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A PSEUDOSPECTRAL METHOD FOR SOLVING NAVIER-STOKES EQUATIONS*

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Abstract

In this paper, we propose a new kind of pseudospectral schemes with a restraint operator to solve the periodic problem of Navier-Stokes equations. The generalized stability of the schemes is analysed and the convergence is proved. Numerical results are presented also.

§1. Introduction

We consider the periodic problem of Navier-Stokes equations as follows:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + (U(x,t)\cdot\nabla)U(x,t) + \nabla P(x,t) - \nu\nabla^2 U(x,t) = f(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t)\in\Omega\times(0,T], \\ \nabla\cdot U(x,t) = 0, \quad (x,t)\in\Omega\times[0,T], \\ U(x,0) = U_0(x), \quad P(x,0) = P_0(x), \quad x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega = [0,2\pi]^n, n = 2$ or 3, $\nu \geq 0$. $U = (U_1, \cdots, U_n)$ is the velocity. P is the ratio of pressure to density. The functions U_0, P_0 and f are given with the period 2π for all the space variables. We require in addition

$$\int_\Omega P(x,t)dx = 0, \qquad \forall t\in[0,T].$$

It is well known that if the genuine solutions of PDEs are infinitely differentiable, then the convergence rates of their spectral/pseudospectral approximations are infinite[1]. Hence they have been widely used in computational fluid dynamics[2]. The pseudospectral methods are easier to implement. But they are not as stable as the spectral ones due to "aliasing". Therefore some authors proposed the filtering techniques[3–5]. In this paper, a new kind of pseudospectral schemes with a restraint operator is constructed to solve (1.1). The generalized stability and the convergence are analysed. In particular, the uniform stability and convergence (independent of the coefficient ν) are obtained in some cases.

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§2. Notations and Lemmas

Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm of $L^2(\Omega)$ respectively. Let $\|\cdot\|_\infty$ be the norm of $L^\infty(\Omega)$. $\|\cdot\|_\mu$ and $|\cdot|_\mu$ denote the norm and semi-norm of $H^\mu(\Omega)$. Define

$$C_p^\infty(\Omega)=\{u/u\in C^\infty(\Omega),\, u \text{ has the period } 2\pi \text{ for the variables } x_q, 1\le q\le n\},$$

and let $H_p^\mu(\Omega)$ be the closure of $C_p^\infty(\Omega)$ in $H^\mu(\Omega)$. Let X be a Banach space. Define

$$L^2(0,T;X)=\{u/u:[0,T]\to X, \|u\|_{L^2(0,T;X)}=\Big(\int_0^T\|u(t)\|_X^2dt\Big)^{1/2}<\infty\},$$

$$C(0,T;X)=\{u/u:[0,T]\to X \text{ is strongly continuous, } |||u|||_X=\max_{0\le t\le T}\|u(t)\|_X\}.$$

Denote by Z the set of integers. For $k=(k_1,\cdots,k_n)\in Z^n$, let $|k|_\infty=\max\limits_{1\le q\le n}|k_q|$ and $|k|=\Big(\sum\limits_{q=1}^n k_q^2\Big)^{1/2}$. For a positive integer N, we define

$$V_N=\mathrm{Span}\{e^{ik\cdot x}/k\in Z^n, |k|_\infty\le N\},\quad W_N=\mathrm{Span}\{e^{ik\cdot x}/k\in Z^n,|k|\le N\}.$$

Let P_N be the orthogonal projection operator from $L^2(\Omega)$ onto W_N. $\tilde P_c$ is the Lagrange interpolation operator from $C(\Omega)$ onto V_N at points $x^{(j)}=2\pi j/(2N+1)$, $j\in Z^n, |j|_\infty\le N$. Let $P_c=P_N\tilde P_c$. It is easy to see that[5]

$$(P_c(uv),w)=(P_c(wv),u),\quad \forall u,v,w\in W_N. \tag{2.1}$$

Lemma 2.1[3]. *If $u,v\in V_N$, then*

$$\text{(i)}\ |u|_1^2\le nN^2\|u\|^2, \tag{2.2}$$

$$\text{(ii)}\ |uv|_1^2\le n(2N+1)^n(\|u\|^2|v|_1^2+\|v\|^2|u|_1^2). \tag{2.3}$$

Lemma 2.2[6]. *If $0\le\mu\le\sigma$, then for all $u\in H_p^\sigma(\Omega)$,*

$$\|P_Nu-u\|_\mu\le CN^{\mu-\sigma}|u|_\sigma.$$

If in addition $\sigma>n/2$, then

$$\|P_cu-u\|_\mu\le CN^{\mu-\sigma}|u|_\sigma.$$

Now we define the restraint operator R_r for $r>1$, i.e., if

$$u(x)=\sum_{|k|\le N}u_ke^{ik\cdot x}$$

then

$$R_ru(x)=\sum_{|k|\le N}\Big(1-\Big(\frac{|k|}{N}\Big)^r\Big)u_ke^{ik\cdot x}.$$

Lemma 2.3. *If $0\le\mu\le\sigma, r\ge\sigma-\mu$, then for all $u\in W_N$,*

$$\|R_ru-u\|_\mu\le CN^{\mu-\sigma}|u|_\sigma.$$

Proof. Since $0\le\mu\le\sigma, r\ge\sigma-\mu$, we have clearly that

$$\|R_ru-u\|_\mu^2\le C\sum_{|k|\le N}\frac{|k|^{2\sigma}}{N^{2(\sigma-\mu)}}\Big(\frac{|k|}{N}\Big)^{2r+2(\mu-\sigma)}|u_k|^2\le CN^{2(\mu-\sigma)}|u|_\sigma^2.$$

Lemma 2.4. *If $u, v \in W_N, w \in H_p^{1+\frac{n}{2}+\gamma}(\Omega), \gamma > 0$, then there exists a positive constant C depending only on r and γ, such that*

$$\left|\left(P_c\Big(\frac{\partial R_r u}{\partial x_q} v\Big), w\right) + \left(P_c\Big((R_r u)\frac{\partial v}{\partial x_q}\Big), w\right)\right| \le C\|u\|\,\|v\|\,\|w\|_{1+\frac{n}{2}+\gamma}.$$

Proof. This Lemma can be proved in the same way as in the proof of Lemma 6 of [5].

§3. The Pseudospectral Schemes

Let τ be the mesh size of the variable t and $S_\tau = \left\{t = k\tau / 0 \le k \le \left[\frac{T}{\tau}\right]\right\}$. Let $u_t(t)$ be the first order forward difference quotient. Clearly

$$2(u_t(t), u(t)) = (\|u(t)\|^2)_t - \tau\|u_t(t)\|^2. \tag{3.1}$$

There are two key points for solving (1.1) numerically. Firstly, the incompressible condition should be treated reasonably. Chorin[7] proposed the artificial compression method and Guo Ben-yu[8] improved it. We follow the idea of [8] to approximate the following equation

$$\beta\frac{\partial P}{\partial t} + \nabla\cdot U - \beta\nu_1\nabla^2 P = 0, \quad \beta, \nu_1 > 0.$$

Secondly, the conservation laws should be simulated. As we know, the solution of (1.1) satisfies

$$\int_\Omega U(x,t)dx = \int_\Omega U_0(x)dx + \int_0^t\int_\Omega f(x,t')dxdt',$$

$$\|U(t)\|^2 + 2\nu\int_0^t |U(t')|_1^2 dt' = \|U_0\|^2 + 2\int_0^t (U(t'), f(t'))dt'.$$

In order to keep the analogous properties for numerical solutions, we define

$$d(w,v) = \frac{1}{2}\sum_{q=1}^n \Big(P_c\Big(v_q\frac{\partial w}{\partial x_q}\Big) + \frac{\partial}{\partial x_q}(P_c(v_q w))\Big), \quad \forall w \in W_N,\ v \in (W_N)^n,$$

$$d(u,v) = (d(u_1,v), d(u_2,v), \cdots, d(u_n,v)), \qquad \forall u, v \in (W_N)^n.$$

From (2.1), we have for all $u, v, w \in (W_N)^n$,

$$(d(u,v), w) + (d(w,v), u) = 0, \tag{3.2}$$

$$(d(u,v), 1) = -\frac{1}{2}\sum_{q=1}^n (\nabla\cdot v, u_q). \tag{3.3}$$

A kind of pseudospectral schemes with the restraint operator R_r for filtering is to find $u(t) \in (W_N)^n$ and $p(t) \in W_N$ for $t \in S_\tau$, such that

$$\begin{cases} u_t(t) + R_r d(R_r(u(t) + \delta\tau u_t(t)), u(t)) + \nabla(p(t) + \sigma\tau p_t(t)) - \nu\nabla^2(u(t) + \theta\tau u_t(t)) \\ \qquad = R_r P_c f(t), \\ \beta p_t(t) + \nabla\cdot(u(t) + \sigma\tau u_t(t)) - \beta\nu_1\nabla^2(p(t) + \theta\tau p_t(t)) = 0, \\ u(0) = P_c U_0, \quad P(0) = P_c P_0, \end{cases} \tag{3.4}$$

where $\delta, \sigma, \theta \geq 0$ are parameters. By (3.2) and (3.3), the solution of (3.4) satisfies the following discrete conservation law:

$$\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx + \tau \sum_{t'=0}^{t-\tau} \int_\Omega \Big[R_r P_c f(x,t') + \frac{1}{2}(\nabla \cdot R_r u(x,t')) \times \Big(\sum_{q=1}^{n} R_r(u_q(x,t') + \delta\tau u_{q_t}(x,t'))\Big)\Big] dx,$$

and if in addition $\delta = \sigma = \theta = 1/2$, then

$$\|u(t)\|^2 + \beta\|p(t)\|^2 + \frac{\tau}{2}\sum_{t'=0}^{t-\tau}(\nu|u(t') + u(t'+\tau)|_1^2 + \beta\nu_1|p(t') + p(t'+\tau)|_1^2) = \|u(0)\|^2 + \beta\|p(0)\|^2 + \tau\sum_{t'=0}^{t-\tau}(R_r P_c f(t'), u(t') + u(t'+\tau)).$$

§4. The Stability

Suppose that the initial values $u(0), p(0)$ in (3.4) have errors $\tilde{u}_0, \tilde{p}_0$ and that the right-hand terms in the first and second equation have errors $\tilde{f}$ and $\tilde{g}$ respectively. Then the errors $\tilde{u}(t), \tilde{p}(t)$ of $u(t)$ and $p(t)$ satisfy

$$\begin{cases} \tilde{u}_t(t) + R_r d(R_r(\tilde{u}(t) + \delta\tau\tilde{u}_t(t)), u(t) + \tilde{u}(t)) + R_r d(R_r(u(t) + \delta\tau u_t(t)), \tilde{u}(t)) \\ \qquad + \nabla(\tilde{p}(t) + \sigma\tau\tilde{p}_t(t)) - \nu\nabla^2(\tilde{u}(t) + \theta\tau\tilde{u}_t(t)) = \tilde{f}(t), \\ \beta\tilde{p}(t) + \nabla\cdot(\tilde{u}(t) + \sigma\tau\tilde{u}_t(t)) - \beta\nu_1\nabla^2(\tilde{p}(t) + \theta\tau\tilde{p}_t(t)) = \tilde{g}(t), \\ \tilde{u}(0) = \tilde{u}_0, \quad \tilde{p}(0) = \tilde{p}_0. \end{cases} \tag{4.1}$$

Consider first the case $\sigma \leq 1/2$. Let $a > 1$ and $\varepsilon > 0$ be constants to be chosen below. Taking inner product of the first equation of (4.1) with $2\tilde{u}(t) + a\tau\tilde{u}_t(t)$, we have from (3.1) and (3.2) that

$$\begin{aligned} &(\|\tilde{u}(t)\|^2)_t + \tau(a-1-\varepsilon)\|\tilde{u}_t(t)\|^2 + (\nabla(\tilde{p}(t) + \sigma\tau\tilde{p}_t(t)), 2\tilde{u}(t) + a\tau\tilde{u}_t(t)) \\ &\quad + 2\nu|\tilde{u}(t)|_1^2 + \nu\tau(\theta + \frac{a}{2})(|\tilde{u}(t)|_1^2)_t + \nu\tau^2(\theta a - \theta - \frac{a}{2})|\tilde{u}_t(t)|_1^2 + \sum_{m=1}^{3} F_m(t) \\ &\leq \|\tilde{u}(t)\|^2 + (1 + \frac{\tau a^2}{4\varepsilon})\|\tilde{f}(t)\|^2, \end{aligned} \tag{4.2}$$

where

$$F_1 = \tau(a - 2\delta)(R_r d(R_r\tilde{u}, u + \tilde{u}), \tilde{u}_t), \quad F_2 = 2(R_r d(R_r(u + \delta\tau u_t), \tilde{u}), \tilde{u}),$$
$$F_3 = a\tau(R_r d(R_r(u + \delta\tau u_t), \tilde{u}), \tilde{u}_t).$$

Taking inner product of the second equation of (4.1) with $2\tilde{p}(t) + a\tau\tilde{p}_t(t)$, we have from (3.1) that

$$\begin{aligned} &\beta(\|\tilde{p}(t)\|^2)_t + \beta\tau(a-1-\varepsilon)\|\tilde{p}_t(t)\|^2 + (\nabla\cdot(\tilde{u}(t) + \sigma\tau\tilde{u}_t(t)), 2\tilde{p}(t) + a\tau\tilde{p}_t(t)) \\ &\quad + 2\beta\nu_1|\tilde{p}(t)|_1^2 + \beta\nu_1\tau(\theta + \frac{a}{2})(|\tilde{p}(t)|_1^2)_t + \beta\nu_1\tau^2(\theta a - \theta - \frac{a}{2})|\tilde{p}_t(t)|_1^2 \\ &\leq \beta\|\tilde{p}(t)\|^2 + \Big(\frac{1}{\beta} + \frac{\tau a^2}{4\beta\varepsilon}\Big)\|\tilde{g}(t)\|^2. \end{aligned} \tag{4.3}$$

Let

$$\eta(t) = |(\nabla(\tilde{p}(t) + \sigma\tau\tilde{p}_t(t)), 2\tilde{u}(t) + a\tau\tilde{u}_t(t)) + (\nabla \cdot (\tilde{u}(t) + \sigma\tau\tilde{u}_t(t)), 2\tilde{p}(t) + a\tau\tilde{p}_t(t))|,$$

and $b > 0$ be an undetermined constant. We have

$$\eta(t) = \tau|a - 2\sigma|\,|(\nabla\tilde{p}(t), \tilde{u}_t(t)) + (\nabla \cdot \tilde{u}(t), \tilde{p}_t(t))| \tag{4.4}$$

and

$$\eta(t) \le \frac{b\tau}{2}(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2) + \frac{n\tau(a-2\sigma)^2}{2b\beta}(|\tilde{u}(t)|_1^2 + \beta|\tilde{p}(t)|_1^2). \tag{4.5}$$

Adding (4.3) to (4.2), we have by (4.5) that

$$\begin{aligned}(\|\tilde{u}(t)\|^2 &+ \beta\|\tilde{p}(t)\|^2)_t + \tau(a - 1 - \varepsilon - \frac{b}{2})(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2) + \Big(2\nu - \frac{n\tau(a-2\sigma)^2}{2b\beta}\Big)|\tilde{u}(t)|_1^2 \\ &+(2\beta\nu_1 - \frac{n\tau(a-2\sigma)^2}{2b})|\tilde{p}(t)|_1^2 + \tau(\theta + \frac{a}{2})(\nu|\tilde{u}(t)|_1^2 + \beta\nu_1|\tilde{p}(t)|_1^2)_t + \tau^2(\theta a - \theta - \frac{a}{2}) \\ &\times(\nu|\tilde{u}_t(t)|_1^2 + \beta\nu_1|\tilde{p}_t(t)|_1^2) + \sum_{m=1}^{3} F_m(t) \le \|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2 + G_1(t),\end{aligned} \tag{4.6}$$

where

$$G_1(t) = (1 + \frac{\tau a^2}{4\varepsilon})(\|\tilde{f}(t)\|^2 + \frac{1}{\beta}\|\tilde{g}(t)\|^2).$$

We denote hereafter by C a general constant independent of ν, N, τ, u and p. Now we are going to estimate $|F_m|$. Firstly,

$$|F_1| \le \varepsilon\tau\|\tilde{u}_t\|^2 + \frac{\tau(a-2\delta)^2}{4\varepsilon}\|R_r d(R_r\tilde{u}, u + \tilde{u})\|^2.$$

Let $\gamma > 0$ be a constant. By using Lemma 2.1, Lemma 2.3 and the embedding theorems, we get

$$\|R_r d(R_r\tilde{u}, u)\| \le C\sum_{m,q=1}^{n}(\|u_q\|_\infty\Big\|\frac{\partial\tilde{u}_m}{\partial x_q}\Big\| + \Big\|\frac{\partial u_q}{\partial x_q}\Big\|_\infty\|\tilde{u}_m\|) \le CN\|u\|_{1+\frac{n}{2}+\gamma}\|\tilde{u}\|,$$

$$\begin{aligned}\|R_r d(R_r\tilde{u}, \tilde{u})\| &\le C\sum_{m,q=1}^{n}\Big(\Big\|P_c(\tilde{u}_q\frac{\partial R_r\tilde{u}_m}{\partial x_q})\Big\| + \Big\|\frac{\partial}{\partial x_q}(P_c(\tilde{u}_q R_r\tilde{u}_m))\Big\|\Big) \\ &\le CN^{n/2}\|\tilde{u}\|\cdot|\tilde{u}|_1.\end{aligned}$$

Hence

$$|F_1| \le \varepsilon\tau\|\tilde{u}_t\|^2 + \frac{C\tau(a-2\delta)^2}{\varepsilon}(N^2\|u\|_{1+\frac{n}{2}+\gamma}^2\|\tilde{u}\|^2 + N^n\|\tilde{u}\|^2|\tilde{u}|_1^2). \tag{4.7}$$

Secondly,

$$|F_2| = 2|(d(R_r(u + \delta\tau u_t), \tilde{u}), R_r\tilde{u})| \le \sum_{m=1}^{3}|I_m|,$$

where

$$I_1 = \sum_{m,q=1}^{n} (P_c(\tilde{u}_q \frac{\partial}{\partial x_q} R_r(u_m + \delta\tau u_{mt})), R_r\tilde{u}_m),$$

$$I_2 = \sum_{m,q=1}^{n} (P_c(\tilde{u}_q \frac{\partial}{\partial x_q} R_r\tilde{u}_m + (R_r\tilde{u}_m)\frac{\partial}{\partial x_q}\tilde{u}_q), R_r(u_m + \delta\tau u_{mt})),$$

$$I_3 = \sum_{m=1}^{n} \Big(P_c\Big((R_r\tilde{u}_m)\cdot\sum_{q=1}^{n}\frac{\partial}{\partial x_q}\tilde{u}_q\Big), R_r(u_m + \delta\tau u_{mt})\Big).$$

Then by using (2.1), lemmas 2.2–2.4 and the embedding theorems, we have

$$|F_2| \le C\|u + \delta\tau u_t\|_{1+\frac{n}{2}+\gamma}(\|\tilde{u}\|^2 + \|\tilde{u}\|\,\|\nabla\cdot\tilde{u}\|) \tag{4.8}$$

and so

$$|F_2| \le \frac{\nu}{2}|\tilde{u}|_1^2 + (\frac{C}{\nu}|||u|||_{1+\frac{n}{2}+\gamma} + C)|||u|||_{1+\frac{n}{2}+\gamma}\|\tilde{u}\|^2. \tag{4.9}$$

Finally, we have from (2.2) that

$$|F_3| \le \varepsilon\tau\|\tilde{u}_t\|^2 + \frac{C\tau a^2N^2}{\varepsilon}|||u|||^2_{1+\frac{n}{2}+\gamma}\|\tilde{u}\|^2. \tag{4.10}$$

By substituting (4.7), (4.9) and (4.10) into (4.6), we get

$$\begin{aligned}&(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2)_t + \tau(a - 1 - 3\varepsilon - \frac{b}{2})(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2) + (\nu - \frac{n\tau(a-2\sigma)^2}{2b\beta})|\tilde{u}(t)|_1^2\\ &+(2\beta\nu_1 - \frac{n\tau(a-2\sigma)^2}{2b})|\tilde{p}(t)|_1^2 + \tau(\theta + \frac{a}{2})(\nu|\tilde{u}(t)|_1^2 + \beta\nu_1|\tilde{p}(t)|_1^2)_t + \tau^2(\theta a - \theta - \frac{a}{2})\\ &\times(\nu|\tilde{u}_t(t)|_1^2 + \beta\nu_1|\tilde{p}_t(t)|_1^2) \le C_1(\|\tilde{u}(t)\|^2 + \beta\|\tilde{p}(t)\|^2) + \xi_1(t)|\tilde{u}(t)|_1^2 + G_1(t),\end{aligned} \tag{4.11}$$

where

$$C_1 = 1 + \frac{C\tau N^2}{\varepsilon}(a^2 + (a-2\delta)^2)|||u|||^2_{1+\frac{n}{2}+\gamma} + (\frac{C}{\nu}|||u|||_{1+\frac{n}{2}+\gamma} + C)|||u|||_{1+\frac{n}{2}+\gamma},$$

$$\xi_1(t) = -\frac{\nu}{2} + \frac{C\tau N^n}{\varepsilon}(a-2\delta)^2\|\tilde{u}(t)\|^2.$$

We assume $\tau = O(N^{-s}), s \ge 2$ and fix the values of positive constants ε, b, r_0. Then we choose the parameter a in three different cases:

(i) If $\theta > 1/2$, we take

$$a \ge a_1 = \max(1 + 3\varepsilon + \frac{b}{2} + r_0, \frac{2\theta}{2\theta - 1})$$

and so

$$\begin{aligned}&\tau(a - 1 - 3\varepsilon - \frac{b}{2})(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2) + \tau^2(\theta a - \theta - \frac{a}{2})(\nu|\tilde{u}_t(t)|_1^2 + \beta\nu_1|\tilde{p}_t(t)|_1^2)\\ &\quad\ge r_0\tau(\|\tilde{u}_t(t)\|^2 + \beta\|\tilde{p}_t(t)\|^2).\end{aligned} \tag{4.12}$$

(ii) If $\theta = 1/2$, then we take

$$a \ge a_2 = 1 + 3\varepsilon + \frac{b}{2} + r_0 + \frac{1}{2}\tau n N^2\max(\nu, \nu_1).$$

By (2.2), we have (4.12) also.

(iii) If $\theta < 1/2$ and $\tau N^2 < \dfrac{2}{n(1-2\theta)\max(\nu,\nu_1)}$, then we take

$$a \geq a_3 = \frac{1+3\varepsilon+b/2+r_0+\theta\tau n N^2\max(\nu,\nu_1)}{1+(\theta-1/2)\tau n N^2\max(\nu,\nu_1)}.$$

Thus by (2.2), (4.12) is still true.

We assume in addition that

$$\beta \geq \frac{n\tau(a-2\sigma)^2}{b\min(\nu,\nu_1)}. \tag{4.13}$$

Then it follows from (4.11) that

$$(\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2)_t + r_0\tau(\|\tilde{u}_t(t)\|^2+\beta\|\tilde{p}_t(t)\|^2) + \frac{1}{2}(\nu|\tilde{u}(t)|_1^2+\beta\nu_1|\tilde{p}(t)|_1^2) + \tau(\theta+\frac{a}{2})$$
$$\times(\nu|\tilde{u}(t)|_1^2+\beta\nu_1|\tilde{p}(t)|_1^2)_t \leq C(\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2) + \xi_1(t)|\tilde{u}(t)|_1^2 + G_1(t). \tag{4.14}$$

Let

$$Q_1(\tilde{u}(t),\tilde{p}(t)) = \|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2 + \tau\sum_{t'=0}^{t-\tau}(r_0\tau(\|\tilde{u}_t(t')\|^2+\beta\|\tilde{p}_t(t')\|^2)$$
$$+\frac{1}{2}(\nu|\tilde{u}(t')|_1^2+\beta\nu_1|\tilde{p}(t')|_1^2)),$$

$$\rho_1(\tilde{u}_0,\tilde{p}_0,G_1,t) = \|\tilde{u}_0\|^2+\beta\|\tilde{p}_0\|^2+\tau(\theta+\frac{a}{2})(\nu|\tilde{u}_0|_1^2+\beta\nu_1|\tilde{p}_0|_1^2)+\tau\sum_{t'=0}^{t-\tau}G_1(t').$$

Summing (4.14) for $t' \leq t-\tau, t' \in S_\tau$, we have

$$Q_1(\tilde{u}(t),\tilde{p}(t)) \leq \rho_1(\tilde{u}_0,\tilde{p}_0,G_1,t) + \tau\sum_{t'=0}^{t-\tau}(C_1Q_1(\tilde{u}(t'),\tilde{p}(t'))+\xi_1(t')|\tilde{u}(t')|_1^2).$$

Finally, by applying Lemma 4 of [3], we get the following result.

Theorem 4.1. *Assume that the following conditions are satisfied:*

(i) $\nu > 0, \tau = O(N^{-s}), s \geq 2$, *and* β *satisfies* (4.13);

(ii) $\sigma \leq 1/2, \theta \geq 1/2$ or $\tau N^2 < \dfrac{2}{n(1-2\theta)\max(\nu,\nu_1)}$;

(iii) there exist $t_0 \in S_\tau$ and suitably small positive constant M_1 independent of τ, N, such that $\rho_1(\tilde{u}_0,\tilde{p}_0,G_1,t_0) \leq M_1 N^{-n+s}$;

Then, there exists a positive constant M_2 depending only on $\nu > 0$ and $|||u|||_{1+\frac{n}{2}+\gamma}$ such that for all $t \in S_\tau, t \leq t_0$,

$$Q_1(\tilde{u}(t),\tilde{p}(t)) \leq \rho_1(\tilde{u}_0,\tilde{p}_0,G_1,t)e^{M_2 t}. \tag{4.15}$$

Remark 4.1. If

$$2\delta \geq \begin{cases} a_1, & \text{if } \theta > 1/2, \\ a_2, & \text{if } \theta = 1/2, \\ a_3, & \text{if } \theta < 1/2, \end{cases}$$

then we can take $a = 2\delta$ in (4.11), and (4.14) holds with $\xi_1(t) = -\nu/2 < 0$. Thus, under the conditions (i) and (ii) of Theorem 4.1, (4.15) holds for all $t \in S_\tau$.

Next we consider the case $\sigma > 1/2$. In this case we can obtain uniform stability for $\nu \geq 0$. Firstly, by putting $a = 2\sigma > 1$ in (4.2)–(4.4), we can get (4.6) in which all terms with the parameter b vanish. Secondly, estimate F_3 by (4.10), and F_1 and F_2 by the following inequalities:

$$|F_1| \le \varepsilon\tau\|\tilde{u}_t\|^2 + \frac{C\tau N^2}{\varepsilon}(\sigma-\delta)^2(|||u|||^2_{1+\frac{n}{2}+\gamma}\|\tilde{u}\|^2 + N^n\|\tilde{u}\|^4),$$
$$|F_2| \le \varepsilon\|\nabla\cdot\tilde{u}\|^2 + C(1+|||u|||^2_{1+\frac{n}{2}+\gamma})\|\tilde{u}\|^2.$$

Thus we get

$$\begin{aligned}&(\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2)_t + \tau(2\sigma-1-3\varepsilon)(\|\tilde{u}_t(t)\|^2+\beta\|\tilde{p}_t(t)\|^2) + 2(\nu|\tilde{u}(t)|_1^2+\beta\nu_1|\tilde{p}(t)|_1^2)\\&+\tau(\theta+\sigma)(\nu|\tilde{u}(t)|_1^2+\beta\nu_1|\tilde{p}(t)|_1^2)_t + \tau^2(2\theta\sigma-\theta-\sigma)(\nu|\tilde{u}_t(t)|_1^2+\beta\nu_1|\tilde{p}_t(t)|_1^2) - \varepsilon\|\nabla\cdot\tilde{u}(t)\|^2\\&\le C_2(\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2) + \xi_2(t)\|\tilde{u}(t)\|^2 + (1+\frac{\tau\sigma^2}{\varepsilon})(\|\tilde{f}(t)\|^2+\frac{1}{\beta}\|\tilde{g}(t)\|^2),\end{aligned} \tag{4.16}$$

where

$$C_2 = 2 + \frac{C\tau N^2}{\varepsilon}((\sigma-\delta)^2+\sigma^2)|||u|||^2_{1+\frac{n}{2}+\gamma} + C(1+|||u|||^2_{1+\frac{n}{2}+\gamma}),$$
$$\xi_2(t) = -1 + \frac{C\tau N^{n+2}}{\varepsilon}(\sigma-\delta)^2\|\tilde{u}(t)\|^2.$$

Thirdly, take inner product of the second equation of (4.1) with $\nabla\cdot\tilde{u}(t+\tau)$. By (2.2), it is not difficult to verify that

$$\begin{aligned}&(\sigma-\frac{1-\sigma}{2}-\varepsilon_1-\frac{\beta}{4b_1\tau}-b_2\nu-\frac{\theta\beta\nu_1 nN^2}{4})\|\nabla\cdot\tilde{u}(t+\tau)\|^2 \le \frac{1-\sigma}{2}\|\nabla\cdot\tilde{u}(t)\|^2\\&+(b_1\beta\tau+\theta\beta\nu_1\tau^2nN^2)\|\tilde{p}_t(t)\|^2 + \frac{\beta^2\nu_1 nN^2}{4b_2}|\tilde{p}(t)|_1^2 + \frac{1}{4\varepsilon_1}\|\tilde{g}(t)\|^2,\end{aligned} \tag{4.17}$$

where $b_1, b_2, \varepsilon_1 > 0$ are constants to be chosen below. Adding (4.16) to (4.17), we obtain

$$\begin{aligned}&(\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2)_t + \tau(2\sigma-1-3\varepsilon-b_1-\theta\nu_1\tau nN^2)(\|\tilde{u}_t(t)\|^2+\beta\|\tilde{p}_t(t)\|^2)\\&+2\nu|\tilde{u}(t)|_1^2+\beta\nu_1(2-\frac{\beta nN^2}{4b_2})|\tilde{p}(t)|_1^2+\tau(\theta+\sigma)(\nu|\tilde{u}(t)|_1^2+\beta\nu_1|\tilde{p}(t)|_1^2)_t\\&+\tau^2(2\theta\sigma-\theta-\sigma)(\nu|\tilde{u}_t(t)|_1^2+\beta\nu_1|\tilde{p}_t(t)|_1^2)+(\sigma-\frac{1-\sigma}{2}-\varepsilon_1-\frac{\beta}{4b_1\tau}-b_2\nu_1\\&-\frac{\theta\beta\nu_1 nN^2}{4})\|\nabla\cdot\tilde{u}(t+\tau)\|^2 \le (\varepsilon+\frac{1-\sigma}{2})\|\nabla\cdot\tilde{u}(t)\|^2 + C_2(\|\tilde{u}(t)\|^2\\&+\beta\|\tilde{p}(t)\|^2)+\xi_2(t)\|\tilde{u}(t)\|^2+G_2(t),\end{aligned} \tag{4.18}$$

where

$$G_2(t) = \left(1+\frac{\tau\sigma^2}{\varepsilon}\right)(\|\tilde{f}(t)\|^2+\frac{1}{\beta}\|\tilde{g}(t)\|^2)+\frac{1}{4\varepsilon_1}\|\tilde{g}(t)\|^2.$$

Suppose that τ and N satisfy the following condition:

$$2\sigma \ge \begin{cases} 1+3\varepsilon+b_1+r_0+\theta\nu\tau nN^2+2\theta\nu_1 nN^2, & \text{if } \theta>1/2,\\ 1+3\varepsilon+b_1+r_0+1/2\nu_1\tau nN^2+1/2\tau nN^2\max(\nu,\nu_1), & \text{if } \theta=1/2,\\ \dfrac{1+3\varepsilon+b_1+r_0+\theta\nu_1\tau nN^2+\theta\tau nN^2\max(\nu,\nu_1)}{(1-2\theta)(2-(1-2\theta)\tau nN^2\max(\nu,\nu_1))}, & \text{if } \theta<1/2.\end{cases} \tag{4.19}$$

Then we have

$$\tau(2\sigma-1-3\varepsilon-b_1-\theta\nu_1\tau nN^2)(\|\tilde{u}_t(t)\|^2+\beta\|\tilde{p}_t(t)\|^2)+\tau^2(2\theta\sigma-\theta-\sigma)$$
$$\times(\nu|\tilde{u}_t(t)|_1^2+\beta\nu_1|\tilde{p}_t(t)|_1^2)\geq r_0\tau(\|\tilde{u}_t(t)\|^2+\beta\|\tilde{p}_t(t)\|^2).$$

Let $\varepsilon, \varepsilon_1, b_1$ be suitably small and $b_2\nu_1 \leq \frac{2\sigma-1}{8}$. Suppose

$$\beta nN^2 \leq 4b_2, \quad \beta \leq 4b_1\tau(\sigma-1/2). \tag{4.20}$$

Then (4.18) can be changed into

$$(\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2)_t+r_0\tau(\|\tilde{u}_t(t)\|^2+\beta\|\tilde{p}_t(t)\|^2)+2\nu|\tilde{u}(t)|_1^2+\beta\nu_1|\tilde{p}(t)|_1^2$$
$$+\tau(\theta+\sigma)(\nu|\tilde{u}(t)|_1^2+\beta\nu_1|\tilde{p}(t)|_1^2)_t+r_0\|\nabla\cdot\tilde{u}(t+\tau)\|^2+\tau(\varepsilon+\frac{1-\sigma}{2}) \tag{4.21}$$
$$\times(\|\nabla\cdot\tilde{u}(t)\|^2)_t \leq C_2(\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2)+\xi_2(t)\|\tilde{u}(t)\|^2+G_2(t).$$

Let

$$Q_2(\tilde{u}(t),\tilde{p}(t))=\|\tilde{u}(t)\|^2+\beta\|\tilde{p}(t)\|^2+\tau\sum_{t'=0}^{t-\tau}[r_0\tau(\|\tilde{u}_t(t')\|^2+\beta\|\tilde{p}_t(t')\|^2)$$
$$+2\nu|\tilde{u}(t')|_1^2+\beta\nu_1|\tilde{p}(t')|_1^2+r_0\|\nabla\cdot\tilde{u}(t'+\tau)\|^2],$$
$$\rho_2(\tilde{u}_0,\tilde{p}_0,G_2^{*},t)=\|\tilde{u}_0\|^2+\beta\|\tilde{p}_0\|^2+\tau(\theta+\sigma)(\nu|\tilde{u}_0|_1^2+\beta\nu_1|\tilde{p}_0|_1^2)$$
$$+\tau(\varepsilon+\frac{1-\sigma}{2})\|\nabla\cdot\tilde{u}_0\|^2+\tau\sum_{t'=0}^{t-\tau}G_2(t).$$

Then we have from (4.21) that

$$Q_2(\tilde{u}(t),\tilde{p}(t))\leq\rho_2(\tilde{u}_0,\tilde{p}_0,G_2,t)+\tau\sum_{t'=0}^{t-\tau}(C_2Q_2(\tilde{u}(t'),\tilde{p}(t'))+\xi_2(t')\|\tilde{u}(t')\|^2). \tag{4.22}$$

Theorem 4.2. *Assume $\sigma > 1/2, \tau = O(N^{-s}), s\geq 2$, and that (4.19), (4.20) are fulfilled. In addition, there exist a $t_0\in S_\tau$ and a suitably small constant $M_3>0$ independent of τ and N such that $\rho_2(\tilde{u}_0,\tilde{p}_0,G_2,t)\leq M_3N^{-n-2+s}$. Then there is a constant M_4 depending on $|||u|||_{1+\frac{n}{2}+\gamma}$ but not on ν such that for all $t\in S_\tau, t\leq t_0$,*

$$Q_2(\tilde{u}(t),\tilde{p}(t))\leq\rho_2(\tilde{u}_0,\tilde{p}_0,G_2,t)e^{M_4t}. \tag{4.23}$$

Remark 4.2. If $\delta=\sigma>1/2$, then $\xi_2(t)\equiv -1<0$ in (4.22). Thus (4.23) holds for all $t\in S_\tau$.

§5. The Convergence

Let U,P be the solution of (1.1) and u,p the solution of (3.4). Define $U^N=P_NU$, $P^N=P_NP$, $e=U^N-u, w=P^N-p$. We derive from (1.1) and (3.4) that

$$\begin{cases} e_t(t)+R_\tau d(R_\tau(e(t)+\delta\tau e_t(t)),U^N(t)+e(t))+R_\tau d(R_\tau(U^N(t)+\delta\tau U_t^N(t)),e(t)) \\ \qquad +\nabla(w(t)+\sigma\tau w_t(t))-\nu\nabla^2(e(t)+\theta\tau e_t(t))=-\Big(\sum_{m=1}^{3}E_m(t)+\nabla E_4(t)+\nu\nabla^2E_5(t)\Big), \\ \beta w_t(t)+\nabla\cdot(e(t)+\sigma\tau e_t(t))-\beta\nu_1\nabla^2(w(t)+\theta\tau w_t(t))=-\beta E_6(t)-\beta\nu_1\nabla^2E_7(t), \\ e(0)=(P_N-P_c)U_0, \quad w(0)=(P_N-P_c)P_0, \end{cases}$$

where

$$E_1 = U_t^N - \frac{\partial U^N}{\partial t}, \quad E_2 = R_r d(R_r(U^N + \delta\tau U_t^N), U^N) - P_N((U\cdot\nabla)U),$$
$$E_3 = (P_N - R_r P_c)f, \quad E_4 = \sigma\tau P_t^N, \quad E_5 = -\theta\tau U_t^N, \quad E_6 = P_t^N,$$
$$E_7 = -(P^N + \theta\tau P_t^N).$$

Suppose that conditions (i) and (ii) in Theorem 4.1 are satisfied. Then we can get the following inequality just as in Section 4,

$$(\|e(t)\|^2 + \beta\|w(t)\|^2)_t + r_0\tau(\|e_t(t)\|^2 + \beta\|w_t(t)\|^2) + \frac{1}{2}(\nu|e(t)|_1^2 + \beta\nu_1|w(t)|_1^2) + \tau(\theta + \frac{a}{2})$$
$$\times(\nu|e(t)|_1^2 + \beta\nu_1|w(t)|_1^2)_t \le C_3(\|e(t)\|^2 + \beta\|w(t)\|^2) + \xi_3(t)|e(t)|_1^2 + G_3(t),$$

where

$$C_3 = 1 + \frac{C\tau N^2}{\varepsilon}((a-2\delta)^2 + a^2)|||U|||^2_{1+\frac{n}{2}+\gamma} + (\frac{C}{\nu}|||U|||_{1+\frac{n}{2}+\gamma} + C)|||U|||_{1+\frac{n}{2}+\gamma},$$

$$\xi_3(t) = -\frac{\nu}{2} + \frac{C\tau N^2}{\varepsilon}(a-2\delta)^2\|e(t)\|^2,$$

$$G_3(t) = (1 + \frac{3\tau a^2}{4\varepsilon})\sum_{m=1}^{3}\|E_m(t)\|^2 + (\frac{\nu}{2} + \frac{\tau n N^2 a^2}{4\varepsilon})\|E_4(t)\|^2 + (\nu + \frac{\tau n N^2 a^2 \nu^2}{4\varepsilon})|E_5(t)|_1^2$$
$$+\beta(1 + \frac{\tau a^2}{4\varepsilon})\|E_6(t)\|^2 + \beta(\nu_1 + \frac{\tau n N^2 a^2 \nu_1^2}{4\varepsilon})|E_7(t)|_1^2,$$

and obtain a conclusion like Theorem 4.1. Thus to get the convergence rates, we need only to estimate $\rho_1(e(0), w(0), G_3, t)$.

Assume $\mu \ge n/2 + \gamma$ and that U, P, U_0, P_0 and f are suitably smooth. Then it is not difficult to show that

$$\|e(0)\|_1 \le CN^{-\mu}\|U_0\|_{\mu+1}, \quad \|w(0)\|_1 \le CN^{s-\mu}\|P_0\|_{\mu-s+1}$$

and

$$\tau\sum_{t'=0}^{t-\tau}\|E_1(t')\|^2 \le C\tau^2\left\|\frac{\partial^2 U}{\partial t^2}\right\|^2_{L^2(0,t;L^2)}.$$

By the second equation in (1.1), we have $\nabla\cdot U^N = P_N(\nabla\cdot U) = 0$. Thus

$$\|E_2(t)\| \le \sum_{m=1}^{4} J_m(t),$$

where

$$J_1(t) = \frac{1}{2}\|R_r(P_c - P_N)((U^N(t)\cdot\nabla)U^N(t))\|,$$

$$J_2(t) = \frac{1}{2}\sum_{q=1}^{n}\|R_r\frac{\partial}{\partial x_q}((P_C - P_N)(U_q^N(t)U^N(t)))\|,$$

$$J_3(t) = \|R_r P_N((U^N(t)\cdot\nabla)U^N(t)) - P_N((U(t)\cdot\nabla)U(t))\|,$$
$$J_4(t) = \delta\tau\|R_r d(R_r U_t^N(t), U^N(t))\|.$$

Using Lemma 2.2, Lemma 2.3 and the embedding theorems, we can show

$$\sum_{m=1}^{3} J_m(t) \le CN^{-\mu}\|U(t)\|^2_{\mu+1}, \quad J_4(t) \le C\tau^{1/2}\Big(\int_t^{t+\tau}\Big\|\frac{\partial U}{\partial \xi}(\xi)\Big\|_1^2 d\xi\Big)^{1/2}\|U(t)\|_{\mu+1}.$$

Thus

$$\tau\sum_{t'=0}^{t-\tau}\|E_2(t')\|^2 \le CN^{-2\mu}|||U|||^4_{\mu+1} + C\tau^2\Big\|\frac{\partial U}{\partial t}\Big\|^2_{L^2(0,t;H^1)}|||U|||^2_{\mu+1}.$$

Also by the lemmas in Section 2, we have

$$\tau\sum_{t'=0}^{t-\tau}(\|E_3(t')\|^2 + \|E_4(t')\|^2 + |E_5(t')|_1^2) \le CN^{-2\mu}|||f|||^2_\mu$$

$$+C\tau^2\Big(\Big\|\frac{\partial P}{\partial t}\Big\|^2_{L^2(0,t;L^2)} + \Big\|\frac{\partial U}{\partial t}\Big\|^2_{L^2(0,t;H^1)}\Big),$$

$$\tau\sum_{t'=0}^{t-\tau}(\|E_6(t')\|^2 + |E_7(t')|_1^2) \le C(\|\frac{\partial P}{\partial t}\|^2_{L^2(0,t;L^2)} + |||P|||_1^2).$$

Consequently, we have $\rho_1(e(0), w(0), G_3, t) \le c(\nu)(\tau^2 + N^{-2\mu} + \beta)$, where $c(\nu)$ is a constant depending on $\nu > 0$. Furthermore assume that

$$\frac{n\tau(a-2\sigma)^2}{b\min(\nu,\nu_1)} \le \beta \le C_0\tau. \tag{5.1}$$

Because $s \ge 2$ and $\mu > 1$, we have $\min(s, 2\mu) > n - s$ and $\rho_1(e(0), w(0), G_3, t) \le c(\nu)N^{-n+s}$. Consequently, we can get the following results.

Theorem 5.1. *Assume that $\mu > n/2$, and $U_0 \in H^{\mu+1}$, $P_0 \in H^{\mu+1-s}$, $f \in C(0,T;H^\mu)$, $U \in C(0,T;H^{\mu+1}) \cap H^1(0,T;H^1) \cap H^2(0,T;L^2)$, $P \in C(0,T;H^1) \cap H^1(0,T;L^2)$. If the conditions* (i) *and* (ii) *in Theorem 4.1, together with (5.1), are fulfilled, then there exists a constant M_5 depending only on U_0, P_0, f, U, P and $\nu > 0$ such that for all $t \in S_\tau$, we have*

$$Q_1(e(t), w(t)) \le M_5(\tau + N^{-2\mu}).$$

Theorem 5.2. *Suppose that U_0, P_0, f, U, P satisfy the corresponding conditions in Theorem 5.1, $\tau = O(N^{-s})$, $s \ge 2$, $\beta = O(\tau^2)$, $\sigma > 1/2$ and that (4.19) and (4.20) are fulfilled. In addition, either $\mu \ge (n+2-s)/2$ or $\delta = \sigma$. Then there exists a constant $M_6 > 0$ depending on U, P, U_0, P_0 and f, but not on ν, such that for all $t \in S_\tau$, we have*

$$Q_2(e(t), w(t)) \le M_6(\tau^2 + N^{-2\mu}).$$

§6. Numerical Results

Take $\delta = 0, \sigma = \theta = 1, N = 8, \tau = 0.01$ in (3.4) and choose the following two sets of test functions:

$$\begin{cases} U_1(x,t) = \cos x_2 \exp(\sin x_1 + \sin x_2 + 0.1t), \\ U_2(x,t) = -\cos x_1 \exp(\sin x_1 + \sin x_2 + 0.1t), \\ P(x,t) = -(\cos 2x_1 + \cos 2x_2)\exp(0.2t); \end{cases} \tag{6.1}$$

$$\begin{cases} U_1(x,t) = -\cos x_1 \sin x_2 \exp(-2\nu t), \\ U_2(x,t) = \sin x_1 \cos x_2 \exp(-2\nu t), \\ P(x,t) = -\frac{1}{4}(\cos 2x_1 + \cos 2x_2)\exp(-4\nu t). \end{cases} \tag{6.2}$$

We consider L^2-normed errors of the numerical solution $u_1(t), u_2(t)$ and $p(t)$, denoted by $E(u_1(t)), E(u_2(t))$ and $E(p(t))$. Tables 1,2, show that the restraint operator R_r and the artificial compression term $\beta\nu_1\nabla^2 p$ can strengthen the computational stability. But ν_1 should not be too large.

Table 1. Example (6.1), $\nu = 10^{-6}, \beta = 10^{-4}, \nu_1 = 10^{-2}$

	$E(u_1(5.0))$	$E(u_2(5.0))$	$E(p(5.0))$
$r=5$	0.3756E-1	0.3849E-1	0.6153E-1
$r=50$	0.1021E+1	0.1065E+1	0.4914E+1
$r=\infty$	0.2347E+1	0.2445E+1	0.2372E+2

Table 2. Example (6.2), $\nu = 0.5, \beta = 10^{-2}, r = 10$

	$E(u_1(1.0))$	$E(u_2(1.0))$	$E(p(1.0))$
$\nu_1 = 0$	0.8079E-3	0.7997E-3	0.1529E-2
$\nu_1 = 1$	0.5606E-3	0.5363E-3	0.7311E-3
$\nu_1 = 10$	0.4794E-2	0.3572E-2	0.5405E-3

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