

A Note on Burkholder Integrals

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Abstract. In this note, for k -quasiconformal mappings of a bounded domain into the complex plane, we give an upper bound of Burkholder integral. Moreover, as an application we obtain an upper bound of the L^p -integral of $\sqrt{|J_f|}$ and $|Df|$ for certain K -quasiconformal mappings.

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1 Introduction

Let Ω and Ω' be two bounded simply connected domains of the complex plane \mathbb{C} . A homeomorphism $f : \Omega \rightarrow \Omega'$ is called k -quasiconformal, if it belongs locally to the Sobolev class $W_{loc}^{1,2}(\Omega)$ and satisfies the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu_f \frac{\partial f}{\partial z} \quad a.e. \quad z \in \Omega,$$

where the Beltrami coefficient has bounded L_∞ norm: $\|\mu_f\|_\infty \leq k < 1$. In particular, a homeomorphism of \mathbb{C} onto itself is called principal solution of the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}} = \mu_f \frac{\partial f}{\partial z},$$

if it satisfies the asymptotical normalization condition

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots, \quad \text{for } |z| \rightarrow \infty.$$

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We denote the formal partial derivatives of f by

$$\partial f = f_z = \frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \quad \text{and} \quad \bar{\partial} f = f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right),$$

and use the notation

$$|Df| = |f_z| + |f_{\bar{z}}| \quad \text{and} \quad J_f = |f_z|^2 - |f_{\bar{z}}|^2.$$

Here the value $|Df|$ is the operator norm for Df and J_f is the Jacobian of f .

A continuous function $E: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is said to be quasiconvex if for every $f \in A + C_0^\infty(\Omega, \mathbb{R}^n)$, we have

$$\mathcal{E}[f] := \int_{\Omega} E(Df) dx \geq \int_{\Omega} E(A) dx = E(A)|\Omega|,$$

where A stands for an arbitrary linear mapping (or its matrix) and $\Omega \subset \mathbb{R}^n$ is any bounded domain. In other words, one requires that compactly supported perturbations of linear maps do not decrease the value of the integral. This notion is very important in the calculus of variations [7]. Another notion is that of rank-one convexity, which requires just that $t \rightarrow E(A + tX)$ is convex for any fixed matrix A and for any rank one matrix X . E is rank-one concave (resp. quasiconcave) if $-E$ is rank-one convex (resp. quasiconvex). The most famous rank-one concave function in dimension two is the Burkholder functional defined for any 2×2 matrix A by

$$B_p(A) = \left[\frac{p}{2} \det A + \left(1 - \frac{p}{2} \right) |A|^2 \right] |A|^{p-2}, \quad p \geq 2. \quad (1.1)$$

where $|A|$ is the operator norm of A , see [3]. Morrey's work [8] implies that quasiconvexity implies rank-one convexity. For the dimension n of \mathbb{R}^n is bigger than 2, Šverák's paper [10] showed that the converse is not true. However, for dimension $n=2$, [5] and [7] gave the evidence to the possibility for a different outcome. So in [1], the authors gave the following conjecture in the spirit of Morrey,

Conjecture 1.1. Rank-one convex functions $E: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ are quasiconvex.

For the $A = Id$, the authors of [1] showed that the Burkholder function is quasiconcave within quasiconformal perturbations of the identity. They showed that when $f: \Omega \rightarrow \Omega$ is a k -quasiconformal map of Ω onto itself with extending to the identity on the boundary, then

$$\int_{\Omega} B_p(Df) dz = \int_{\Omega} \left(1 - \frac{p|\mu(z)|}{1+|\mu(z)|} \right) (|f_z(z)| + |f_{\bar{z}}(z)|)^p dz \leq \int_{\Omega} B_p(Id) dz = |\Omega|, \quad (1.2)$$

where $2 \leq p \leq 1 + \frac{1}{k}$.

In this paper, we first use the method learned from [1] to prove the following result:

Theorem 1.1. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be the principal solution of a Beltrami equation

$$f_{\bar{z}}(z) = \mu(z)f_z(z),$$

for a given $\mu(z)$ with $|\mu(z)| \leq k\chi_D(z)$, $0 \leq k < 1$, $D = \{z : |z| < 1\}$ and it has the expansion

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots.$$

Then, for all $2 \leq p \leq 1 + \frac{1}{k}$, we have

$$\int_D B_p(Df) = \int_D \left(1 - \frac{p|\mu(z)|}{1+|\mu(z)|}\right) (|f_z(z)| + |f_{\bar{z}}(z)|)^p dz \leq \int_D B_2(z + b_1\bar{z}) dz = \pi(1 - |b_1|^2). \quad (1.3)$$

For the arbitrary bounded domain Ω , we have the following result:

Theorem 1.2. Let $f : \Omega \rightarrow L(\Omega)$ be a k -quasiconformal map of the bounded Ω onto its affine image, extending continuously up to the boundary, where it coincides with the affine mapping L . Here $L(z) = z + \frac{b_1}{R^2}\bar{z}$ with $|b_1| < 1$ is a linear mapping and

$$R = \begin{cases} 1, & \text{if } \text{diam}(\Omega) \leq 2, \\ \frac{\text{diam}\Omega}{2}, & \text{if } \text{diam}(\Omega) \geq 2. \end{cases}$$

Then, for all $2 \leq p \leq 1 + \frac{1}{k}$, we have

$$\int_{\Omega} B_p(Df) dz \leq \int_{\Omega} B_2\left(z + \frac{b_1}{R^2}\bar{z}\right) dz = \left(1 - \frac{|b_1|^2}{R^4}\right).$$

This paper is organized as follows. In Section 2, we will introduce some preliminary lemmas used in this paper. In Section 3 we give the proofs of Theorem 1.1 and Theorem 1.2. In the final section, we discuss some applications of the main theorems.

2 Preliminary lemmas

Let $\mathcal{M}(\Omega, \sigma)$ denote the class of complex valued σ -measurable functions on a measure space (Ω, σ) . The following lemmas will be used in this note.

Lemma 2.1. (Interpolation Lemma for the disk [1]) Let $0 < p_0, p_1 \leq \infty$, and $\{\Phi_\lambda : |\lambda| < 1\} \subset \mathcal{M}(\Omega, \sigma)$ be an analytic and nonvanishing family with complex parameter λ in the unit disc. Suppose

$$M_0 := \|\Phi_0\|_{p_0} < \infty, \quad M_1 := \sup_{|\lambda| < 1} \|\Phi_\lambda\|_{p_1} < \infty \quad \text{and} \quad M_r := \sup_{|\lambda|=r} \|\Phi_\lambda\|_{p_r} < \infty,$$

where

$$\frac{1}{p_r} = \frac{1-r}{1+r} \frac{1}{p_0} + \frac{2r}{1+r} \frac{1}{p_1}.$$

Then, for every $0 \leq r < 1$, we have

$$M_r \leq M_0^{\frac{1-r}{1+r}} M_1^{\frac{2r}{1+r}} < \infty. \quad (2.1)$$

Lemma 2.2. (Area Theorem [2]) *Suppose that $f \in W_{loc}^{1,2}(\mathbb{C})$ is analytic outsider the disk $D(0,r)$ and has the expansion*

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots, \quad \text{for } |z| \rightarrow \infty .$$

Then

$$\int_{D(0,r)} J(z,f) dz = \pi \left(r^2 - \sum_{n=1}^{\infty} n |b_n|^2 r^{-2n} \right). \tag{2.2}$$

Lemma 2.3. ([4]) *If f is a k -quasiconformal mapping of a domain $\Omega \subset \mathbb{C}$ into \mathbb{C} , then for every $p \in [2, 1 + \frac{1}{k})$,*

$$\frac{(1 - (p-1)k)(1+k)^{p-1}}{(\sqrt{1-k^2})^p} \left(\sqrt{J_f} \right)^p \leq B_p(Df). \tag{2.3}$$

3 Proofs of Theorem 1.1 and Theorem 1.2

First we will use holomorphic interpolation method and holomorphic motion to give the proof of Theorem 1.1 inspired by [1].

Proof of Theorem 1.1

We fix an exponent $2 \leq p \leq 1 + \frac{1}{\|\mu\|_{\infty}}$ and look for holomorphic deformation of the given function f . To do this, we will construct an analytic family of Beltrami equations together with their principal solutions as the authors did in [1].

Let $F^{\lambda}(z)$ be the principal solutions of the following equations

$$F_z^{\lambda} = \mu_{\lambda}(z) F_z^{\lambda}, \quad \mu_{\lambda}(z) = \tau_{\lambda}(z) \frac{\mu(z)}{|\mu(z)|}.$$

Here $\tau_{\lambda}(z)$ is an analytic function in λ with

$$\frac{\tau_{\lambda}(z)}{1 + \tau_{\lambda}(z)} = p \frac{|\mu(z)|}{1 + |\mu(z)|} \frac{\lambda}{1 + \lambda},$$

for $p = 1 + \frac{1}{\lambda_0}$. More explicitly,

$$\tau_{\lambda}(z) = \frac{p\lambda|\mu(z)|}{(1 + \lambda)(1 + |\mu(z)|) - p\lambda|\mu(z)|}.$$

It is easy to see that $\tau_0(z) = 0$ and $F^0(z) = z$ as $\lambda = 0$. There exists some complex number $\lambda = \lambda_0$ such that when $\lambda = \lambda_0$, we have $\tau_{\lambda_0}(z) = |\mu(z)|$, so that $F^{\lambda_0}(z) = f(z)$. Here and the following, we let $p = 1 + \frac{1}{\lambda_0}$, $p_0 = \infty$ and $p_2 = 2$. We choose a nonvanishing analytic family $\{\Phi_{\lambda}\}_{|\lambda| < 1}$ by

$$\Phi_{\lambda}(z) = F_z^{\lambda}(z)(1 + \tau_{\lambda}(z)).$$

It is easy to see that for $\lambda = \lambda_0$,

$$|\Phi_{\lambda_0}(z)| = (1 + |\mu(z)|)|f_z| = |f_z| + |f_{\bar{z}}| = |Df|.$$

We shall then apply Lemma 2.1 (the Interpolation Lemma) in the measure space $\mathcal{M}(D, \sigma)$ over the unit disk, where

$$d\sigma(z, \lambda) = \frac{1}{\pi(1 - |b_1^\lambda|^2)} \left(1 - \frac{p|\mu(z)|}{1 + |\mu(z)|} \right) dz,$$

and b_1^λ is the coefficient of $\frac{1}{z}$ for the expansion of $F^\lambda(z)$. In particular, for $\lambda = \lambda_0$, we have

$$d\sigma(z, \lambda_0) = \frac{1}{\pi(1 - |b_1|^2)} \left(1 - \frac{p|\mu(z)|}{1 + |\mu(z)|} \right) dz,$$

where b_1 is the coefficient of $\frac{1}{z}$ for the expansion of f .

When $\lambda = 0, F^0(z) = z$, hence $F_z^0 \equiv 1, \Phi_0 \equiv 1$ and $M_0 = \|\Phi_0\|_\infty = 1$. In the follows we will estimate $M_1 = \sup_{\lambda \in D} \|\Phi_\lambda\|_2 \leq 1$.

In view of Lemma 2.2

$$\int_D J(z, F^\lambda) dz \leq \pi(1 - |b_1^\lambda|^2),$$

with equality if and only if $F^\lambda = z + \frac{e^{i\theta} b_1^\lambda}{z}$, for some $\theta \in (0, 2\pi)$, outside the unit disk.

We find that

$$\begin{aligned} J(z, F^\lambda) &= |F_z^\lambda(z)|^2 (1 - |\mu_\lambda(z)|^2) = |\Phi_\lambda(z)|^2 \left(1 - 2\operatorname{Re} \frac{\tau_\lambda(z)}{1 + \tau_\lambda(z)} \right) \\ &= |\Phi_\lambda(z)|^2 \left(1 - p \frac{|\mu(z)|}{1 + |\mu(z)|} \operatorname{Re} \frac{2\lambda}{1 + \lambda} \right) \geq |\Phi_\lambda(z)|^2 \left(1 - p \frac{|\mu(z)|}{1 + |\mu(z)|} \right). \end{aligned}$$

Hence

$$|\Phi_\lambda(z)|^2 d\sigma(z, \lambda) \leq \frac{1}{\pi(1 - |b_1^\lambda|^2)} J(z, F^\lambda) dz,$$

therefore

$$M_1 = \sup_{|\lambda| < 1} \int_D |\Phi_\lambda(z)|^2 d\sigma(z, \lambda) \leq \sup_{|\lambda| < 1} \frac{1}{\pi(1 - |b_1^\lambda|^2)} \int_D J(z, F^\lambda) dz \leq 1.$$

For every $0 \leq r < 1$, in view of the Interpolation Lemma, we have

$$M_r = \sup_{|\lambda| = r} \{ \|\Phi_\lambda\|_{\frac{1+r}{1-r}} \} \leq M_0^{\frac{1-r}{1+r}} M_1^{\frac{2r}{1+r}} \leq 1.$$

After the substitution $r = \frac{1}{p-1} = \lambda_0$, the inequality is immediate

$$\int_D \left(1 - \frac{p|\mu(z)|}{1 + |\mu(z)|} \right) |Df|^p = \pi(1 - |b_1|^2) \int_D |\Phi_{\lambda_0}|^{\frac{1+r}{r}} d\sigma(z, \lambda_0) \leq \pi(1 - |b_1|^2) = \int_D B_2(z + b_1 \bar{z}) dz.$$

So the proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2

Since

$$R = \begin{cases} 1, & \text{if } \text{diam}(\Omega) \leq 2, \\ \frac{\text{diam}\Omega}{2}, & \text{if } \text{diam}(\Omega) \geq 2, \end{cases}$$

we get $D_R \supset \Omega$. In view of the proof of Theorem 1.1, we just let

$$d\sigma(z, \lambda) = \frac{1}{\pi R^2 (1 - \frac{|b_1|^2}{R^4})} \left(1 - \frac{p|\mu(z)|}{1 + |\mu(z)|}\right) dz,$$

and can obtain following

$$\int_{D_R} B_p(Df) dz \leq \int_{D_R} B_2(z + \frac{b_1}{R^2} \bar{z}) dz = (1 - \frac{|b_1|^2}{R^4}) |D_R|. \tag{3.1}$$

Let

$$\tilde{f}(z) = \begin{cases} f(z), & \text{if } z \in \Omega, \\ z + \frac{b_1}{R^2} \bar{z}, & \text{if } z \in D_R - \Omega, \\ z + \frac{b_1}{R^2} \frac{1}{z}, & \text{if } z \in \mathbb{C} - D_R. \end{cases}$$

By inequality (3.1), this yields

$$\int_{D_R} B_p(D\tilde{f}) \leq \int_{D_R} B_2(z + \frac{b_1}{R^2} \bar{z}). \tag{3.2}$$

And since

$$\int_{D_R - \Omega} B_p(D\tilde{f}) = \int_{D_R - \Omega} B_2(z + \frac{b_1}{R^2} \bar{z}), \tag{3.3}$$

in view of (3.2) and (3.3), we get

$$\int_{\Omega} B_p(Df) dz \leq \int_{\Omega} B_2(z + \frac{b_1}{R^2} \bar{z}) dz = |\Omega| (1 - \frac{|b_1|^2}{R^4}).$$

So the proof of Theorem 1.2 is complete. □

4 Upper bounds of the L^p -integrals of $\sqrt{J_f}$ and $|Df|$

In this section, as an application of Theorem 1.1 or Theorem 1.2, we obtain upper bounds of the L^p -integrals of $\sqrt{J_f}$ and $|Df|$.

Theorem 4.1. *Let $f : \Omega \rightarrow L(\Omega)$ be a k -quasiconformal map of the bounded Ω onto its affine image, extending continuously up to the boundary, where it coincides with the affine mapping L . Here we assume that the mapping L satisfies $L(z) = z + \frac{b_1}{R^2} \bar{z}$ with some $|b_1| < 1$ and*

$$R = \begin{cases} 1, & \text{if } \text{diam}(\Omega) \leq 2, \\ \frac{\text{diam}\Omega}{2}, & \text{if } \text{diam}(\Omega) \geq 2. \end{cases}$$

Then, for all $2 \leq p \leq 1 + \frac{1}{k}$, we have

$$\int_{\Omega} \left(\sqrt{J_f}\right)^p dz \leq \frac{(\sqrt{1-k^2})^p}{(1-(p-1)k)(1+k)^{p-1}} \cdot \left(1 - \frac{|b_1|^2}{R^4}\right) |\Omega|.$$

Proof. By the Lemma 2.3, for every $2 \leq p \leq 1 + \frac{1}{k}$, we may write

$$\int_{\Omega} \left(\sqrt{J_f}\right)^p dz \leq \frac{(\sqrt{1-k^2})^p}{(1-(p-1)k)(1+k)^{p-1}} \int_{\Omega} B_p(Df) dz. \tag{4.1}$$

By Theorem 1.2 and (4.1), we have

$$\int_{\Omega} \left(\sqrt{J_f}\right)^p dz \leq \frac{(\sqrt{1-k^2})^p}{(1-(p-1)k)(1+k)^{p-1}} \cdot \left(1 - \frac{|b_1|^2}{R^4}\right) |\Omega|.$$

□

Corollary 4.1. Let $f : \Omega \rightarrow L(\Omega)$ be a k -quasiconformal map of the bounded Ω onto its affine image, extending continuously up to the boundary, where it coincides with the affine mapping L . Here $L(z) = z + \frac{b_1}{R^2} \bar{z}$ with $|b_1| < 1$ is a linear mapping and

$$R = \begin{cases} 1, & \text{if } \text{diam}(\Omega) \leq 2, \\ \frac{\text{diam}\Omega}{2}, & \text{if } \text{diam}(\Omega) \geq 2. \end{cases}$$

Then, for all $2 \leq p \leq 1 + \frac{1}{k}$, we have

$$\int_{\Omega} |Df|^p dz \leq \frac{1+k}{1-(p-1)k} \cdot \left(1 - \frac{|b_1|^2}{R^4}\right) |\Omega|.$$

Proof. By the equality

$$\left(\sqrt{J_f}\right)^p = (|f_z| + |f_{\bar{z}}|)^p \left(\frac{|f_z| - |f_{\bar{z}}|}{|f_z| + |f_{\bar{z}}|}\right)^{\frac{p}{2}}, \tag{4.2}$$

we get from Theorem 4.1 that

$$\left(\frac{1-k}{1+k}\right)^{\frac{p}{2}} \int_{\Omega} |Df|^p dz \leq \int_{\Omega} \left(\sqrt{J_f}\right)^p dz \leq \frac{(\sqrt{1-k^2})^p}{(1-(p-1)k)(1+k)^{p-1}} \cdot \left(1 - \frac{|b_1|^2}{R^4}\right) \cdot |\Omega|.$$

Hence,

$$\int_{\Omega} |Df|^p dz \leq \frac{1+k}{1-(p-1)k} \cdot \left(1 - \frac{|b_1|^2}{R^4}\right) \cdot |\Omega|.$$

The claim of Corollary 4.2 follows.

□

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