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ESTIMATE OF CONDITION NUMBER FOR SOME DISCRETE ILL-POSED EQUATIONS*

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Abstract

This paper extends the numerical method and the estimate of the condition number for some discrete ill-posed equations with positive definite matrix in [2] to the case with generalized positive definite matrix.

§1. Introduction

This paper considers the following discrete ill-posed problem:

$$Ax = b \tag{1.1}$$

where A belongs to the set

$$D = \{A \in R^{n\times n} |(Ax, x) > 0, x \in R^n \setminus 0\}, \tag{1.2}$$

and the data b and solution x lie in the n-dimensional vector space R^n. In practice, this problem is ill-posed as A has a large condition number, which is defined as the radio of the largest to the smallest singular values. Here a bounded inverse of A does exist in theory, but the solution $x = A^{-1}b$ is numerically unstable.

Let b_δ be approximate or measured data satisfying

$$\|b_\delta - b\|_2 \le \delta, \tag{1.3}$$

where $\|\cdot\|_2$ is Euclid's norm, $\delta \ge 0$. Here we can replace equation (1.1) by an approximate equation with prior estimate (1.3)

$$Ax = b_\delta. \tag{1.4}$$

For the ill-posed problem (1.4), Tikhonov [1] et al. have developed several very useful numerical methods based on the least-squares principle:

$$\|Ax - b_\delta\|_2^2 + \alpha^2\|x\|_2^2 = \min. \tag{1.5}$$

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The minimal solution is

$$x_\alpha = (A^* A + \alpha^2 I)^{-1} A^* b_\delta. \tag{1.6}$$

In [2], J.N. Franklin has analyzed a different numerical method. (1.4) is replaced by the following approximate equation

$$(A + \alpha I)x_\alpha = b_\delta, \tag{1.7}$$

in which $\alpha > 0$, A is a positive definite matrix and I is the unit matrix. The solution by Franklin's method has a simple form

$$x_\alpha = (A + \alpha I)^{-1} b_\delta. \tag{1.8}$$

Above-mentioned Franklin's method is generally less applicable than Tikhonov's, because it applies only to the ill-posed problem $Ax = b$ in which A is a positive definite operator. In this paper, Franklin's method will be extended to the discrete ill-posed problem (1.4) in which A belongs to set D. And we will give an estimate of the condition number for (1.6) and (1.8) for $A \in D$.

We now define $\|\cdot\|_2$ to be the spectral norm, and $K(A)$ the spectral condition number of matrix A. Moreover, we define several sets:

$$D_1 = \{B \in R^{n\times n} | B \text{ is a positive definite matrix}\},$$
$$D_2 = \{B \in R^{n\times n} | B^* = B\}.$$

§2. Estimate of Condition Number

Although Franklin's method is less applicable than Tikhonov's, its simple form of solution has many advantages in numerical analysis. [2] gives the following estimate of the condition number for the two methods:

$$\frac{1}{2} \le \frac{K^2(A + \alpha I)}{K(A^* A + \alpha^2 I)} \le 2, \tag{2.1}$$

where $A \in D_1$.

(2.1) implies that Franklin's method is not only simpler but also stabler than Tikhonov's as A belongs to D_1. In the following, we will give an estimate of the condition number similar to (2.1) when A belongs to D.

Let $\{\lambda_i(A)\}, \{\sigma_i(A)\}, i = 1, 2, \cdots, n$, be the eigenvalues and the singular values of matrix A, respectively. And we assume

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n. \tag{2.2}$$

If $A \in D_2$, then suppose

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n. \tag{2.3}$$

Lemma 1. *If $A \in D$, then $A + A^* \in D_1$.*

Lemma 2 (Weyl). *Let $A, B \in D_2$. Then the following inequality holds:*

$$\lambda_i(A) + \lambda_n(B) \le \lambda_i(A+B) \le \lambda_i(A) + \lambda_1(B). \tag{2.4}$$

Lemma 3. *If $A \in R^{n\times n}$, then*

$$\lambda_i(A+A^*) \le 2\sigma_i(A). \tag{2.5}$$

Our main results are the following

Theorem 1. *Let $A \in D$. Then we have the estimate:*

$$\frac{1}{1+S(\alpha)K(A^*A)} \le \frac{K^2(A+\alpha I)}{K(A^*A+\alpha^2 I)} \le \frac{2}{1+\dfrac{\alpha\lambda_n(A+A^*)}{\lambda_n(A^*A)+\alpha^2}} < 2, \tag{2.6}$$

where $S(\alpha) = 2\alpha/\sigma_1(A)$.

Proof. From Lemmas 1 and 2, we can obtain

$$\sigma_1^2(A+\alpha I) \le \lambda_1(A^*A+\alpha^2 I) + \alpha\lambda_1(A+A^*), \tag{2.7}$$

$$\sigma_n^2(A+\alpha I) \ge \lambda_n(A^*A+\alpha^2 I) + \alpha\lambda_n(A+A^*). \tag{2.8}$$

Again by Lemma 3 and the arithmetic mean inequality, we have

$$\lambda_i(A^*A+\alpha^2 I) = \lambda_i(A^*A) + \alpha^2 \ge 2\alpha\lambda_i^{1/2}(A^*A) \ge \alpha\lambda_i(A+A^*), \tag{2.9}$$

i.e.,

$$\alpha\lambda_i(A+A^*)/\lambda_i(A^*A+\alpha^2 I) \le 1, \quad i = 1,2,\cdots,n. \tag{2.10}$$

Thus, using inequaliies (2.7)–(2.10), we have

$$\begin{aligned} K^2(A+\alpha I) &= \frac{\sigma_1^2(A+\alpha I)}{\sigma_n^2(A+\alpha I)} \le \frac{\lambda_1(A^*A+\alpha^2 I)+\alpha\lambda_1(A+A^*)}{\lambda_n(A^*A+\lambda^2 I)+\alpha\lambda_n(A+A^*)} \\ &= K(A^*A+\alpha^2 I)\frac{1+\alpha\lambda_1(A+A^*)/\lambda_1(A^*A+\alpha^2 I)}{1+\alpha\lambda_n(A+A^*)/\lambda_n(A^*A+\alpha^2 I)} \\ &\le K(A^*A+\alpha^2 I)\frac{2}{1+\alpha\lambda_n(A+A^*)/\lambda_n(A^*A+\alpha^2 I)} \\ &< 2K(A^*A+\alpha^2 I). \end{aligned} \tag{2.11}$$

Similarly to the above proof, we also have

$$\sigma_1^2(A+\alpha I) \ge \lambda_1(A^*A+\alpha^2 I), \tag{2.12}$$

$$\begin{aligned} \sigma_n^2(A+\alpha I) &\le \lambda_n(A^*A+\alpha^2 I) + \alpha\lambda_1(A+A^*) \\ &\le \lambda_n(A^*A+\alpha^2 I) + 2\alpha\lambda_1^{1/2}(A^*A). \end{aligned} \tag{2.13}$$

Thus

$$\begin{aligned} K^2(A+\alpha I) &= \frac{\sigma_1^2(A+\alpha I)}{\sigma_n^2(A+\alpha I)} \ge \frac{\lambda_1(A^*A+\alpha^2 I)}{\lambda_n(A^*A+\alpha^2 I)+2\alpha\lambda_1^{1/2}(A^*A)} \\ &\ge \frac{K(A^*A+\alpha^2 I)}{1+2\alpha K(A^*A)/\sigma_1(A)}. \end{aligned} \tag{2.14}$$

Clearly, (2.11) and (2.14) imply (2.6).

Theorem 1 proves that algorithm (1.7) is very stable numerically.

Using the method in this paper, we can also prove that if $A \in D_1$, then

$$\frac{K^2(A+\alpha I)}{K(A^*A+\alpha^2 I)} \leq \frac{2}{1+2\alpha\lambda_n(A)/(\lambda_n^2(A)+\alpha^2)} < 2. \tag{2.15}$$

Obviously, (2.15) improves inequality (2.1).

Theorem 2. *If $A \in D$ and $K(A^*A) \geq K(A+A^*)$, then there is the estimate*

$$K(A+\alpha I) \leq K(A). \tag{2.16}$$

Proof. From (2.7) and (2.8), we have inequality

$$\frac{\sigma_1^2(A+\alpha I)}{\sigma_n^2(A+\alpha I)} \leq \frac{\lambda_1(A^*A)+\alpha\lambda_1(A+A^*)+\alpha^2}{\lambda_n(A^*A)+\alpha\lambda_n(A+A^*)+\alpha^2} \leq \frac{\lambda_1(A^*A)+\alpha\lambda_1(A+A^*)}{\lambda_n(A^*A)+\alpha\lambda_n(A+A^*)}. \tag{2.17}$$

Again by using the assumption , we have

$$\frac{\lambda_1(A^*A)+\alpha\lambda_1(A+A^*)}{\lambda_n(A^*A)+\alpha\lambda_n(A+A^*)} \leq \frac{\lambda_1(A^*A)}{\lambda_n(A^*A)}. \tag{2.18}$$

Hence

$$K(A+\alpha I) = \frac{\sigma_1(A+\alpha I)}{\sigma_n(A+\alpha I)} \leq \sqrt{\frac{\lambda_1(A^*A)}{\lambda_n(A^*A)}} = K(A).$$

Remark. The estimate (2.16) implies that the condition of matrix $A+\alpha I$ is not inferior to that of matrix A. If $A \in D_1$, then $K(A^*A) = \lambda_1^2(A)/\lambda_n^2(A) \geq 2\lambda_1(A)/2\lambda_n(A)$, so the condition $K(A^*A) \geq K(A+A^*)$ holds naturally.

Moreover, our numerical experiences indicate that Franklin's method can also be extended to solve some ill-posed equations $Ax = b$ in which A is a non-symmetrical matrix and has positive eigenvalues or other characteristics. For example, it is well known that the discrete algebraic equations for integral equations of the first kind with smooth nonself-conjugate kernel belong to ill-posed problems [3], [4]. We have obtained satisfactory numerical results by using the method of this paper. But, in those cases, it is very difficult to make an estimate of the condition number. Some research results will be discussed in another paper.

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