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The Distortion Theorems for Harmonic Mappings with Negative Coefficient Analytic Parts

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Abstract. Some sharp estimates for coefficients, distortion and the growth order are obtained for harmonic mappings $f\in TL_H^{\alpha}$, which are locally univalent harmonic mappings in the unit disk $\mathbb{D}:=\{z:|z|<1\}$. Moreover, denoting the subclass TS_H^{α} of the normalized univalent harmonic mappings, we also estimate the growth of $|f|, f\in TS_H^{\alpha}$, and their covering theorems.

AMS subject classifications: 30D15, 30D99

Key words: Harmonic mapping, coefficient estimate, distortion theorem, covering problem

1 Introduction

Let S denote the class of functions of the form $F(z)=z+\sum_{n=2}^{\infty}a_nz^n$, that are analytic and univalent in the unit disk $\mathbb{D}:=\{z:|z|<1\}$. Denoting T to be the subclass of S consisting of functions whose nonzero coefficients, from the second on, are negative. That is, an univalent analytic function $F\in T$ if and only if it can be written in the form

$$F(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\ \ z\in\mathbb{D}. \tag{1.1}$$

A complex-valued harmonic function f in the unit disk $\mathbb{D}$ has a canonical decomposition

$$f(z)=h(z)+\overline{g(z)} \tag{1.2}$$

where h and g are analytic in $\mathbb{D}$ with $g(0)=0$. Usually, we call h the analytic part of f and g the co-analytic part of f. A complete and elegant account of the theory of planar harmonic mappings is given in Duren′s monograph [1].

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In [2], Ikkei Hotta and Andrzej Michalski denoted the class L_H of all normalized locally univalent and sense-preserving harmonic functions in the unit disk with $h(0)=g(0)=h'(0)-1=0$. Which means every function $f\in L_H$ is uniquely determined by coefficients of the following power series expansions

$$h(z)=z+\sum_{n=2}^{\infty}a_nz^n,\ \ g(z)=\sum_{n=1}^{\infty}b_nz^n,\ \ z\in\mathbb{D},\tag{1.3}$$

where $a_n,\ b_n\in\mathbb{C},n=2,3,4,\ldots$ Clunie and Sheil-small introduced in [3] the class S_H of all normalized univalent harmonic mappings in $\mathbb{D}$, obviously, $S_H\subset L_H$.

Lewy [4] proved that a necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathbb{D}$ is $J_f(z)>0$, where

$$J_f(z)=|h'(z)|^2-|g'(z)|^2,\ \ z\in\mathbb{D}.\tag{1.4}$$

To such a function f, not identically constant, let

$$\omega(z)=\frac{g'(z)}{h'(z)},\ \ z\in\mathbb{D},\tag{1.5}$$

then $\omega(z)$ is analytic in $\mathbb{D}$ with $|\omega(z)|<1$, it is called the second complex dilatation of f.

In [5], Silverman investigated the subclass of T which denoted by $T^*(\beta)$, starlike of order $\beta(0\le\beta<1)$. That is, a function $F(z)\in T^*(\beta)$ if $\mathrm{Re}\{zF'(z)/F(z)\}>\beta$, $z\in\mathbb{D}$. It was proved in [5] that

Corollary 1.1.

$$T=T^*(0).$$

In [7-8], Dominika Klimek and Andrzej Michalski studied the cases when the analytic parts h is the identity mapping or a convex mapping, respectively. The paper [2] was devoted to the case when the analytic h is a starlike analytic mapping. In [9], Qin Deng got sharp results concerning coefficient estimate, distortion theorems and covering theorems for functions in T. The main idea of this paper is to characterize the subclasses of L_H and S_H when $h\in T$.

In order to establish our main results, we need the following theorems and lemmas.

Theorem 1.1. ([8]) *A function $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ is in T if and only if*

$$\sum_{n=2}^{\infty}n|a_n|\le 1,\ \ z\in\mathbb{D}.\tag{1.6}$$

Lemma 1.1. ([10]) *If $f(z)=a_0+a_1z+\ldots+a_nz^n+\ldots$ is analytic and $|f(z)|\le 1$ on $\mathbb{D}$, then*

$$|a_n|\le 1-|a_0|^2,\ \ n=1,2,\ldots.\tag{1.7}$$

Theorem 1.2. ([8]) *If $f\in T$, then*

$$1-|z|\le|f'(z)|\le 1+|z|,\ \ z\in\mathbb{D}, \tag{1.8}$$

with equality for

$$f(z)=z-\frac{1}{2}z^2,\ \ z\in\mathbb{D}.$$

Theorem 1.3. ([8]) *If $f\in T$, then*

$$|z|-\frac{1}{2}|z|^2\le|f(z)|\le|z|+\frac{1}{2}|z|^2,\ \ z\in\mathbb{D}, \tag{1.9}$$

with equality for

$$f(z)=z-\frac{1}{2}z^2,\ \ z\in\mathbb{D}.$$

2 Main results and their proofs

Similar with the papers [2], [7], [8] and [12], we consider the following function sets.

Definition 2.1. For $\alpha\in[0,1)$, let

$$TL_H^\alpha:=\{f(z)=h(z)+\overline{g(z)}\in L_H: h(z)\in T, |b_1|=\alpha\}$$

and

$$TL_H:=\bigcup_{\alpha\in[0,1)} TL_H^\alpha.$$

Definition 2.2. For $\alpha\in[0,1)$, let

$$TS_H^\alpha:=\{f(z)=h(z)+\overline{g(z)}\in S_H: h(z)\in T, |b_1|=\alpha\}$$

and

$$TS_H:=\bigcup_{\alpha\in[0,1)} TS_H^\alpha.$$

For $f\in TL_H^\alpha$, applying Theorem 1.1 and Lemma 1.1, we can prove the following theorem.

Theorem 2.1. *If $f\in TL_H^\alpha$, then $|a_n|\le 1/n$, $n=2,3,4,\ldots$, and*

$$|b_2|\le\frac{1+\alpha-\alpha^2}{2},\ \textit{where}\ \ |b_1|=\alpha. \tag{2.1}$$

It is sharp estimate for $|b_2|$*, the extremal functions are*

$$f_0(z)=\begin{cases} z-\frac{1}{2}z^2+\overline{(-1-\frac{1}{\alpha}+\frac{1}{\alpha^2})z+\frac{1}{2\alpha}z^2+(1-\frac{1}{\alpha}-\frac{1}{\alpha^2}+\frac{1}{\alpha^3})\ln(1-\alpha z)}, & \alpha\neq 0, \\ z-\frac{1}{2}z^2+\overline{\frac{1}{2}z^2-\frac{1}{3}z^3}, & \alpha=0. \end{cases} \tag{2.2}$$

And

$$|b_n|\leq\frac{2+\alpha-2\alpha^2}{n},\ \ n=3,4,5,\ldots. \tag{2.3}$$

Proof. If $f(z)=h(z)+\overline{g(z)}=z-\sum_{n=2}^{\infty}|a_n|z^n+\overline{\sum_{n=1}^{\infty}b_nz^n}$, $z\in\mathbb{D}$, then $|a_n|\leq 1/n$ by Theorem 1.1. Let $g'(z)=\omega(z)h'(z)$, where $\omega(z)$ is the dilatation of f. Since $\omega(z)$ is analytic in $\mathbb{D}$, it has a power series expansion

$$\omega(z)=\sum_{n=0}^{\infty}c_nz^n,\ \ z\in\mathbb{D}, \tag{2.4}$$

where $c_n\in\mathbb{C}$, $n=0,1,2,\ldots$, and $|c_0|=|\omega(0)|=|g'(0)|=|b_1|=\alpha$. Recall that $|\omega(z)|<1$ for all $z\in\mathbb{D}$, then by Lemma 1.1, we have

$$|c_n|\leq 1-|c_0|^2,\ \ n=1,2,3,\ldots. \tag{2.5}$$

Together with the formula (1.3), (1.5) and (2.4), we give

$$\sum_{n=1}^{\infty}nb_nz^{n-1}=\sum_{n=0}^{\infty}c_nz^n\sum_{n=1}^{\infty}n\tilde{a}_nz^{n-1},\ \ z\in\mathbb{D},$$

where $\tilde{a}_1=1$, $\tilde{a}_n=-|a_n|$, $n=2,3,4,\ldots$, which leads to

$$\sum_{n=0}^{\infty}(n+1)b_{n+1}z^n=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)\tilde{a}_{k+1}c_{n-k}\right)z^n,\ \ z\in\mathbb{D}.$$

Hence, we obtain

$$(n+1)b_{n+1}=\sum_{k=0}^{n}(k+1)\tilde{a}_{k+1}c_{n-k},\ \ n=0,1,2,\ldots.$$

Applying the formula (1.6) and (2.5), and by simple calculation, we have

$$\begin{aligned} 2|b_2|&\leq|\tilde{a}_1||c_1|+2|\tilde{a}_2||c_0| \\ &\leq 1-|c_0|^2+|c_0| \\ &=1+\alpha-\alpha^2. \end{aligned}$$

Now, we will prove the estimate is sharp. For $\alpha\in[0,1)\subset\mathbb{D}$. Consider a function $f_0(z)=h_0(z)+\overline{g_0(z)}$, such that $h_0(z)=z-\frac{1}{2}z^2\in T$, and suppose that the dilatation of f_0 satisfies

$$\omega_0(z)=\frac{z-\alpha}{1-\alpha z},\quad z\in\mathbb{D}.$$

Applying the formula (1.5), we obtain

$$g_0'(z)=-\frac{\alpha-(1+\alpha)z+z^2}{1-\alpha z}=-\alpha+(1+\alpha-\alpha^2)z+(-1+\alpha+\alpha^2-\alpha^3)z^2+\ldots,\quad z\in\mathbb{D},$$

which implies the estimate of (2.1) is sharp. Since $g_0(0)=0$, by integration, we uniquely deduce

$$g_0(z)=\begin{cases}(-1-\frac{1}{\alpha}+\frac{1}{\alpha^2})z+\frac{1}{2\alpha}z^2+(1-\frac{1}{\alpha}-\frac{1}{\alpha^2}+\frac{1}{\alpha^3})\ln(1-\alpha z),\ \alpha\neq 0,\\ \\ \frac{1}{2}z^2-\frac{1}{3}z^3,\ \alpha=0,\end{cases}\qquad z\in\mathbb{D}.$$

Obviously, $|\omega_0(z)|<1$, $z\in\mathbb{D}$, which means $f_0(z)=h_0(z)+\overline{g_0(z)}\in TL_H^{\alpha}$.

In the same way,

$$\begin{aligned}n|b_n|&\leq|\tilde{a}_1||c_{n-1}|+2|\tilde{a}_2||c_{n-2}|+\ldots+(n-1)|\tilde{a}_{n-1}||c_1|+n|\tilde{a}_n||c_0|\\&\leq\left(1+\sum_{k=2}^{n-1}k|\tilde{a}_k|\right)(1-|c_0|^2)+|c_0|\\&\leq 2+\alpha-2\alpha^2,\quad n=3,4,5,\ldots\end{aligned}$$

Hence, the proof is completed. □

Corollary 2.1. If $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\overline{\sum_{n=1}^{\infty}b_nz^n}\in TL_H$, then

$$|b_n|\leq\frac{17}{8n},\quad n=3,4,5\ldots.$$

Proof. By simple calculation, we have

$$\sup_{\alpha\in[0,1)}|b_n|=\frac{2+\alpha-2\alpha^2}{n}|_{\alpha=1/4}=\frac{17}{8n},\quad n=3,4,5,\ldots.$$

then the corollary follows immediately from Theorem 2.1. □

Since the analytic part h of $f\in TL_H^{\alpha}$ belongs to T, we have the following distortion estimate of h by Theorem 1.2.[9]

$$1-|z|\leq|h'(z)|\leq 1+|z|,\quad z\in\mathbb{D}.\tag{2.6}$$

Our next aim is to give the distortion estimate of the co-analytic part g of $f\in TL_H^{\alpha}$.

Theorem 2.2. *If* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\overline{\sum_{n=1}^{\infty}b_nz^n}\in TL_H^{\alpha}$, *then*

$$|g'(z)|\geq\frac{(\alpha-|z|)(1-|z|)}{1-\alpha|z|},\quad z\in\mathbb{D},\tag{2.7}$$

and

$$|g'(z)|\leq\frac{(\alpha+|z|)(1+|z|)}{1+\alpha|z|},\quad z\in\mathbb{D}.\tag{2.8}$$

These inequalities are sharp. The equalities hold for the harmonic function $f_0(z)$ *which is defined in* (2.2)*.*

Proof. Let $b_1=g'(0)=\alpha e^{i\psi}$. Consider the function

$$F(z):=\frac{e^{-i\psi}\omega(z)-\alpha}{1-\alpha e^{-i\psi}\omega(z)},\quad z\in\mathbb{D}.\tag{2.9}$$

It satisfies assumptions of the Schwarz lemma, which gives

$$|e^{-i\psi}\omega(z)-\alpha|\leq|z||1-\alpha e^{-i\psi}\omega(z)|,\quad z\in\mathbb{D}.\tag{2.10}$$

It is equivalent to

$$|e^{-i\psi}\omega(z)-\frac{\alpha(1-|z|^2)}{1-\alpha^2|z|^2}|\leq\frac{|z|(1-\alpha^2)}{1-\alpha^2|z|^2},\quad z\in\mathbb{D},\tag{2.11}$$

and the equality holds only for the functions satisfying

$$\omega(z)=e^{i\psi}\frac{e^{i\varphi}z+\alpha}{1+\alpha e^{i\varphi}z},\quad z\in\mathbb{D}.\tag{2.12}$$

where $\varphi\in\mathbb{R}$.

Hence, applying the triangle inequalities and the formula (2.11) we have

$$\frac{\alpha-|z|}{1-\alpha|z|}\leq|\omega(z)|\leq\frac{\alpha+|z|}{1+\alpha|z|},\quad z\in\mathbb{D}.\tag{2.13}$$

Finally, applying the formula (2.6) together with (2.13) to the identity $g'=\omega h'$, we obtain (2.7) and (2.8). The function $f_0(z)$ defined in (2.2) shows that inequalities (2.7) and (2.8) are sharp. The proof is completed. □

Corollary 2.2. If $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\overline{\sum_{n=1}^{\infty}b_nz^n}\in TL_H$, then

$$|g'(z)|\leq 1+|z|,\quad z\in\mathbb{D}.\tag{2.14}$$

Proof. Let α tend to 1 in the estimate (2.8), then the corollary follows from theorem 2.2. immediately. $\square$

From the Theorem 1.3[9], we can get the growth estimate of the analytic part h of $f\in TL_H^{\alpha}$

$$|z|-\frac{1}{2}|z|^2\le |h(z)|\le |z|+\frac{1}{2}|z|^2,\quad z\in\mathbb{D}.$$

Next results, we give the growth estimate of co-analytic part g of $f\in TL_H^{\alpha}$.

Theorem 2.3. *If* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\overline{\sum_{n=1}^{\infty}b_nz^n}\in TL_H^{\alpha}$, *then*

$$|g(z)|\le\begin{cases}(1+\frac{1}{\alpha}-\frac{1}{\alpha^2})|z|+\frac{1}{2\alpha}|z|^2+(1-\frac{1}{\alpha}-\frac{1}{\alpha^2}+\frac{1}{\alpha^3})\ln(1+\alpha|z|),\ \alpha\neq 0,\\ \\ \frac{1}{2}|z|^2+\frac{1}{3}|z|^3,\ \alpha=0,\end{cases}\quad z\in\mathbb{D}.\tag{2.15}$$

The inequality is sharp. The equality hold for the harmonic function $f_0(z)$ *which is defined in* (2.2).

Proof. Let $\Gamma:=[0,z]$, applying the estimate (2.8), we have

$$|g(z)|=|\int_{\Gamma}g'(\zeta)d\zeta|\le\int_{\Gamma}|g'(\zeta)||d\zeta|\le\int_0^{|z|}\frac{(\alpha+s)(1+s)}{1+\alpha s}ds.$$

By integration, we obtain the estimate (2.15). The function $f_0(z)$ defined (2.2) shows that the inequality (2.15) is sharp. $\square$

Using the distortion estimates in Theorem 1.2[9] and Theorem 2.2, we can easily deduce the following Jacobian estimates of $f\in TL_H^{\alpha}$.

Theorem 2.4. *If* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n+\overline{\sum_{n=1}^{\infty}b_nz^n}\in TL_H^{\alpha}$, *then*

$$J_f(z)\ge\frac{(1-\alpha^2)(1+|z|)(1-|z|)^3}{(1+\alpha|z|)^2},\quad z\in\mathbb{D},\tag{2.16}$$

and

$$J_f(z)\le\begin{cases}\frac{(1-\alpha^2)(1-|z|)(1+|z|)^3}{(1-\alpha|z|)^2},\ \alpha>|z|,\\ \\ (1+|z|)^2,\ \alpha\le|z|,\end{cases}\quad z\in\mathbb{D}.\tag{2.17}$$

Proof. Observe that if $f\in TL_H^{\alpha}$, then h' does not vanish in $\mathbb{D}$. We can give the Jacobian of f in the form

$$J_f(z)=|h'(z)|^2(1-|\omega(z)|^2),\quad z\in\mathbb{D},\tag{2.18}$$

where ω is the dilatation of f. Applying (2.6) and (2.13) to the (2.17) we obtain

$$J_f(z)\geq(1-|z|)^2\frac{(1-\alpha^2)(1-|z|^2)}{(1+\alpha|z|)^2},\ \ z\in\mathbb{D},$$

and

$$J_f(z)\leq\begin{cases}(1+|z|)^2\frac{(1-\alpha^2)(1-|z|^2)}{(1-\alpha|z|)^2},\ \alpha>|z|,\\ \\ (1+|z|)^2,\ \alpha\leq|z|,\end{cases}\qquad z\in\mathbb{D},$$

this completes the proof. □

Since every univalent function is locally univalent, we can give the growth estimate of $f\in TS_H^\alpha$.

Theorem 2.5. *If $f(z)=z-\sum_{n=2}^\infty|a_n|z^n+\overline{\sum_{n=1}^\infty b_nz^n}\in TS_H^\alpha$, then*

$$|f(z)|\geq\begin{cases}(2-\frac{1}{\alpha}-\frac{1}{\alpha^2})|z|+\frac{1-\alpha}{2\alpha}|z|^2-(1+\frac{1}{\alpha}-\frac{1}{\alpha^2}-\frac{1}{\alpha^3})\ln(1+\alpha|z|),\ \alpha\neq0,\\ \\ |z|-|z|^2+\frac{1}{3}|z|^3,\ \alpha=0,\end{cases}\qquad z\in\mathbb{D},\tag{2.19}$$

and

$$|f(z)|\leq\begin{cases}(2+\frac{1}{\alpha}-\frac{1}{\alpha^2})|z|+\frac{1+\alpha}{2\alpha}|z|^2+(1-\frac{1}{\alpha}-\frac{1}{\alpha^2}+\frac{1}{\alpha^3})\ln(1+\alpha|z|),\ \alpha\neq0,\\ \\ |z|+|z|^2+\frac{1}{3}|z|^3,\ \alpha=0,\end{cases}\qquad z\in\mathbb{D}.\tag{2.20}$$

Proof. For any point $z\in\mathbb{D}$ and suppose $r:=|z|$, we denote $\mathbb{D}_r:=\mathbb{D}(0,r)=\{z\in\mathbb{D}:|z|<r\}$, and let

$$R:=\min_{z\in\mathbb{D}_r}|f(\mathbb{D}_r)|,$$

then $\mathbb{D}(0,R)\subseteq f(\mathbb{D}_r)\subseteq f(\mathbb{D})$. Hence, there exists $z_r\in\partial\mathbb{D}_r$ such that $R=|f(z_r)|$. Let $\Gamma(t):=tf(z_r)$, $t\in[0,1]$, then $\gamma(t):=f^{-1}(\Gamma(t))$, $t\in[0,1]$ is a well-defined Jordan arc. Since $f(z)=h(z)+\overline{g(z)}$, then we can write

$$R=|f(z_r)|=\int_\Gamma|dw|=\int_\gamma|df|=\int_\gamma|h'(\zeta)d\zeta+\overline{g'(\zeta)}d\overline{\zeta}|\geq\int_\gamma(|h'(\zeta)|-|g'(\zeta)|)|d\zeta|.$$

By $g'=\omega h'$ and the formula (2.6) and (2.13). We obtain

$$|h'(\zeta)|-|g'(\zeta)|=|h'(\zeta)|(1-|\omega(\zeta)|)\geq(1-|\zeta|)\left(1-\frac{\alpha+|\zeta|}{1+\alpha|\zeta|}\right)=\frac{(1-\alpha)(1-|\zeta|)^2}{1+\alpha|\zeta|}.$$

Hence, we have

$$R \geq \int_{\gamma} \frac{(1-\alpha)(1-|\zeta|)^2}{1+\alpha|\zeta|}|d\zeta| = \int_0^1 \frac{(1-\alpha)(1-|\gamma(t)|)^2}{1+\alpha|\gamma(t)|}dt \geq \int_0^{|z|} \frac{(1-\alpha)(1-\rho)^2}{1+\alpha\rho}d\rho.$$

Integrating, we obtain the estimate (2.19). To prove (2.20) we simply use the inequality

$$|f(z)| = |h(z)+\overline{g(z)}| \leq |h(z)|+|g(z)|.$$

Then, by the formula (1.8) and (2.15) with simple calculation we have (2.20), this completes the proof. □

Finally, the growth estimate of $f \in TS_H^{\alpha}$ yields a covering estimate.

Theorem 2.6. *If* $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in TS_H^{\alpha}$, *then*

$$D(0,R) \subset f(D),$$

where

$$R := \begin{cases} \frac{3}{2} - \frac{1}{2\alpha} - \frac{1}{\alpha^2} - (1+\frac{1}{\alpha} - \frac{1}{\alpha^2} - \frac{1}{\alpha^3})\ln(1+\alpha), & \alpha \neq 0, \\ \frac{1}{3}, & \alpha = 0. \end{cases}$$

The images of $\alpha \in [0,1) \mapsto R$ *are shown in Fig. 1. This figure is drawn by using Mathematica.*

Proof. If we let $|z|$ tend to 1 in the estimate (2.19), then the Theorem 2.6. follows immediately from the argument principle for harmonic mappings. □

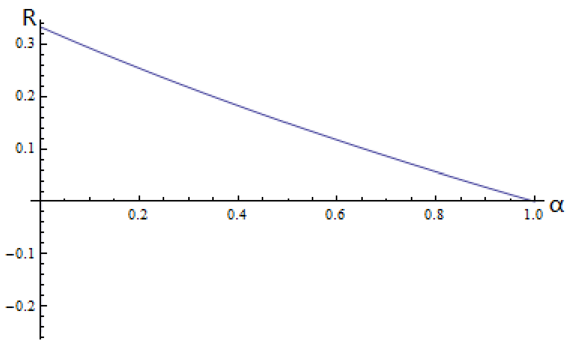

Figure 1: The image of $\alpha \in [0,1) \mapsto R$.

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