A CLASS OF MULTIVARIATE RATIONAL INTERPOLATION FORMULAS*

XU LI-ZHI (徐利治 HSU, L. C.) YANG JIA-XIN (杨家新)
(*Institute of Applied Mathematics, Dalian Institute of Technology, Dalian, China*)

Abstract

The object of this paper is to construct a class of multivariate rational interpolation formulas that can be used to solve interpolation problems with function data given at equidistant knots of various directed lines in the higher dimensional Euclidean space. Our formulas are built up of some explicit multivariate rational functions involving three sets of free parameters so that they enjoy sufficient flexibility for interpolating functions of several variables possessing certain kinds of singularities (poles). The method adopted is an extension and modification of that described in our previous papers (cf. [3], [5]).

§ 1. Rational Interpolation $S_m(f:z)$ on a Directed Line

Denote by $\mathbb{R}^n$ and $\mathbb{C}^n$ the n-dimensional real Euclidean space and complex Euclidean space respectively. We shall adopt the following usual notations:

$$x=(x_1, \cdots, x_n)\in\mathbb{R}^n, \quad z=(z_1, \cdots, z_n)\in\mathbb{C}^n,$$

$$\langle z, \xi\rangle=\sum_{i=1}^{n} z_i\bar{\xi}_i, \quad (z, \xi\in\mathbb{C}^n),$$

$$|z|=\langle z, z\rangle^{\frac{1}{2}}=\left(\sum_{i=1}^{n} z_i\bar{z}_i\right)^{\frac{1}{2}}.$$

In particular, we write $\mathbb{C}$ instead of $\mathbb{C}'$.

Given a set of points $A_k\in\mathbb{C}^n (k=0, 1, 2, \cdots, m)$, and a function $f: \mathbb{C}^n\to\mathbb{C}$. It is easily observed that Gould-Hsu's inversion formulas[2] can be put in the following form

$$g(A_s)=\sum_{k=0}^{s}(-1)^k\binom{s}{k}\psi(k, s)f(A_k), \tag{1.1}$$

$$f(A_s)=\sum_{k=0}^{s}(-1)^k\frac{a_{k+1}+k\cdot b_{k+1}}{\psi(s, k+1)}\binom{s}{k}g(A_k), \tag{1.2}$$

where $s=0, 1, \cdots, m$, and $\{a_k\}$, $\{b_k\}\in\mathbb{C}$ are sequences of parameters and $\psi(x, k)$ is a sequence of polynomials defined by

$$\psi(x, k)=\prod_{i=1}^{k}(a_i+b_i x), \quad \psi(x, 0)\equiv 1, \qquad x\in\mathbb{C},$$

in which $\{a_k\}$ and $\{b_k\}$ are chosen such that $\psi(x, k)\neq 0$ for $x, k=0, 1, 2, \cdots$.

In what follows let the set of points $\{A_j\}_0^m$ take the form $A_j=A_0+jh (j=0, 1, \cdots, m)$ with $h=(h_1, \cdots, h_n)\in\mathbb{C}^n$ and $h_i (i=1, \cdots, n)$ being complex constants. Suppose that $\varphi(u, v_j)$: $\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ is a homogeneous function of u with $v_j\in\mathbb{C}^n$ $(j=0, 1, \cdots,$

* Received August 13, 1983.

m), viz. $\varphi(\lambda u, v_j)=\lambda\varphi(u, v_j)$ for real λ. Then, applying the process of construction for interpolation formulas as described in [3], substituting (1.1) into the equation (1.2) with $s=m$, and replacing the discrete variable m by the continuous parameter

$$\varphi(z-A_0, v_j)/\varphi(h, v_j)\in\mathbb{C}$$

in each term of the resultant summation, we obtain

$$S_m(f; z)=\sum_{j=0}^{m} f(A_j)\sum_{k=j}^{m}\lambda_{jk}\begin{pmatrix}\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}\\ k\end{pmatrix}\psi\left(\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}, k+1\right)^{-1}, \tag{1.3}$$

where $\lambda_{jk}=(-1)^{j+k}(a_{k+1}+kb_{k+1})\binom{k}{j}\psi(j, k),\quad j, k=0, 1, \cdots, m.$

That (1.3) is actually a multivariate interpolation formula for $f(z)$ can be proved as a theorem.

We may rewrite (1.3) in the form

$$S_m(f; z)=\sum_{j=0}^{m} f(A_j)\cdot l_j^{(m)}(z) \tag{1.4}$$

with

$$l_j^{(m)}(z)=\sum_{k=j}^{m}\lambda_{jk}\begin{pmatrix}\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}\\ k\end{pmatrix}\psi\left(\frac{\varphi(z-A_0, v_j)}{\varphi(h, v_j)}, k+1\right)^{-1}. \tag{1.5}$$

Theorem 1. *The summation $S_m(f; z)$ defined by (1.4) satisfies the interpolation conditions*

$$S_m(f; A_r)=f(A_r),\qquad r=0, 1, \cdots, m. \tag{1.6}$$

Proof. In the formula (1.4) (or (1.3)) put $z=A_r$. Since $\varphi(u, \cdot)$ is a homogeneous function of u we have

$$\frac{\varphi(A_r-A_0, v_j)}{\varphi(h, v_j)}=\frac{\varphi(rh, v_j)}{\varphi(h, v_j)}=r.$$

Thus we may evaluate $l_j^{(m)}(A_r)$ as follows

$$\begin{aligned}l_j^{(m)}(A_r)&=\sum_{k=j}^{r}\lambda_{jk}\binom{r}{k}\psi(r, k+1)^{-1}\\&=\sum_{k=j}^{r}(-1)^{k+j}c_k\binom{r}{k}\binom{k}{j}\psi(j, k)\psi(r, k+1)^{-1}\quad(c_k=a_{k+1}+kb_{k+1})\\&=\binom{r}{j}\sum_{k=j}^{r}(-1)^{k+j}c_k\binom{r-j}{k-j}\psi(j, k)\psi(r, k+1)^{-1}\\&=\binom{r}{j}\sum_{k=0}^{r-j}(-1)^{k}c_{k+j}\binom{r-j}{k}\psi(j, k+j)\psi(r, k+j+1)^{-1}\\&=\binom{r}{j}\delta_{rj}=\delta_{rj}.\end{aligned}$$

Here the final result is attained by making use of the orthogonality relation (3.2) of the paper [2]. Consequently we have

$$S_m(f; A_r)=\sum_{j=0}^{m} f(A_j)\, l_j^{(m)}(A_r)=\sum_{j=0}^{m} f(A_j)\,\delta_{rj}=f(A_r).$$

It is clear that for a given sequence of points $\{A_j\}$ the formula (1.3) (or (1.4)) is a multivariate interpolation formula in which the sequences $\{a_k\}$, $\{b_k\}$, $\{v_k\}$ and the auxiliary function $\varphi: \mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}$ may be assigned arbitrarily. As the proof shows, the fact that (1.3) satisfies the interpolation condition (1.6) is independent of the choices of the sequences $\{a_k\}$, $\{b_k\}$, $\{v_k\}$ and φ. Thus (1.3) involves a large class of multivariate interpolation formulas.

In particular we can get a class of rational interpolation formulas of the form

$$S_m(f;z)=\sum_{j=0}^{m} f(A_j)\sum_{k=j}^{m}\lambda_{jk}\begin{pmatrix}\dfrac{\langle z-A_0,\ v_j\rangle}{\langle h,\ v_j\rangle}\\ k\end{pmatrix}\psi\left(\frac{\langle z-A_0,\ v_j\rangle}{\langle h,\ v_j\rangle},\ k+1\right)^{-1}. \tag{1.7}$$

This follows from (1.3) by taking $\varphi(u, v)=\langle u, v\rangle$.

For the case $z\in\mathbb{C}$ and $\{A_j\}$ and $\{v_j\}$ being sequences of points of $\mathbb{C}$ we have

$$\langle z-A_0,\ v_j\rangle/\langle h,\ v_j\rangle=(z-A_0)\cdot\bar{v}_j/h\cdot\bar{v}_j=(z-A_0)/h.$$

Thus it is clear that (1.7) may be regarded as an extension of the one-dimensional generalized Newton interpolation formula (cf. [4], [5]).

Remark 1. Using geometrical terminology, the knots of interpolation $A_j=A_0+jh(j=0, 1, \cdots, m)$ are all lying on the directed line L which passes through A_0 with preassigned direction $h=(h_1, \cdots, h_n)\in\mathbb{C}^n$. Hence (1.3) just represents an interpolation process along the fixed directed line L, so that it can only be used for approximate computation of the function $f(z)$ on L or near L. However, in order to compute the approximate values of $f(z)$ in a given domain of $\mathbb{C}^n$, one should make use of an interpolation formula with interpolation knots distributed on various directed lines piercing through the given domain. Such formulas will be constructed in § 3.

Remark 2. The proof of Theorem 1 shows that $l_j^{(m)}(A_r)=\delta_{rj}\ (r=0, 1, \cdots, m)$ does not depend on m. Consequently the summation (1.3) may be extended into a formal series, namely we have

$$S(f;z)\overset{\text{def}}{=\!=}\sum_{j=0}^{\infty}f(A_j)l_j(z),$$

where

$$l_j(z)\overset{\text{def}}{=\!=}\sum_{k=j}^{\infty}\lambda_{jk}\begin{pmatrix}\dfrac{\varphi(z-A_0,\ v_j)}{\varphi(h,\ v_j)}\\ k\end{pmatrix}\psi\left(\frac{\varphi(z-A_0,\ v_j)}{\varphi(h,\ v_j)},k+1\right)^{-1}.$$

Obviously, $S(f;z)$ satisfies the interpolation conditions

$$S(f;A_r)=f(A_r),\qquad r=0, 1, 2, \cdots.$$

Remark 3. For the rational interpolation formula (1.7) with $n>2$ it is seen that the equations

$$\psi\left(\frac{\langle z-A_0,\ v_j\rangle}{\langle h,\ v_j\rangle},k+1\right)=0,\qquad j, k=0, 1, 2, \cdots$$

represent some hyperplanes in the space $\mathbb{C}^n$, on which every point is a singularity (pole) of $S_m(f;z)$. The hyperplanes may be called singularity regions. If we know such regions relating to $f(z)$ in advance, we may suitably choose $\{a_k\}$, $\{b_k\}$ and $\{v_j\}$ in order to make the formula $S_m(f;z)$ reflects those singularity regions as much as possible.

§ 2. A Connection with Kergin's Interpolation

As is known, Kergin's interpolation is essentially a kind of extension of Lagrange's polynomial interpolation in $\mathbb{C}^n$, so that it is quite different from the interpolation considered in this paper. However there is some connection between their methodology.

As an illustration, let us take $v_j=A_j (j=0, 1, \cdots, m)$, so that (1.7) gives

$$S_m(f; z) = \sum_{j=0}^{m} f(A_j) l_j^{(m)}(z) \tag{2.1}$$

with

$$l_j^{(m)}(z) = \sum_{k=j}^{m} \lambda_{jk} \binom{\frac{\langle z-A_0, A_j\rangle}{\langle h, A_j\rangle}}{k} \psi\left(\frac{\langle z-A_0, A_j\rangle}{\langle h, A_j\rangle}, k+1\right)^{-1}. \tag{2.2}$$

We may show that (2.1) can also be obtained by using the idea of Kergin's interpolation (cf. § 1.8 of [1]).

Suppose that $f(z)$: $\mathbb{C}^n \to \mathbb{C}$ is an analytic function. Then we have the Cauchy-Szego formula

$$f(z) = \frac{R\cdot(n-1)!}{2\pi^n} \int_{S(R)} \frac{f(\xi)}{(R^2-\langle z, \xi\rangle)^n} d\tau(\xi), \quad |z|<R, \tag{2.3}$$

where $S(R)$ is the spherical surface of radius R with center at the origin of $\mathbb{C}^n$, and $d\tau(\xi)$ is the area element of $S(R)$. Cauchy-Szego's kernel function

$$K(z, \xi) = \frac{R\cdot(n-1)!}{2\pi^n} \frac{1}{(R^2-\langle z, \xi\rangle)^n}$$

may be considered as a composite function

$$K(t) = \frac{R\cdot(n-1)!}{2\pi^n} \frac{1}{t^n}, \qquad t=R^2-\langle z, \xi\rangle.$$

Assume that the points $A_j (j=0, 1, \cdots, m)$ are all inside the sphere $S(R)$. Now apply the generalized Newton interpolation formula to $K(t)$ with $t_j=R^2-\langle A_j, \xi\rangle (j=0, 1, \cdots, m)$ as interpolation knots, we get

$$\begin{aligned} S_m(K;t) &= \frac{R\cdot(n-1)!}{2\pi^n} \sum_{k=0}^{m} \frac{c_k}{\psi((t-t_0)/h^*, k+1)} \\ &\quad \cdot \binom{(t-t_0)/h^*}{k} \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} \frac{\psi(j, k)}{t_j^n} \\ &= \frac{R\cdot(n-1)!}{2\pi^n} \sum_{k=0}^{m} \sum_{j=0}^{k} \lambda_{jk} \binom{(t-t_0)/h^*}{k} \psi((t-t_0)/h^*, k+1)^{-1} t_j^{-n}, \end{aligned}$$

where $h^*=\langle -h, \xi\rangle$. Substituting the quantities $t=R^2-\langle z, \xi\rangle$, $t_j=R^2-\langle A_j, \xi\rangle$ into the above expression and integrating over $S(R)$, we obtain

$$\begin{aligned} S_m(f; z) &= \int_{S(R)} S_m(K(z, \xi); z) f(\xi) d\tau(\xi) \\ &= \sum_{k=0}^{m} \sum_{j=0}^{k} \lambda_{jk} \frac{R\cdot(n-1)!}{2\pi^n} \int_{S(R)} G(z, A_j, k, \xi) d\tau(\xi), \end{aligned}$$

where

$$G(z, A_j, k, \xi)=\binom{\langle z-A_0, \xi\rangle/\langle h, \xi\rangle}{k}\psi\left(\frac{\langle z-A_0, \xi\rangle}{\langle h, \xi\rangle}, k+1\right)^{-1}\cdot(R^2-\langle A_j, \xi\rangle)^{-n}f(\xi).$$

Thus an application of Cauchy-Szego's integral formula to the above integral gives

$$S_m(f; z)=\sum_{k=0}^{m}\sum_{j=0}^{k}\lambda_{jk}f(A_j)\binom{\langle z-A_0, A_j\rangle/\langle h, A_j\rangle}{k}\psi\left(\frac{\langle z-A_0, A_j\rangle}{\langle h, A_j\rangle}, k+1\right)^{-1}.$$

This is actually equivalent to (2.1).

§ 3. Rational Interpolation on an Arbitrary Set of Directed Lines

Given p directed lines $L_1, \cdots, L_p$ in $\mathbb{C}^n$ with m_r+1 knots on $L_r(1\leqslant r\leqslant p)$, namely

$$A_j^{(r)}=A_0^{(r)}+jh^{(r)}, \quad j=0, 1, \cdots, m_r;\ r=1, 2, \cdots, p, \tag{3.1}$$

where the direction of L_r is determined by $h^{(r)}=(h_1^{(r)}, \cdots, h_n^{(r)})\in\mathbb{C}^n$. We shall assume that all the directions $h^{(r)}(r=1, \cdots, p)$ are different, and moreover, all the knots $A_j^{(r)}$ $(0\leqslant j\leqslant m_r,\ 1\leqslant r\leqslant p)$ are distinct from each other. It can be shown that there exists a kind of multivariate rational interpolation formulas with all the points given by (3.1) as interpolation knots.

For each fixed r, $\{A_j^{(r)}\}$ are lying on a directed line segment, so that one may employ (1.7) to construct a rational interpolation formula $S_{m_r}^{(r)}(f; z)$ on that directed line L_r.

In accordance with Lagrange's interpolation process, one may introduce the following

$$\omega(z)=\prod_{\mu=1}^{p}\prod_{s=0}^{m_\mu}\langle z-A_s^{(\mu)}, A\rangle, \quad A=(1, 1, \cdots, 1)\in\mathbb{C}^n,$$

$$\omega^*(r, j)=\prod_{\mu=1}^{p}\prod_{s=0}^{m_\mu}{}'\langle A_j^{(r)}-A_s^{(\mu)}, A\rangle,$$

where $\prod\prod'$ means omitting the factor with $\mu=r$, $s=j$ in the above product. Then it is easily observed that the polynomials defined by

$$\beta_r(z)=\sum_{j=0}^{m_r}\frac{\omega(z)}{\langle z-A_j^{(r)}, A\rangle\omega^*(r, j)}$$

satisfy the conditions

$$\beta_r(A_j^{(r)})=1, \quad 0\leqslant j\leqslant m_r,$$
$$\beta_r(A_j^{(q)})=0, \quad q\neq r.$$

Consequently we may state the following

Theorem 2. *A formula of the form*

$$\mathfrak{S}_m^{(p)}(f; z)=\sum_{r=1}^{p}\beta_r(z)S_{m_r}^{(r)}(f; z) \tag{3.2}$$

satisfies the interpolation conditions

$$\mathfrak{S}_m^{(p)}(f; A_j^{(r)})=f(A_j^{(r)}), \tag{3.3}$$

where $r=1, 2, \cdots, p$; $j=0, 1, \cdots, m_r$; $m=p+m_1+\cdots+m_p$.

Of course this theorem can easily be justified by means of Theorem 1.

Remark 4. Though the formula (3.2) is much more complicated than the ordinary Lagrange interpolation formula in higher dimensions, it may be useful for the cases where the interpolated functions have some poles (singularities) distributed on some lines, planes or hyperplanes in $\mathbb{C}^n$.

Remerk 5. For the case where the domain of interpolation $\mathscr{D}\subset\mathbb{C}^n$ is a simply connected one, we may choose a set of various directed lines $\{L_r\}$ piercing through $\mathscr{D}$ and take sufficiently many knots of the form (3.1) on these directed line segments inside $\mathscr{D}$ so that the formula (3.2) may be used as an interpolation formula for interpolating $f(z)$ defined on $\mathscr{D}$.

References

[1] T. Bloom, Kergin interpolation of entire functions on C^n, *Duke Math. J.*, **48**: **1** (1981), 69—83.

[2] H. W. Gould, L. C. Hsu, Some new inverse series relations, *Duke Math. J.*, **40**: 4 (1973), 885—891.

[3] L. C. Hsu, J. X. Yang, On a method of constructing interpolation formulas via inverse series relations, *J. Math. Research & Exposition*, **2**: 2 (1982), 113—126.

[4] L. C. Hsu, J. X. Yang, Bao Xue-song, A class of generalized complex Newton interpolation series (in Chinese, with English abstract), *ibid*, **1** (1981), Initial issue, 69—72.

[5] L. C. Hsu, J. X. Yang, A Class of Generalized Newton Interpolation Series with Some Applications, *Numerical Mathematics—A Journal of Chinese Universities*, **3**: 1 (1981), 88—96.