rho \in R$ and, moreover,

$$\|v\|_\rho = \sup_{0 \neq w \in H^{-\rho}} \Big\{ \frac{(v, w)}{\|w\|_{-\rho}} \Big\}.$$

In [3] one finds several examples of systems of boundary integral equations belonging to our class. In particular, the Fredholm integral equation of the first kind with logarithmic kernel is discussed in [8], [9], [17], [29] and has the order $\alpha = -1$. Singular integral equations of Cauchy's type are those with $\alpha = 0$ including the classical Fredholm integral equations of the second kind. For $\alpha > 0$ we have hypersingular integral equations as in [13], [18] and [29]. All these types of equations are used to solve many different problems in applications as, e.g. in elasticity; some of the applications with $\alpha = -1, \alpha = 0$, and $\alpha = 1$ are listed in [3], [17] and [29].

For the approximate solution of (1) we consider approximating spaces $S_h = S_h^{k,t}, 0 \leq k < t$, a family of 1-periodic finite element spaces associated with a family of partitions $\mathcal{D}$ of maximal meshwidth h. For S_h we require the following properties:

i)

$$S_h = S_h^{k,t} \subseteq H^k(\Gamma), \tag{2}$$

ii)

$$\inf_{\chi \in S_h} \|v - \chi\|_l \leq c h^{m-l} \|v\|_m, \quad 0 \leq l \leq m \leq t, \quad l \leq k \tag{3}$$

for all $v \in H^m(\Gamma)$, where the constant c is independent of v and h.

iii) The family of meshes consists of K-meshes, where $\mathcal{D}$ is called a K-mesh if for any two subintervals $\Delta = \bar{\Delta}$ and $\Delta' = \bar{\Delta}'$ of the partition $\mathcal{D}$ with $\Delta \cap \Delta' \neq \emptyset$ there holds

$$K^{-1} \leq \frac{|\Delta'|}{|\Delta|} \leq K. \tag{4}$$

with fixed $K > 1$ for the whole family and $|\Delta|, |\Delta'|$ the length of Δ and Δ', respectively. Mostly, adaptive meshes can be constructed as K-meshes. In what follows we shall always assume that the meshes $\mathcal{D}$ of our family are K-meshes which is obtained from some originally uniform mesh by consecutive finite subdivisions into two subintervals.

In (2) and (3), t refers to the local approximation order of S_h, which usualy consists of piecewise polynomial functions of degree $t-1$. k indicates the global smoothness of these functions. As usual, a typical finite element function space will satisfy (2), (3) and some 'inverse relation'. Here however, we only assume S_h to satisfy (2) and (3), since we want to apply our results to adaptive meshes with (4), on which S_h usually can not satisfy the inverse relation.

For the approximate solution of (1) we consider a conforming Galerkin boundary element method, i.e.,

$$S_h \subset H^\gamma(\Gamma), \quad \gamma = \frac{\alpha}{2} \leq k$$

and find $u_h \in S_h$ such that

$$(Au_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in S_h. \tag{5}$$

We further assume that the Galerkin method converges asymptotically and globally of optimal order, i.e. that

$$\|u - u_h\|_l \leq ch^{m-l}\|u\|_m, \tag{6}$$

for $u \in H^m(\Gamma), \alpha - t \leq l \leq \gamma \leq m \leq t, -t \leq \gamma \leq k$, and $0 < h \leq h_0$ with some suitable constant $h_0 > 0$.

Due to Schmidt's results in [25], for a uniform family of partitions, this would imply that A is a strongly elliptic pseudo-differential operator. Conversely, for strongly elliptic A, (6) was shown in [16], see also [3], [17], [22], [23], [29].

For feedback mesh refinement, the estimate (6) does not provide information on the local error at some point or on some closed subinterval $I_1 \subset \Gamma$. In [8], [9], [24] local error estimates are shown. These, however require regular families of mesh refinements and the inverse relation. Here, for $|\alpha| \leq 2$, we will show local error estimates which are based on the local behavior of the residual, defined by the computable expression

$$R = f - Au_h \tag{7}$$

on the subinterval I_2 with $I_1 \subset I_2^o \subset \Gamma$, I_2^o the open interior of I_2, and higher order remainders which are negligible for sufficiently small h. The main results are summarized in the following theorem.

Theorem 2.1. *Let A be a strongly elliptic pseudo-differential operator of order α with $-2 \leq \alpha \leq 2$. Let $t - 1 \geq k \geq |\alpha| + \min\{0, \frac{\alpha}{2}\}$ and let the boundary elements be defined on a family of k-meshes. Let $u \in H^k(I_2) \cap H^\alpha(\Gamma) \cap L_2(\Gamma), I_1 \subset I_2^o \subset \Gamma$. Then for $0 < h \leq h_0$, the local error of the Galerkin boundary element method satisfies the estimates,*

$$\|u - u_h\|_{L_2(I_1)} \leq C\{h^{k+1}\|u\|_0 + h^k\|R\|_{-\alpha} + \|R\|_{H^{-\alpha}(I_2)}\} \tag{8}$$

for $-2 \le \alpha < 0$ *and*

$$\|u - u_h\|_{L_2(I_1)} \le C\{h^{\min(k+\alpha,t)-\gamma+1}\|u\|_\gamma + h^{\min(k+\alpha,t)}\|R\|_0 + h^\alpha\|R\|_{L_2(I_2)}\} \quad (9)$$

for $0 \le \alpha \le 2$.

Remark. The boundary integral operators in most applications have operators of orders $\alpha = -1$ [15],[29], $\alpha = 0$ [3],[29] and $\alpha = 1$ [13],[18],[29]. For these cases, (8) and (9), respectively, specialize to the following estimates:

$$\alpha = -1 : \|u - u_h\|_{0,I_1} \le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{1,\Gamma} + \|R\|_{1,I_2}\},$$

$$\alpha = 0 : \|u - u_h\|_{0,I_1} \le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{0,\Gamma} + \|R\|_{0,I_2}\},$$

$$\alpha = 1 : \|u - u_h\|_{0,I_1} \le C\{h^{k+\frac{3}{2}}\|u\|_{\frac{1}{2},\Gamma} + h^{k+1}\|R\|_{0,\Gamma} + h\|R\|_{0,I_2}\}.$$

The results (8) and (9) show us that for strongly elliptic boundary integral equations of order α with $|\alpha| \le 2$ and for K-meshes, the local error of their Galerkin approximate solution can be estimated by calculating the residual in a local norm. These estimates can be used to design an adaptive boundary element method and for a-posteriori error estimates providing information on the error distribution over the mesh. According to this information, the mesh of the approximation structure can be changed automatically so as to improve the quality of the numerical solution with less computational work than by uniform refinement. This is the basic idea of feedback adaptive methods. Numerical examples with the residual as indicator can be found in [14] and [30].

§3. Some Fundamental Properties of the Operators Involved

Lemma 3.1. *If* $A : H^\gamma(\Gamma) \to H^{-\gamma}(\Gamma)$ *is a strongly elliptic pseudo-differential operator of order* $\alpha = 2\gamma$ *then the adjoint operator* A^* *is also a strongly elliptic pseudo-differential operator of order* α. *A and* A^* *are continuous. Uniqueness of* (1) *then implies that* A^{-1} *and* A^{*-1} *are also strongly elliptic pseudo-differential operators of order* $-\alpha$ *and continuous.*

For the proof see [27] and [29].

Now let $I_1 \subset\subset I_2' \subset\subset I_2 \subseteq \Gamma$ and $J_i, 1 \le i \le 9$, satisfy

$$I_1 = J_1 \subset\subset J_2 \subset\subset \cdots \subset\subset J_9 \subseteq I_2',$$

where I_1, J_i, I_2', I_2 are subarcs of Γ. By $I_1 \subset\subset I_2'$ we denote the properties $\bar{I}_1 = I_1 \subset (I_2')^o$. Let further $\omega_i \in C_0^\infty(\Gamma), 1 \le i \le 8$ be cut-off functions associated with J_i and satisfying

i) $\omega_i = 1$ in J_i

ii) $\mathrm{supp}\,\omega_i \subset\subset J_{i+1}$ and

iii) $0 \le \omega_i \le 1$, $i = 1, \cdots, 8$.

For simplicity we denote

$$\hat{\omega}_i = 1 - \omega_i.$$

Then we have the following two lemmas. For their proofs see [27] and [4, Lemma 3,2].

Lemma 3.2. *For any distribution v on Γ we have*

$$\hat{\omega}_{i+1} A^* \omega_i v \in C^\infty(\Gamma),$$

and for arbitrary real l and m and $v \in H^m(\Gamma)$,

$$\|\hat{\omega}_{i+1} A^* \omega_i v\|_l \leq C\|v\|_m,$$

where C is independent of v.

Lemma 3.3. *The commutator $\omega A^* - A^*\omega$ is a pseudo-differential operator of order $\alpha - 1$, for every $\omega \in C_0^\infty(\Gamma)$.*

§4. Local Approximation on K-Meshes

For any $\Delta \in \mathcal{D}$ we define the influence region I_Δ for Δ by using the interpolation basis functions ψ_p corresponding to any node p,

$$I_\Delta = \text{interior } of\Big(\cup_{p \text{ with } \Delta \in \mathcal{D}_p} \operatorname{supp} \psi_p \Big)$$

where

$$\mathcal{D}_p = \{\Delta \in \mathcal{D} | \Delta \subseteq supp\, \psi_p\}.$$

With these definitions it is easily seen that the inequality

$$|I_\Delta| \leq C(K)|\Delta|$$

holds with a constant $C(K)$ depending only on the constant K of the K-mesh.

Lemma 4.1. *Let I_{Δ^*} be the influence region for the standard element $\Delta^* = [0,1]$, then there is a constant $C = C(K) > 0$ such that for $z \in H^m(I_{\Delta^*})$ with $m \geq 1$, and $z = 0$ at the ends of I_{Δ^*}, then*

$$\inf_{q \in Q} \|z - q\|_{m, I_{\Delta^*}} \leq C|z|_{m, I_{\Delta^*}},$$

where Q is the set of all polynomials of degree $m-1$ on I_{Δ^*} which vanish at the ends of I_{Δ^*}.

Here and in the following, we use the m-th order seminorm

$$|z|_{m, I_{\Delta^*}} = \Big\{ \int_{I_{\Delta^*}} |\frac{d^m z}{ds^m}|^2 ds \Big\}^{\frac{1}{2}}.$$

Proof. Suppose the lemma were not true, then we could find $z_n \in H^m(I_{\Delta^*}), n = 1, 2, \cdots$, such that

$$\inf_{q \in Q} \|z_n - q\|_{m, I_{\Delta^*}} \geq n|z_n|_{m, I_{\Delta^*}}. \tag{10}$$

Without loss of generality we may as well suppose that

$$\inf_{q \in Q} \|z_n - q\|_{m, I_{\Delta^*}} = 1 \tag{11}$$

(by taking suitable multiples of the original z_n if necessary) and

$$\|z_n\|_{m, I_{\Delta^*}} \leq 2 \tag{12}$$

(by adding a suitable $q_n \in Q$ to each of the original z_n if necessary). From (12) we conclude, by using the Rellich Lemma, that a sequence of the $\{z_n\}$ converges in

$H^{m-1}(I_{\Delta^*})$. We can suppose that this subsequence is the entire original sequence (by deleting members of the original sequence if necessary). From (10) and (11) we find

$$\lim_{n\to\infty} |z_n|_{m,I_{\Delta^*}} = 0.$$

Thus z_n converges in $H^m(I_{\Delta^*})$. Let z_∞ be the limit. Obviously,

$$|z_\infty|_{m,I_{\Delta^*}} = 0,$$

and as I_{Δ^*} is connected, z_∞ is on I_{Δ^*} a polynomial of degree $m-1$. Moreover, $z_\infty = 0$ at the ends of I_{Δ^*}. Then $z_\infty \in Q$ in contradiction to (11). Since there are only finitely many possibilities of standard influence regions I_{Δ^*} for fixed K, the constant C in this lemma is only dependent on K.

Lemma 4.2. *There is a constant $C = C(K) > 0$ such that for any $z \in H^m(I)$, $1 \le m \le t$, $I \subseteq \Gamma$, with $z = 0$ at the ends of I, and for any K-mesh $\mathcal{D}$, there exists a function $\Pi z \in S_h(I)$ with $\Pi z = 0$ at the ends of I, such that*

$$|z - \Pi z|_{l,\Delta} \le C|\Delta|^{m-l}|z|_{m,I_\Delta}, \quad 0 \le l \le m \le t, \quad \Delta \in \mathcal{D}(I),$$

where I_Δ is the influence region for the element Δ.

Proof. Let $\Phi(\xi)$ be a polynomial in one variable satisfying

$$\int_0^1 \Phi(\xi)d\xi = 1 \tag{13}$$

and

$$\int_0^1 \xi^j\Phi(\xi)d\xi = 0 \quad \text{for} \quad j = 1,2,\cdots,t-1. \tag{14}$$

For any $\varepsilon > 0$ define

$$\Phi_\varepsilon = \begin{cases} \Phi(\xi\varepsilon^{-1}) & \text{for } \xi \in [0,\varepsilon], \\ \Phi(-\xi\varepsilon^{-1}) & \text{for } \xi \in [-\varepsilon,0], \\ 0 & \text{for } |\xi| > \varepsilon. \end{cases}$$

For any node p of $\mathcal{D}$ let

$$\varepsilon_p = \frac{1}{2}\min_{\Delta\in\mathcal{D}_p} |\Delta|, \qquad S_p = \{x \in I \,|\, |x-p| < \varepsilon_p\}.$$

Obviously $S_p \subset \operatorname{supp} \psi_p$. For any $y \in H^m(I)$ let

$$Y_p = \frac{\int_{S_p} y(x)\Phi_{\varepsilon_p}(x-p)dx}{\int_{S_p} dx}$$

and define

$$\Pi y = \sum_{p\in I^o} Y_p\psi_p, \tag{15}$$

where I^o = *Interior of I*. ψ_p is the interpolation basis function corresponding to p. Then $\Pi y \in S_h$ and $\Pi y = 0$ at the ends of I. Suppose, $\Delta \in \mathcal{D}(I)$. Then

$$|\Pi y|_{l,\Delta} \le \sum_{p\in I^o} |Y_p||\psi_p|_{l,\Delta} \le C \sum_{p\in I^o,\Delta\in\mathcal{D}_p} |Y_p||\Delta|^{\frac{1}{2}-l}, \quad l \ge 0.$$

Moreover,

$$|Y_p| \le \frac{\|y\|_{0,S_p}\|\Phi_{\epsilon_p}\|_{0,S_p}}{\displaystyle\int_{S_p} dx} \le C\epsilon_p^{-\frac{1}{2}}\|y\|_{0,S_p}.$$

Thus

$$|\Pi y|_{l,\Delta} \le C \sum_{p\in I^o,\Delta\in\mathcal{D}_p} |\Delta|^{-l}\|y\|_{0,S_p} \le C|\Delta|^{-l}\|y\|_{0,I_\Delta}, \tag{16}$$

where $C = C(K)$ is a constant. Upon rescaling Δ to the unit interval, i.e. $\tau_\Delta : \Delta \to [0,1] \equiv \Delta^*$, (16) becomes

$$|\Pi y \circ \tau_\Delta^{-1}|_{l,[0,1]} \le C\|y \circ \tau_\Delta^{-1}\|_{0,I_{\Delta^*}},$$

hence

$$|(y - \Pi y) \circ \tau_\Delta^{-1}|_{l,[0,1]} \le C\|y \circ \tau_\Delta^{-1}\|_{m,I_{\Delta^*}}, \quad \text{for} \quad m \ge l \ge 0. \tag{17}$$

If ζ is on I_Δ a polynomial of degree $m-1$ which vanishes at the ends of I_Δ, then ζ can trivially be extended by zero to all of I. Direct calculation shows that

$$\Pi\zeta = \sum_{p\in I^o}\Big\{\int_{S_p}\zeta(x)\Phi_{\epsilon_p}(x-p)dx\Big/\int_{S_p}dx\Big\}\psi_p = \sum_{p\in I^o}\zeta(p)\psi_p = \zeta \quad \text{for } m \le t$$

because of the definition (13) and (14) of $\Phi(\xi)$. Hence, for any given $z \in H^m(I)$ with $z = 0$ at the ends of I, let

$$y = z - \zeta.$$

Then we obtain from (17)

$$|z \circ \tau_\Delta^{-1} - \Pi z \circ \tau_\Delta^{-1}|_{l,[0,1]} = \inf_\zeta |(z-\zeta)\circ\tau_\Delta^{-1} - \Pi(z-\zeta)\circ\tau_\Delta^{-1}|_{l,[0,1]}$$

$$\le C\inf_\zeta \|(z-\zeta)\circ\tau_\Delta^{-1}\|_{m,I_{\Delta^*}}, \quad \text{for} \quad 0 \le l \le m \le t.$$

Clearly, as ζ ranges over $H^m(I)$ vanishing at the endpoints of I, $\zeta \circ \tau_\Delta^{-1}$ ranges over all polynomials of degree $m-1$ on I_{Δ^*} vanishing at the endpoints of I_{Δ^*}. Thus, by Lemma 4.1,

$$|z \circ \tau_\Delta^{-1} - \Pi z \circ \tau_\Delta^{-1}|_{l,[0,1]} \le C|z \circ \tau_\Delta^{-1}|_{m,I_{\Delta^*}}, \quad \text{for} \quad 0 \le l \le m \le t.$$

Rescaling back to Δ gives the desired result

$$|z - \Pi z|_{l,\Delta} \le C|\Delta|^{m-l}|z|_{m,I_\Delta} \quad \text{for} \quad 0 \le l \le m \le t, \Delta \in \mathcal{D}, \tag{18}$$

where the constant $C = C(K)$ is mesh independent.

Now let

$$E = u - u_h \tag{19}$$

be the error, i.e. the difference between the solution of the equation (1) and the solution of problem (5). Consider the auxiliary problem

$$A^* w = \omega_1 E, \tag{20}$$

where $\omega_1 \in C_0^\infty(\Gamma)$ is defined as above with $supp\,\omega_1 \subset\subset J_2$. For $u \in H^k(J_2)$, we have $\omega_1 E \in H^k(\Gamma)$, and therefore with Lemma 3.1,

$$w = A^{*-1}(\omega_1 E) \in H^{k+\alpha}(\Gamma).$$

Lemma 4.3. *There exists a constant $C = C(K) > 0$ such that for any $u \in H^k(J_2) \cap H^\gamma(\Gamma)$ and the associated solution w of problem (20), there exists $\zeta \in S_h^{k,t}$ with $supp\,\zeta \subseteq J_3 \subset\subset J_4$ such that*

$$\|\omega_2 w - \zeta\|_{l,J_4} \le ch^{m-l}\|w\|_{m,J_4}$$

for $-q \le l \le \min(k,m), 2-q \le m \le \min(k+\alpha,t), 0 < h \le h_0$. Here $q \ge 2$ is an arbitrary fixed positive integer.

Proof. From the definition of w and ω_2, we know that

$$\omega_2 w \in H^{k+\alpha}(\Gamma) \quad \text{and} \quad \operatorname{supp}(\omega_2 w) \subset\subset J_3 \subset\subset J_4.$$

Define

$$D^{-1}f(x) = \int_a^x f(\xi)d\xi, \quad a \le x \le b,$$

where $J_4 = [a,b]$. Let $q \ge 2$ be an arbitrary fixed positive integer and

$$v = D^{-q}(\omega_2 w).$$

We have $v \in H^{k+\alpha+q}(J_4)$ with $v(a) = 0$. Obviously, in $J_4 \backslash J_3$ v is a polynomial of degree $q-1$. Now, we can easily choose a polynomial p_1 of degree 1 and show that $p_1(a) = 0$ and $p_1(b) = v(b)$. Then

$$v - p_1 \in H^{k+\alpha+q}(J_4) \quad \text{and} \quad v(a) - p_1(a) = v(b) - p_1(b) = 0,$$

and $v - p_1$ is a polynomial of degree $q-1$ in $J_4 \backslash J_3$. By Lemma 4.2 there exists a function $\Pi(v-p_1) \in S_h^{k+q,t+q}(J_4)$, which vanishes at the ends of J_4, such that

$$|(v-p_1) - \Pi(v-p_1)|_{l,\Delta} \le C|\Delta|^{m+q-l}|v-p_1|_{m+q,I_\Delta}$$

for all $0 \le l \le m+q \le \min(k+\alpha+q, t+q), m+q \ge 1, \Delta \in \mathcal{D}(J_4)$. From the definition of Π in the proof of Lemma 4.2 we know that

$$\Pi(v-p_1) = v - p_1 \quad \text{in } J_4 \backslash J_3.$$

Because $\cup_{\Delta \in J_3} I_\Delta \subset\subset J_4$ when h is small enough, we have

$$|(v-p_1) - \Pi(v-p_1)|_{l,J_4} = |(v-p_1) - \Pi(v-p_1)|_{l,J_3} \le ch^{m+q-l}|v-p_1|_{m+q,J_4}$$

for $0 \le l \le m+q \le \min(k+\alpha+q, t+q), l \le k+q, m+q \ge 1$. Observe that $p_1 = \Pi p_1$ and $D^{m+q}p_1 = 0$ when $m+q \ge 2$, hence we obtain

$$|v - \Pi v|_{l,J_4} \le ch^{m+q-l}|v|_{m+q,J_4}$$

for $0 \le l \le m+q \le \min(k+\alpha+q, t+q), l \le k+q, m+q \ge 2$. Obviously, in the above inequality the left seminorm $|\cdot|_{l,J_4}$ can be replaced by the norm $\|\cdot\|_{l,J_4}$:

$$\|v - \Pi v\|_{l,J_4} \le ch^{m+q-l}|v|_{m+q,J_4} \tag{21}$$

for $0 \le l \le m+q \le \min(k+\alpha+q, t+q), l \le k+q, m+q \ge 2$. Let $\zeta = D^q(\Pi v)$, then $\zeta \in S_h^{k,t}(J_4), \zeta = 0$ in $J_4 \backslash J_3$ and so supp $\zeta \subseteq J_3$. Thus we have

$$\|\omega_2 w - \zeta\|_{-q,J_4} = \sup_{0 \ne f \in H_0^q(J_4)} \frac{|(\omega_2 w - \zeta, f)_{J_4}|}{\|f\|_{q,J_4}} = \sup_{0 \ne f \in H_0^q(J_4)} \frac{|(D^q(v - \Pi v), f)_{J_4}|}{\|f\|_{q,J_4}}.$$

Integration by parts yields

$$\|\omega_2 w - \zeta\|_{-q,J_4} = \sup_{0\neq f\in H_0^q(J_4)} \frac{|(v - \Pi v, D^q f)_{J_4}|}{\|f\|_{q,J_4}} \le \|v - \Pi v\|_{0,J_4}.$$

From (21) with $l = 0$ there follows

$$\begin{aligned}\|\omega_2 w - \zeta\|_{-q,J_4} &\le \|v - \Pi v\|_{0,J_4} \le ch^{m+q}|v|_{m+q,J_4} = ch^{m+q}\|D^{m+q}v\|_{0,J_4} \\ &= ch^{m+q}\|D^{m+q}(\omega_2 w)\|_{0,J_4} \le ch^{m+q}\|w\|_{m,J_4}\end{aligned} \tag{22}$$

for $2 - q \le m \le \min(k+\alpha, t)$. Moreover, from (21) with $l = \min(m+q, k+q)$ we have

$$\|v - \Pi v\|_{\min(m+q,k+q),J_4} \le ch^{m+q-\min(m+q,k+q)}|v|_{m+q,J_4},$$

i.e.

$$\|\omega_2 w - \zeta\|_{\min(m,k)J_4} \le ch^{m-\min(m,k)}|\omega_2 w|_{m,J_4} \le ch^{m-\min(m,k)}\|w\|_{m,J_4} \tag{23}$$

for $2 - q \le m \le \min(k+\alpha, t)$. Interpolation between (22) and (23) yields the desired estimate

$$\|\omega_2 w - \zeta\|_{l,J_4} \le ch^{m-l}\|w\|_{m,J_4} \tag{24}$$

for $-q \le l \le \min(m,k), 2 - q \le m \le \min(k+\alpha, t)$.

Now we consider

$$\phi = \phi_1 + \hat{\phi}_1,$$

where $\phi_1 \in S_h$ is the local approximation ζ of $\omega_2 w$, which is defined by Lemma 4.3, supp $\phi_1 \subseteq J_3$, and $\hat{\phi}_1 \in S_h$ is the solution of the auxiliary problem

$$\begin{cases} \text{Find} \quad \hat{\phi}_1 \in S_h \quad \text{such that} \\ (A(\hat{\omega}_2 w - \hat{\phi}_1), \psi) = 0 \quad \text{for all} \quad \psi \in S_h, \end{cases}$$

i.e. $\hat{\phi}_1$ is the Galerkin approximation of $\hat{\omega}_2 w$. Let

$$\varepsilon_1 = \omega_2 w - \phi_1, \quad \hat{\varepsilon}_1 = \hat{\omega}_2 w - \hat{\phi}_1, \quad \varepsilon = w - \phi \tag{25}$$

then

$$\varepsilon = (\omega_2 w - \phi_1) + (\hat{\omega}_2 w - \hat{\phi}_1) = \varepsilon_1 + \hat{\varepsilon}_1.$$

Lemma 4.4. *There exist positive constants $C = C(K)$ and h_0 such that for any $u \in H^k(J_2) \cap H^\gamma(\Gamma), t \ge \alpha, t - 1 \ge k \ge |\gamma|, q \ge \max(2, 2-\alpha), 0 \le h \le h_0$,*

i)
$$\|\varepsilon_1\|_l \le ch^{\alpha - l}\|E\|_{\omega_1} \quad \text{for} \quad -q \le l \le \min(k, \alpha), \tag{26}$$

ii)
$$\|\hat{\varepsilon}_1\|_l \le ch^{\min(k+\alpha,t)-l}\|E\|_{\omega_1} \quad \text{for} \quad \alpha - t \le l \le \gamma \le k, \tag{27}$$

where

$$\|E\|^2_{\omega_1} = (\omega_1 E, E). \tag{28}$$

Proof. (i) Using Lemma 4.3, we have

$$\|\varepsilon_1\|_l = \|\omega_2 w - \phi_1\|_{l,J_4} \le ch^{m-l}\|w\|_{m,J_4}$$

for all $-q \le l \le \min(k, m), 2-q \le m \le \min(k+\alpha, t)$. Taking $m = \alpha$ and using Lemma 3.1, we obtain

$$\|\varepsilon_1\|_l \le ch^{\alpha-l}\|w\|_{\alpha,J_4} \le ch^{\alpha-l}\|A^*w\|_{0,J_4} = ch^{\alpha-l}\|\omega_1 E\|_0 \le ch^{\alpha-l}\|E\|_{\omega_1}$$

for all $-q \le l \le \min(k, \alpha)$.

(ii) Using the global error estimate (6) for the Galerkin approximation of $\hat{\omega}_2 w$, and taking $m = \min(t, k+\alpha)$, we have

$$\begin{aligned}\|\hat{\varepsilon}_1\|_l &= \|\hat{\omega}_2 w - \hat{\phi}_1\|_l \le ch^{\min(t,k+\alpha)-l}\|\hat{\omega}_2 w\|_{\min(t,k+\alpha)} \\ &= ch^{\min(t,k+\alpha)-l}\|\hat{\omega}_2 A^{*-1}\omega_1 E\|_{\min(t,k+\alpha)} \quad \text{for} \quad \alpha - t \le l \le \gamma \le k.\end{aligned}$$

Then we obtain by the use of Lemma 3.2 the desired estimate

$$\|\hat{\varepsilon}_1\|_l \le ch^{\min(t,k+\alpha)-l}\|\omega_1 E\|_0 \le ch^{\min(k+\alpha,t)-l}\|E\|_{\omega_1} \quad \text{for} \quad \alpha - t \le l \le \gamma \le k.$$

§5. A-Posteriori Local Error Estimates for Equations of Negative Order α with $-2 \le \alpha < 0$

In this section we discuss equation (1) and prove Theorem 2.1 for $-2 \le \alpha < 0$.

Lemma 5.1. *If $u \in H^k(I_2) \cap L_2(\Gamma)$ is the solution of equation (1) for $-2 \le \alpha < 0$, $t - 1 \ge k \ge -\gamma = -\frac{\alpha}{2}$, then for $0 < h \le h_0$, the local error of the finite element solution u_h of (6) on every subarc $I_1 \subset\subset I_2' \subset\subset I_2 \subseteq \Gamma$ can be estimated by*

$$\|E\|_{0,I_1} \le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma} + \|R\|_{-\alpha,I_2} + h\|E\|_{0,I_2'}\},$$

with a constant $C = C(K) > 0$ which is independent of h, u, u_h.

Proof. Let J_i, ω_i and $\hat{\omega}_i$ be defined as before. Then

$$\begin{aligned}\|E\|^2_{0,I_1} \le \|E\|^2_{\omega_1} &= (\omega_1 E, E) = (A^*w, E) = (w, AE) = (w - \phi, AE) \\ &= (\varepsilon, AE) = (E, A^*\varepsilon) = (E, \hat{\omega}_5 A^*\varepsilon) + (E, \omega_5 A^*\varepsilon) \\ &= (E, \hat{\omega}_5 A^*\omega_4\varepsilon) + (E, \hat{\omega}_5 A^*\hat{\omega}_4\varepsilon) + (E, \omega_5 A^*\varepsilon) \\ &= T_1 + T_2 + T_3, \qquad (29)\end{aligned}$$

where

$$T_1 = (E, \hat{\omega}_5 A^*\omega_4\varepsilon), \quad T_2 = (E, \hat{\omega}_5 A^*\hat{\omega}_4\varepsilon), \quad T_3 = (E, \omega_5 A^*\varepsilon). \qquad (30)$$

Using Lemma 3.2, Lemma 4.4 and the global error estimate (6), and observing that $\min(k+\alpha, t) = k+\alpha$ for $\alpha < 0$, we have

$$\begin{aligned}|T_1| &= (E, \hat{\omega}_5 A^*\omega_4\varepsilon) \le \|E\|_{-t+\alpha}\|\hat{\omega}_5 A^*\omega_4\varepsilon\|_{t-\alpha} \le C\|E\|_{-t+\alpha}\|\varepsilon\|_{-t+\alpha} \\ &\le C\|E\|_{-t+\alpha}(\|\varepsilon_1\|_{-t+\alpha} + \|\hat{\varepsilon}_1\|_{-t+\alpha}) \\ &\le Ch^{t-\alpha}\|u\|_{0,\Gamma}(h^t\|E\|_{\omega_1} + h^{k+t}\|E\|_{\omega_1}) \le Ch^{2t-\alpha}\|u\|_{0,\Gamma}\|E\|_{\omega_1}. \qquad (31)\end{aligned}$$

Using Lemma 3.3, Lemma 4.4, the global error estimate (6), and observing that $\hat{\omega}_4\varepsilon_1 = 0$ due to supp $\varepsilon_1 \subseteq J_3$ and $\hat{\omega}_4 = 0$ in J_4, we have

$$\begin{aligned}
|T_2| &= |(E, \hat{\omega}_5 A^*\hat{\omega}_4\varepsilon)| = |(E, \hat{\omega}_5 A^*\hat{\omega}_4(\varepsilon_1 + \hat{\varepsilon}_1))| \\
&= |(E, \hat{\omega}_5 A^*\hat{\omega}_4\hat{\varepsilon}_1)| = |(E, (\hat{\omega}_5 A^* - A^*\hat{\omega}_5)\hat{\omega}_4\hat{\varepsilon}_1) + (E, A^*\hat{\omega}_5\hat{\omega}_4\hat{\varepsilon}_1)| \\
&\le C\{\|E\|_{-1}\|(\hat{\omega}_5 A^* - A^*\hat{\omega}_5)\hat{\omega}_4\hat{\varepsilon}_1\|_1 + |(AE, \hat{\omega}_5\hat{\omega}_4\hat{\varepsilon}_1)|\} \\
&\le C\{\|E\|_{-1}\|\hat{\varepsilon}_1\|_\alpha + \|AE\|_{-\alpha}\|\hat{\varepsilon}_1\|_\alpha\} \\
&\le C\{h\|u\|_{0,\Gamma} + \|R\|_{-\alpha,\Gamma}\}h^k\|E\|_{\omega_1} \\
&\le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma}\}\|E\|_{\omega_1}. \qquad (32)
\end{aligned}$$

Using Lemma 3.3, Lemma 3.2, Lemma 4.4 and the global error estimate (6), we have

$$\begin{aligned}
|T_3| &= |(E, \omega_5 A^*\varepsilon)| = |(E, (\omega_5 A^* - A^*\omega_5)\varepsilon) + (E, A^*\omega_5\varepsilon)| \\
&= |(E, \omega_6(\omega_5 A^* - A^*\omega_5)\varepsilon) + (E, \hat{\omega}_6(\omega_5 A^* - A^*\omega_5)\varepsilon) + (E, A^*\omega_5\varepsilon)| \\
&= |(\omega_6 E, (\omega_5 A^* - A^*\omega_5)\varepsilon) - (E, \hat{\omega}_6 A^*\omega_5\varepsilon) + (\omega_5 AE, \varepsilon)| \\
&\le \|E\|_{\omega_6}\|(\omega_5 A^* - A^*\omega_5)\varepsilon\|_0 + \|E\|_{-t+\alpha}\|\hat{\omega}_6 A^*\omega_5\varepsilon\|_{t-\alpha} + \|\omega_5 AE\|_{-\alpha}\|\varepsilon\|_\alpha \\
&\le C\{\|E\|_{\omega_6}\|\varepsilon\|_{\alpha-1} + \|E\|_{-t+\alpha}\|\varepsilon\|_{-t+\alpha} + \|R\|_{-\alpha,I_2}\|\varepsilon\|_\alpha\} \\
&\le C\{\|E\|_{\omega_6}(h\|E\|_{\omega_1} + h^{k+1}\|E\|_{\omega_1}) + h^{t-\alpha}\|u\|_{0,\Gamma}(h^t\|E\|_{\omega_1} + h^{k+t}\|E\|_{\omega_1}) \\
&\qquad + \|R\|_{-\alpha,I_2}(\|E\|_{\omega_1} + h^k\|E\|_{\omega_1})\} \\
&\le C\{h\|E\|_{0,I_2'} + h^{2t-\alpha}\|u\|_{0,\Gamma} + \|R\|_{-\alpha,I_2}\}\|E\|_{\omega_1}. \qquad (33)
\end{aligned}$$

Since here we have the restriction $-2 \le \alpha < 0$ and $t-1 \ge k \ge -\gamma$, we can apply Lemma 4.4 and the global error estimate (6) in all estimates (31),(32) and (33). Combining (29-33), we obtain

$$\begin{aligned}
\|E\|_{0,I_1}^2 &\le \|E\|_{\omega_1}^2 \le |T_1| + |T_2| + |T_3| \\
&\le C\{(h^{2t-\alpha} + h^{k+1})\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma} + \|R\|_{-\alpha,I_2} + h\|E\|_{0,I_2'}\}\|E\|_{\omega_1} \\
&\le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma} + \|R\|_{-\alpha,I_2} + h\|E\|_{0,I_2'}\}\|E\|_{\omega_1},
\end{aligned}$$

i.e.

$$\|E\|_{0,I_1} \le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma} + \|R\|_{-\alpha,I_2} + h\|E\|_{0,I_2'}\}.$$

Proof of Theorem 2.1 for $-2 \le \alpha < 0$. In this case our assumptions simplify to $u \in H^k(I_2) \cap L_2(\Gamma)$ and $t-1 \ge k \ge -\gamma$. Let

$$I_1 \subset\subset I_2' \subset\subset I_3' \subset\subset \cdots \subset\subset I_{k+2}' \subset\subset I_2 \subseteq \Gamma.$$

Using Lemma 5.1 $(k+1)$-times, we obtain

$$\|u - u_h\|_{0,I_1} \le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma} + \|R\|_{-\alpha,I_2} + h^{k+1}\|E\|_{0,I_2}\}$$

$$\le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma} + \|R\|_{-\alpha,I_2} + h^{k+1}\|R\|_{-\alpha,\Gamma}\}$$

$$\le C\{h^{k+1}\|u\|_{0,\Gamma} + h^k\|R\|_{-\alpha,\Gamma} + \|R\|_{-\alpha,I_2}\}.$$

§6. A-Posteriori Local Error Estimates for Equations of Positive and Zero Order with $0 \le \alpha \le 2$

Lemma 6.1. *If $u \in H^k(I_2) \cap H^\alpha(\Gamma)$ is the solution of equation (1) for $0 \le \alpha \le 2$, $t-1 \ge k \ge \alpha = 2\gamma$, then for $0 < h \le h_0$, the local error of the finite element solution u_h of (1) on every subarc $I_1 \subset\subset I_2' \subset\subset I_2 \subseteq \Gamma$ can be estimated by*

$$\|E\|_{0,I_1} \le C\{h^{\min(k+\alpha,t)+1-\gamma}\|u\|_{\gamma,\Gamma} + h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma} + h^\alpha\|R\|_{0,I_2} + h\|E\|_{0,I_2'}\},$$

with a constant $C = C(K) > 0$ which is independent of h, u, u_h.

Proof. Let J_i, ω_i and $\hat\omega_i$ be defined as above. Then similar to the proof of Lemma 5.1 we have

$$\|E\|_{0,I_1}^2 \le \|E\|_{\omega_1}^2 \le T_1 + T_2 + T_3,$$

where $T_i, i = 1,2,3$, are given by (30). Using Lemma 3.2, Lemma 4.4 and the global estimate (6), we have

$$\begin{aligned}|T_1| &= |(E, \hat\omega_5 A^*\omega_4\varepsilon)| \le \|E\|_{-t+\alpha}\|\hat\omega_5 A^*\omega_4\varepsilon\|_{t-\alpha} \\ &\le C\|E\|_{-t+\alpha}\|\varepsilon\|_{-t+\alpha} \le Ch^{t-\gamma}\|u\|_\gamma(\|\varepsilon_1\|_{-t+\alpha} + \|\hat\varepsilon_1\|_{-t+\alpha}) \\ &\le Ch^{t-\gamma}\|u\|_\gamma(h^t\|E\|_{\omega_1} + h^{\min(k+\alpha,t)+t-\alpha}\|E\|_{\omega_1}) \\ &\le Ch^{2t-\gamma}\|u\|_\gamma\|E\|_{\omega_1}. \end{aligned} \tag{34}$$

Using Lemma 3.3, Lemma 4.4, the global error estimate (6) and observing that $\hat\omega_4\varepsilon_1 = 0$, we have

$$\begin{aligned}|T_2| &= |(E, \hat\omega_5 A^*\hat\omega_4\varepsilon)| = |(E, \hat\omega_5 A^*\hat\omega_4(\varepsilon_1 + \hat\varepsilon_1))| \\ &= |(E, \hat\omega_5 A^*\hat\omega_4\hat\varepsilon_1)| = |(E, (\hat\omega_5 A^* - A^*\hat\omega_5)\hat\omega_4\hat\varepsilon_1) + (E, A^*\hat\omega_5\hat\omega_4\hat\varepsilon_1)| \\ &\le \|E\|_0\|(\hat\omega_5 A^* - A^*\hat\omega_5)\hat\omega_4\hat\varepsilon_1\|_0 + \|AE\|_0\|\hat\varepsilon_1\|_0 \\ &\le C(\|E\|_{0,\Gamma}\|\hat\varepsilon_1\|_{\alpha-1} + \|R\|_{0,\Gamma}\|\hat\varepsilon_1\|_0) \\ &\le C(h^\gamma\|u\|_{\gamma,\Gamma}h^{\min(k+\alpha,t)-(\alpha-1)}\|E\|_{\omega_1} + \|R\|_{0,\Gamma}h^{\min(k+\alpha,t)}\|E\|_{\omega_1}) \\ &= C(h^{\min(k+\alpha,t)+1-\gamma}\|u\|_{\gamma,\Gamma} + h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma})\|E\|_{\omega_1}. \end{aligned} \tag{35}$$

Using Lemma 3.3, Lemma 3.2, Lemma 4.4 and the global error estimate (6), we find

$$\begin{aligned}|T_3| &= |(E, \hat\omega_5 A^*\varepsilon)| = |(E, \omega_6(\omega_5 A^* - A^*\omega_5)\varepsilon) + (E, -\omega_6 A^*\omega_5\varepsilon) + (E, A^*\omega_5\varepsilon)| \\ &= |(\omega_6 E, (\omega_5 A^* - A^*\omega_5)\varepsilon) - (E, \hat\omega_6 A^*\omega_5\varepsilon) + (\omega_5 AE, \varepsilon)| \\ &\le \|E\|_{\omega_6}\|(\omega_5 A^* - A^*\omega_5)\varepsilon\|_0 + \|E\|_{-t+\alpha}\|\hat\omega_6 A^*\omega_5\varepsilon\|_{t-\alpha} + \|\omega_5 AE\|_0\|\varepsilon\|_0 \end{aligned}$$

$$\leq C(\|E\|_{\omega_6}\|\varepsilon\|_{\alpha-1} + \|E\|_{-t+\alpha}\|\varepsilon\|_{-t+\alpha} + \|R\|_{\omega_5}\|\varepsilon\|_0)$$

$$\leq C\{\|E\|_{\omega_6}(h\|E\|_{\omega_1} + h^{\min(k+\alpha,t)-\alpha+1}\|E\|_{\omega_1})$$

$$+ h^{t-\gamma}\|u\|_{\gamma,\Gamma}(h^t\|E\|_{\omega_1} + h^{\min(k+\alpha,t)+t-\alpha}\|E\|_{\omega_1})$$

$$+ \|R\|_{\omega_5}(h^\alpha\|E\|_{\omega_1} + h^{\min(k+\alpha,t)}\|E\|_{\omega_1})\}$$

$$\leq C\{h\|E\|_{\omega_6} + h^{2t-\gamma}\|u\|_{\gamma,\Gamma} + h^\alpha\|R\|_{\omega_5}\}\|E\|_{\omega_1}$$

$$\leq C\{h\|E\|_{0,I_2'} + h^{2t-\gamma}\|u\|_{\gamma,\Gamma} + h^\alpha\|R\|_{0,I_2}\}\|E\|_{\omega_1}. \tag{36}$$

Because of the restriction $0 \leq \alpha \leq 2$ and $t-1 \geq k \geq \alpha$ we can apply Lemma 4.4 and the global error estimate (6) in each of the above three estimates by using $q \geq t-\alpha$ in Lemma 4.4 in order to obtain (34) and (36). Combining (34–36), from

$$\|E\|_{0,I_1}^2 \leq \|E\|_{\omega_1}^2 \leq |T_1| + |T_2| + |T_3| \leq C\{(h^{2t-\gamma} + h^{\min(k+\alpha,t)+1-\gamma})\|u\|_{\gamma,\Gamma}$$

$$+ h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma} + h^\alpha\|R\|_{0,I_2} + h\|E\|_{0,I_2'}\}\|E\|_{\omega_1}$$

$$\leq C\{h^{\min(k+\alpha,t)+1-\gamma}\|u\|_{\gamma,\Gamma} + h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma}$$

$$+ h^\alpha\|R\|_{0,I_2} + h\|E\|_{0,I_2'}\}\|E\|_{\omega_1}$$

we obtain

$$\|E\|_{0,I_1} \leq C\{h^{\min(k+\alpha,t)+1-\gamma}\|u\|_{\gamma,\Gamma} + h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma} + h^\alpha\|R\|_{0,I_2} + h\|E\|_{0,I_2'}\}.$$

Proof of Theorem 2.1 for $0 \leq \alpha \leq 2$. Here our assumptions simplify to $u \in H^k(I_2) \cap H^\alpha(\Gamma), t-1 \geq k \geq \alpha$. Let

$$I_1 \subset\subset I_2' \subset\subset I_3' \subset\subset \cdots \subset\subset T_{k+2}' \subset\subset I_2 \subseteq \Gamma.$$

Using Lemma 6.1 $(k+1)$-times, we have

$$\|u-u_h\|_{0,I_1} \leq C\{h^{\min(k+\alpha,t)+1-\gamma}\|u\|_{\gamma,\Gamma} + h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma}$$

$$+ h^\alpha\|R\|_{0,I_2} + h^{k+1}\|E\|_{0,I_2}\}.$$

Since

$$h^{k+1}\|E\|_{0,I_2} \leq Ch^{k+1+\gamma}\|u\|_{\gamma,\Gamma}$$

and

$$k+1+\gamma \geq \min(k+1+\gamma, t+1-\gamma) = \min(k+\alpha,t)+1-\gamma,$$

we obtain the desired estimate

$$\|u-u_h\|_{0,I_1} \leq C\{(h^{\min(k+\alpha,t)+1-\gamma} + h^{k+1+\gamma})\|u\|_{\gamma,\Gamma} + h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma} + h^\alpha\|R\|_{0,I_2}\}$$

$$\leq C\{h^{\min(k+\alpha,t)+1-\gamma}\|u\|_{\gamma,\Gamma} + h^{\min(k+\alpha,t)}\|R\|_{0,\Gamma} + h^\alpha\|R\|_{0,I_2}\}.$$

Remark. If $u \in H^l(\Gamma)$ instead of $u \in H^\alpha(\Gamma) \cap L_2(\Gamma)$ in Theorem 2.1, $\max(\alpha,0) \leq l \leq k$, then the local error satisfies the estimates,

$$\|u-u_h\|_{0,I_1} \leq C\{h^{k+l+1}\|u\|_l + h^k\|R\|_{-\alpha} + \|R\|_{-\alpha,I_2}\} \tag{37}$$

for $-2 \leq \alpha < 0$ and

$$\|u - u_h\|_{0,I_1} \leq C\{h^{\min(k+\alpha,t)+l-\alpha+1}\|u\|_l + h^{\min(k+\alpha,t)}\|R\|_0 + h^{\alpha}\|R\|_{0,I_2}\} \quad (38)$$

for $0 \leq \alpha \leq 2$ (See [38]).

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