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NUMERICAL STUDIES OF A CLASS OF COMPOSITE PRECONDITIONERS*

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Abstract

In this paper, we study a composite preconditioner that combines the modified tangential frequency filtering decomposition with the ILU(0) factorization. Spectral property of the composite preconditioner is examined by the approach of Fourier analysis. We illustrate that condition number of the preconditioned matrix by the composite preconditioner is asymptotically bounded by $\mathcal{O}(h_p^{-\frac{2}{3}})$ on a standard model problem. Performance of the composite preconditioner is compared with other preconditioners on several problems arising from the discretization of PDEs with discontinuous coefficients. Numerical results show that performance of the proposed composite preconditioner is superior to other relative preconditioners.

Mathematics subject classification: 65F10, 65N22.
Key words: Preconditioner, ILU, Tangential frequency filtering decomposition, GMRES.

1. Introduction

We consider preconditioning techniques for solving systems of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1.1}$$

with

$$\mathbf{A} = \begin{bmatrix} D_1 & U_1 & & \\ L_1 & D_2 & \ddots & \\ & \ddots & \ddots & U_{n_x-1} \\ & & L_{n_x-1} & D_{n_x} \end{bmatrix} \in \mathcal{R}^{N\times N}, \quad b \in \mathcal{R}^N,$$

where n_x denotes the number of grid points in the x-direction, $D_i \in \mathcal{R}^{n_i\times n_i}$, $L_i \in \mathcal{R}^{n_{i+1}\times n_i}$, $U_i \in \mathcal{R}^{n_i\times n_{i+1}}$, and $N = \sum_{i=1}^{x} n_i$ is the total number of grid points. Such kinds of problems frequently arise from numerical solution of partial differential equations [31, 41, 42, 47]. Due to prohibitive memory requirement, direct methods are generally not acceptable, especially for 3D problems. In recent years, Krylov subspace methods combining with appropriate preconditioners have become the natural choice [6, 27, 37, 43, 47]. It is known that convergence rate of preconditioned Krylov subspace methods heavily depend on the spectrum distribution of the preconditioned matrix, and preconditioning plays a key role in making spectrum distribution

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avail for fast convergence [43,46]. Therefore, a great deal of effort has been put into the development of efficient preconditioners. The multigrid methods are well known for their efficiency of reducing the high-frequency and low-frequency components of the error by complementary schemes [52]. However, these approaches may become ineffective when dealing with general sparse systems of linear equations. Another type of efficient preconditioning techniques are constructed by making use of spectrum information of the preconditioned matrix [23,29,38,44,45]. They are able to get rid of the influence of eigenvalues close to zero, which is generally difficult to handle by conventional incomplete factorization preconditioners.

It is well known that a wide class of preconditioners are based on incomplete factorizations, e.g., BILU, ILUT, SOR, HSS [4,6,10,14,15,28,30,31,43]. In the present work, we mainly focus on a class of incomplete factorization $\mathbf{M}$ that enables a so called right filtering property

$$(\mathbf{M}-\mathbf{A})\mathbf{f}=0,$$

or left filtering property

$$\mathbf{f}^T(\mathbf{M}-\mathbf{A})=0,$$

for a vector $\mathbf{f}$. The idea of filtering is longstand technique, and it is utilized in a nested factorization [3] by J. R. Appleyard, and popularized in recent years by G. Wittum and his successors in [1,2,19,21,22,48–51]. In particular, Tangential Frequency Filtering Decomposition (TFFD) preconditioner proposed by Achdou and Nataf [2] is an efficient filtering preconditioner. It is constructed by using the eigenvector associated with the smallest eigenvalue as a filtering vector. As the ILU(0) preconditioner is efficient in reducing the influence of the higher part of the spectrum [24] and the TFFD is efficient in removing the influence of low part, it is suggested in [2] to combine the TFFD with the classical ILU(0) in a multiplicative way. The combination results in a composite preconditioner which is efficient on some challenging problems with highly discontinuous coefficients. The selection of the filtering vector is an important issue. The choice of the filtering vectors is investigated in [32], and several different kinds of filtering methods are compared in [32]. The results reveal that $\mathbf{e}^T=[1,\ldots,1]^T$ is a reasonable choice for a wide range of problems. It has been illustrated that the use $\mathbf{e}$ as a filtering vector is robust, and can save the cost in forming the filtering vector [32].

Fourier analysis is a classical scheme for analyzing both differential equations and discrete solution methods for time dependent problems [20, 42]. It is popularized by T.F. Chan etc [24, 33, 40] for analyzing algebraic preconditioners and classical iterative methods. On some point-wise incomplete factorization type preconditioners, for example, ILU(0) [34], modified ILU (MILU) [30] and relaxed ILU (RILU) [5], Fourier analysis has been carried out in [24–26]. The Fourier analysis of block ILU and MILU factorization preconditioners is considered in [40] on a time-dependent hyperbolic PDE problem. In [2] the TFFD preconditioner is analyzed by the approach of Fourier method, and an optimal modification of TFFD preconditioner is derived. The preconditioner is called Modified Tangential Frequency Filtering Decomposition (MTFFD) preconditioner. It is illustrated that the condition number of the MTFFD preconditioned matrix is asymptotically bounded by $\mathcal{O}(h^{-\frac{2}{3}})$. Compared with the asymptotic bounds of the condition number by using some classical incomplete factorizations like ILU(0), BILU and MILU ($\mathcal{O}(h^{-2})$ or $\mathcal{O}(h^{-1})$) [4,25,30,40], we can see that the bound obtained by MTFFD is considerably better. Numerical tests on some discontinuous problems also illustrate that MTFFD can improve the performance of TFFD.

In this paper, we investigate a composite preconditioner, which is constructed by combing the MTFFD preconditioner with the ILU(0) preconditioner in a multiplicative way. The composite preconditioner is analyzed by the approach of Fourier analysis on a model problem of Poisson equation posed on unit square. We illustrate that the spectral property of the preconditioned matrix can be well predicted by the results of Fourier analysis. We also show that the condition number of the preconditioned matrix by composite preconditioner is asymptotically bounded by $\mathcal{O}(h^{-\frac{2}{3}})$, which is better than the ILU(0) and the MILU preconditioner. We also observe that the eigenvalues of the preconditioned matrix by the composite preconditioner is more clustered than that of the MTFFD preconditioned matrix. Although the precise proof of the superiority of composite preconditioner over the MTFFD preconditioner is difficult, better cluster property of the eigenvalues of the preconditioned matrix can be observed on the model problem. These results indicate that the composite preconditioner should produce better preconditioning effect. We evaluate the performance of the composite preconditioner on several problems arising from the discretization of PDEs with discontinuous coefficients. Numerical results show that the composite preconditioner proposed in this paper is superior to other relative preconditioner discussed in [2].

The paper is arranged as follows. In Section 2, we briefly review the model problem which will be used for performing Fourier analysis. In Section 3, we analyze the composite preconditioner, which is formed by combining MTFFD with ILU(0). In Section 4, performance of the composite preconditioner is examined on several problems arising from the discretization of PDEs with discontinuous coefficients. Finally, some concluding remarks are given in Section 5.

2. Preliminary

2.1. Notations

We use $ctrid_m(\alpha,\beta,\gamma)$ and $circ_m(\gamma_1,\dots,\gamma_m)$ to denote the circulant tridiagonal matrix and circulant matrix of order m, i.e.,

$$ctrid_m(\alpha,\beta,\gamma)=\begin{bmatrix}\beta & \gamma & & \alpha\\ \alpha & \ddots & \ddots & \\ & \ddots & \ddots & \gamma\\ \gamma & & \alpha & \beta\end{bmatrix},$$

$$circ_m(\gamma_1,\dots,\gamma_m)\begin{bmatrix}\gamma_1 & \gamma_2 & \cdots & \gamma_{m-1} & \gamma_m\\ \gamma_m & \gamma_1 & \gamma_2 & \cdots & \gamma_{m-1}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \gamma_3 & \cdots & \gamma_m & \gamma_1 & \gamma_2\\ \gamma_2 & \gamma_3 & \cdots & \gamma_m & \gamma_1\end{bmatrix}.$$

We also use $trid_m(\alpha,\beta,\gamma)$ and $Btrid_m(L,T,U)$ to denote the $m\times m$ tridiagonal and $mk\times mk$ block tridiagonal matrix with each diagonal block of size $k\times k$ respectively, i.e.,

$$trid_m(\alpha,\beta,\gamma)=\begin{bmatrix}\beta & \gamma & & \\ \alpha & \ddots & \ddots & \\ & \ddots & \ddots & \gamma\\ & & \alpha & \beta\end{bmatrix}\quad\text{and}\quad Btrid_m(L,T,U)=\begin{bmatrix}T & U & & \\ L & \ddots & \ddots & \\ & \ddots & \ddots & U\\ & & L & T\end{bmatrix}.$$

2.2. The model problem

Consider the 2-D Poisson equation [5, 24, 26]

$$-\Delta u = f \tag{2.1}$$

posed on the unite square $\Omega = 0 \le x, y \le 1$ with periodic boundary conditions

$$u(x,0) = u(x,1), \quad u(0,y) = u(1,y).$$

Discretizing problem (2.1) by the standard second order finite difference scheme on a uniform $(n+1) \times (n+1)$ grid, we have system of linear equations

$$Au = b \tag{2.2}$$

where

$$\begin{aligned} A &= I_n \otimes D + \kappa_2 S \otimes I_{n+1}, \\ D &= ctrid_n(-1, 4, -1), \\ S &= ctrid_n(-1, 0, 1). \end{aligned}$$

Let $u_{j,k} = u(jh_p, kh_p)$ be values of function u on mesh points (jh_p, kh_p), $1 \le j, k \le n$, with $h_p = \frac{1}{n}$. The subscript $_p$ is used for parameters in the periodic case. The Fourier eigenvalues of the coefficient matrix A is given by the following equation [24]

$$Au^{(j,k)} = \lambda_A u^{(j,k)},$$

where $u^{(j,k)}$ is defined by

$$u_{s,t}^{(j,k)} = e^{\mathtt{i}s\theta_j} e^{\mathtt{i}t\phi_k}, \quad 1 \le s \le n, \quad 1 \le t \le n$$

with

$$\theta_j = \frac{2\pi j}{n+1}, \qquad \phi_k = \frac{2\pi k}{n+1},$$

and $\mathtt{i}^2 = -1$. The Fourier eigenvalues of A corresponding to the Fourier eigenvector $u_{s,t}^{(j,k)}$ is given by [24]

$$\lambda_A(j,k) = 4\left(\sin^2\frac{\theta_j}{2} + \sin^2\frac{\phi_k}{2}\right). \tag{2.3}$$

The above results will be used later in analyzing properties of the preconditioned matrix.

2.3. Modified tangential frequency filtering preconditioner

For the block tridiagonal coefficient matrix $\mathbf{A}$, the *Modified Tangential Frequency Filtering Decomposition* (MTFFD) preconditioner $\mathbf{M}$ is given by [39]

$$\mathbf{M} = \begin{bmatrix} T_1 & & & \\ L_1 & T_2 & & \\ & \ddots & \ddots & \\ & & L_{m-1} & T_m \end{bmatrix} \begin{bmatrix} T_1^{-1} & & & \\ & T_2^{-1} & & \\ & & \ddots & \\ & & & T_m^{-1} \end{bmatrix} \begin{bmatrix} T_1 & U_1 & & \\ & T_2 & \ddots & \\ & & \ddots & U_{m-1} \\ & & & T_m \end{bmatrix},$$

where off-diagonal blocks L_i and U_i are defined in (1.1), and diagonal blocks T_i in MTFFD are generated by a recursion formula

$$T_i = \begin{cases} D_1 + c_d \Lambda_1 h_d^q, & i = 1, \\ D_i - L_{i-1}(2G_i - G_i T_{i-1} G_i)U_{i-1} + c_d \Lambda_i h_d^q, & 2 \le i \le m. \end{cases} \tag{2.4}$$

Here Λ_i, $1 \le i \le m$, are diagonal matrices with positive diagonal elements, parameter q is the order of modification, and c_d is a nonnegative relaxation parameter (the subscript $_d$ implies that the parameter belongs to the Dirichlet case). The optimal choice of q and c and the existence of MTFFD preconditioner has been discussed in [39].

The matrix G_i is a diagonal approximation to T_{i-1}^{-1}, which can be determined by making the constructed TFFD preconditioner satisfying a right filtering condition. In this paper, we consider preconditioner $\mathbf{M}$ that preserves an *approximate* right filtering condition

$$(\mathbf{M} - \mathbf{A})f = c_d h_d^q \Lambda f$$

with $\Lambda = Bdiag(\Lambda_1, \ldots, \Lambda_m)$ be a diagonal matrix and f be a selected filtering vector. As

$$(\mathbf{M} - \mathbf{A}) = Bdiag(N_1, \ldots, N_m) + c_d \Lambda$$

is a block diagonal matrix, with $N_1 = 0$ and

$$N_i = L_{i-1}(G_i T_{i-1} - I) T_{i-1}^{-1} (T_{i-1} G_i - I) U_{i-1}, \quad \text{for} \quad 2 \le i \le m.$$

Based on Lemma 2.1 in [2], the diagonal matrices G_i can be determined by

$$G_i = Diag((T_{i-1}^{-1} U_{i-1} f)./(U_{i-1} f)), \tag{2.5}$$

where ./ denotes the point-wise vector division and $Diag(v)$ is the diagonal matrix constructed from vector v. From (2.5) we can see that G_i exists as long as $U_{i-1}f$ has no zero entries. For filtering vector $f = \mathbf{e} = [1, 1 \ldots, 1]^T$, this property always holds for the model problem (2.2).

Remarks:

- If the *approximate* left filtering condition

  $$f^T(\mathbf{M} - \mathbf{A}) = c_d h_d^q f^T \Lambda$$

  is required, then diagonal matrix G_i can be determined by [32]

  $$G_i = Diag((f^T L_{i-1} T_{i-1}^{-1})./(f^T L_{i-1})).$$

  For a symmetric coefficient matrix, the left filtering preconditioner is coincide with right filtering preconditioner.

- It is well known that MILU preconditioner satisfies a row-sum equality condition (also known as mass preserve property) along with a possible perturbation term [24, 30],

  $$\mathbf{M}_{MILU} e = \mathbf{A} e + c h^2 e$$

  where $c \ge 0$, $e = [1, \ldots, 1]^T$. In the case of $c = 0$, it is obvious that the row-sum equality property is actually a right filtering property. For TFFD preconditioner $\mathbf{M}_{TFFD}$ [2], if the filtering vector chosen as e, then $\mathbf{M}_{TFFD}$ will also has the same right filtering property. However, $\mathbf{M}_{TFFD}$ is able to choose filtering vector in a more flexibly way. For example, in constructing similar type of preconditioners, certain sine functions have been adopted as filtering vector in [48, 49], adaptive vectors was used as filtering vector in [50], and Ritz vectors was used as filtering vector in [2]. We remark that it is also possible that $\mathbf{M}_{MILU}$ be constructed in a relaxed way such that the preconditioner holds a predetermined filtering property.

- It is proved [24] that the condition number of the preconditioned matrix by $\mathbf{M}_{MILU}$ is bounded by $\mathcal{O}(h^{-1})$ if $c > 0$ and $\mathcal{O}(h^{-2})$ if $c = 0$. Note that in the first case, the preconditioner just satisfies an approximate right filtering property. The modified tangential frequency filtering decomposition preconditioner $\mathbf{M}_{MTFFD}$ possess a similar approximate filtering property in the sense that

$$(\mathbf{M}_{MILU} - \mathbf{A})f = ch^2\Lambda f,$$

where $c \geq 0$ and Λ is a diagonal matrix. The asymptotic bound of the condition number of $\mathbf{M}_{MTFFD}$ preconditioned matrix is proved to be $\mathcal{O}(h^{-\frac{2}{3}})$ [39], which improves that of $\mathbf{M}_{MILU}$.

In what follows, we will have a brief review of theoretical results which will be used for analysis. According to theory developed in [24], Fourier analysis can only be performed on constant coefficient problems with periodic boundary conditions [24]. Therefore, the MTFFD preconditioner $\hat{M}$ for model problem (2.2) should be forced to have constant diagonals, i.e., the MTFFD preconditioner $\hat{M}$ for periodic system (2.2) is assumed to be in form of

$$\hat{M} = (\hat{L} + \hat{T})\hat{T}^{-1}(\hat{T} + \hat{U}),$$

where

$$\hat{L} = \begin{bmatrix} 0 & & & -I_n \\ -I_n & \ddots & & \\ & \ddots & \ddots & \\ & & -I_n & 0 \end{bmatrix}, \quad \hat{U} = \begin{bmatrix} 0 & -I_n & & \\ & \ddots & \ddots & \\ & & \ddots & -I_n \\ -I_n & & & 0 \end{bmatrix}, \quad \hat{T} = I_n \otimes \hat{T}_0,$$

with $\hat{T}_0 = circ_n(\hat{d}, -\hat{\kappa}_1, 0, \ldots, 0, -\hat{\kappa}_1)$ determined by $\hat{d}$ and $\hat{\kappa}_1$, and $circ_n(\gamma_1, \ldots, \gamma_n)$ defined by

$$circ_n(\gamma_1, \ldots, \gamma_n) = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} & \gamma_n \\ \gamma_n & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \gamma_3 & \cdots & \gamma_n & \gamma_1 & \gamma_2 \\ \gamma_2 & \gamma_3 & \cdots & \gamma_n & \gamma_1 \end{bmatrix}.$$

For the purpose of analysis, we fix $\Lambda_i = I_n$, and choose $\mathbf{e} = [1, \ldots, 1]^T$ as a filtering vector. Then following lemmas hold.

Lemma 2.1. ([39]) *The (j,k)-th Fourier eigenvalues of $\hat{M}^{-1}A$ is given by*

$$\lambda_{j,k}^{-1}(\hat{M}^{-1}A) = 1 + \frac{4(\delta_h^{-1}+1)(\delta_h^{-1}-1)\sin^2(\frac{\theta_j}{2}) + \frac{\eta_h^2}{\sin^2(\frac{\theta_j}{2})}}{16\hat{\kappa}_1\sin^2(\frac{\theta_j}{2}) + 16\hat{\kappa}_1\sin^2(\frac{\phi_k}{2}) + 4(1+\eta_h) + 4(1+\eta_h)\frac{\sin^2(\frac{\phi_k}{2})}{\sin^2(\frac{\theta_j}{2})}},$$

where

$$\eta_h = \frac{1}{2}c_p h_p^q + \frac{1}{2}\sqrt{(4+c_p h_p^q)c_p h_p^q}, \quad \delta_h = \frac{\sqrt{(4+c_p h_p^q)c_p h_p^q}}{c_p h_p^q + 2}.$$

Lemma 2.2. ([24]) *The (j,k)-th Fourier eigenvalues of M_{ilu} is given by*

$$\lambda_{j,k}(M_{ilu}) = 4\left(\sin^2(\frac{\theta_j}{2}) + \sin^2(\frac{\phi_k}{2})\right) + \frac{2}{2+\sqrt{2}}\cos(\theta_j - \phi_k).$$

Therefore, the (j,k)-th Fourier eigenvalues of $M_{ilu}^{-1}A$ is given by

$$\lambda_{j,k}(M_{ilu}^{-1}A) = \frac{4(\sin^2(\frac{\theta_j}{2}) + \sin^2(\frac{\phi_k}{2}))}{4(\sin^2(\frac{\theta_j}{2}) + \sin^2(\frac{\phi_k}{2})) + \frac{2}{2+\sqrt{2}}\cos(\theta_j - \phi_k)}.$$

For model problem (2.2), $\lambda_{j,k}(M_{ilu}^{-1}A)$ asymptotically fall into interval $(\frac{(8+4\sqrt{2})\pi h_p^2}{9+4\sqrt{2}}, 4(2+\sqrt{2}))$.

Lemma 2.3. ([39]) *For model problem (2.2), the optimal choice of modification order of MTFFD is $q = \frac{4}{3}$, the optimal relaxation parameter is $c_p = (4\pi^2)^{\frac{2}{3}}$. With $q = \frac{4}{3}$ and fixed c_p, then $\lambda_{j,k}(\hat{M}^{-1}A)$ asymptotically fall into interval $(\frac{2(2\pi h_p)^{\frac{2}{3}}}{1+2(2\pi h_p)^{\frac{2}{3}}}, 1)$.*

3. Fourier Analysis of the Composite Preconditioner

It is known that a judicious combination of different types of preconditioners may make the iterative solver more efficient and robust [11–13, 16–18, 23, 38, 44, 45]. Since the ILU preconditioner is efficient in removing the influence of the larger part of the spectrum [24] and MTFFD is efficient in dealing with the low part of the spectrum, we propose to combine the ILU(0) preconditioner with the MTFFD preconditioner in this paper. The multiplicative combination results in a composite preconditioner M_c defined by

$$M_c = (M_{ilu}^{-1} + \hat{M}^{-1} - \hat{M}^{-1}AM_{ilu}^{-1})^{-1}. \tag{3.1}$$

From (3.1), we have

$$I - M_c^{-1}A = (I - \hat{M}^{-1}A)(I - M_{ilu}^{-1}A). \tag{3.2}$$

Similar types of composite preconditioners have been discussed in [23, 29, 35, 36, 38, 44, 45]. Here we mention that the preconditioners based on alternative direction iterations [14, 41, 46] can also be classified as such kind of composite preconditioners.
From equality (3.2) it follows that

Theorem 3.1. *The (j,k)-th Fourier eigenvalues of $M_c^{-1}A$ is given by*

$$\begin{aligned}&\lambda_{j,k}(M_c^{-1}A)\\ =& 1 - \frac{4(\delta_h^{-2}-1)s_j^4 + \eta_h^2}{4(s_j^2+s_k^2)(1+\eta_h+4\hat{\kappa}_1 s_j^2) + 4(\delta_h^{-2}-1)s_j^4 + \eta_h^2}\frac{\cos(\theta_j - \phi_k)}{2(2+\sqrt{2})(s_j^2+s_k^2) + \cos(\theta_j - \phi_k)},\end{aligned}$$

where $s_j = \sin(\frac{\theta_j}{2})$, $s_k = \sin(\frac{\theta_k}{2})$.

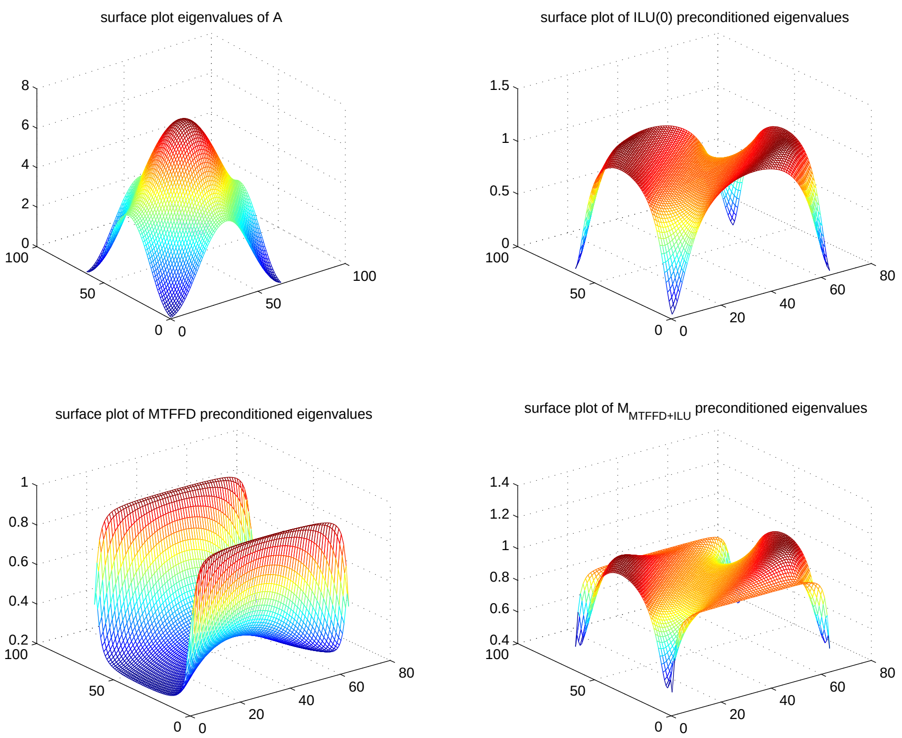

Fig. 3.1. The surface plot of Fourier eigenvalues, with $c = 4$ and $h_p = \frac{1}{64}$.

The proof is based on

$$1 - \lambda_{j,k}(M_c^{-1}A) = (1 - \lambda_{j,k}(\hat{M}^{-1}A))(1 - \lambda_{j,k}(M_{ilu}^{-1}A))$$

and simple arithmetic computation, so we omitted here.

Subsequently, we analyze condition number of the preconditioned matrix by M_c, which is defined by

$$\kappa(M_c^{-1}A) = \frac{\max_{j,k} \lambda_{j,k}(M_c^{-1}A)}{\min_{j,k} \lambda_{j,k}(M_c^{-1}A)}.$$

For convenience of discussions, we consider the continuous function defined by

$$f(\mu, \nu) = 1 - (1 - \mu)(1 - \nu) \tag{3.3}$$

with $\mu \in (\frac{2(2\pi h_p)^{\frac{2}{3}}}{1+2(2\pi h_p)^{\frac{2}{3}}}, 1)$ and $\nu \in (\frac{(8+4\sqrt{2})\pi h_p^2}{9+4\sqrt{2}}, 4(2+\sqrt{2}))$. Since

$$f'_\nu(\mu, \nu) = (1 - \mu) \geq 0,$$

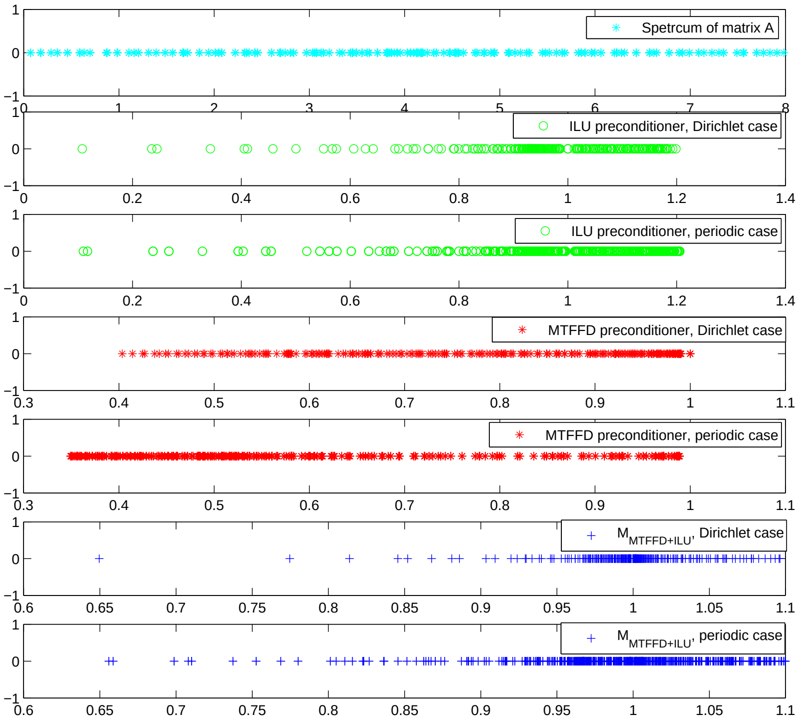

Fig. 3.2. Eigenvalue distribution comparison, with $h_p = \frac{1}{32}$.

so $f(\mu, \nu)$ is monotonously increase with respect to ν. Therefore,

$$\begin{aligned}
\max_{\mu,\nu} f(\mu,\nu) &= \max_{\mu}\{1-(1-\mu)(1-4(2+\sqrt{2}))\} \\
&= \max_{\mu}\{4(2+\sqrt{2})-(7+4\sqrt{2})\mu\} \\
&= 4(2+\sqrt{2})-(7+4\sqrt{2})\frac{2(2\pi h_p)^{\frac{2}{3}}}{1+2(2\pi h_p)^{\frac{2}{3}}} \\
&\approx 4(2+\sqrt{2}) = \mathcal{O}(1), \\
\min_{\mu,\nu} f(\mu,\nu) &= \min_{\mu}\{1-(1-\mu)(1-\frac{(8+4\sqrt{2})\pi h_p^2}{9+4\sqrt{2}})\} \\
&= \min_{\mu}\{\frac{(8+4\sqrt{2})\pi h_p^2}{9+4\sqrt{2}}+(1-\frac{(8+4\sqrt{2})\pi h_p^2}{9+4\sqrt{2}})\mu\} \\
&= \frac{(8+4\sqrt{2})\pi h_p^2}{9+4\sqrt{2}}+(1-\frac{(8+4\sqrt{2})\pi h_p^2}{9+4\sqrt{2}})\frac{2(2\pi h_p)^{\frac{2}{3}}}{1+2(2\pi h_p)^{\frac{2}{3}}} \\
&\approx \frac{2(2\pi h_p)^{\frac{2}{3}}}{1+2(2\pi h_p)^{\frac{2}{3}}} = \mathcal{O}(h_p^{\frac{2}{3}}).
\end{aligned}$$

Based on above analysis, we have

Theorem 3.2. *For composite preconditioner M_c, the condition number of preconditioned matrix $M_c^{-1}A$ is asymptotically bounded by $\mathcal{O}(h_p^{-\frac{2}{3}})$.*

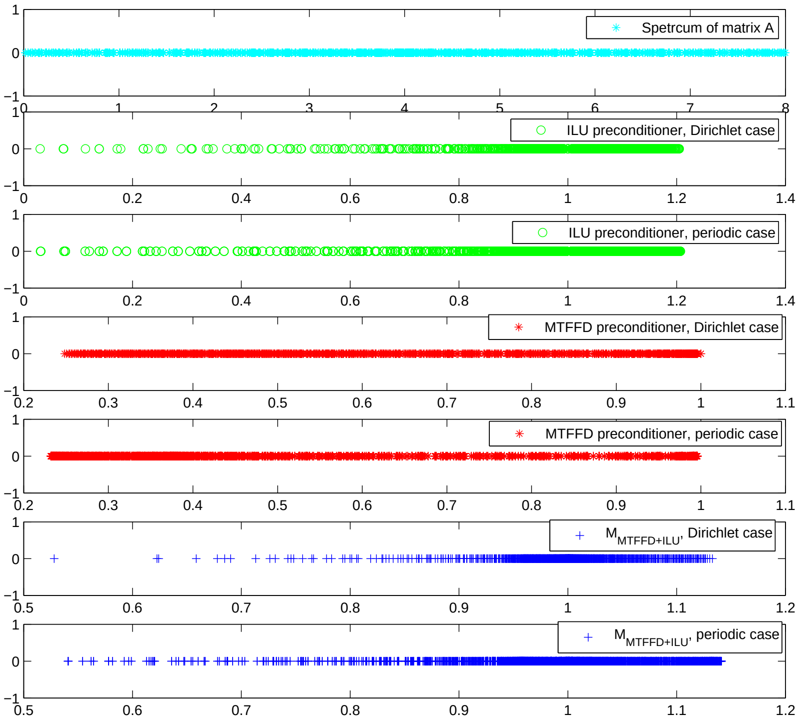

Fig. 3.3. Eigenvalue distribution comparison, with $h_p = \frac{1}{64}$.

In Fig. 3.1, surface plots of eigenvalues of several preconditioned matrices are displayed. From these figures, we can observe that surface of eigenvalues of $M_{ilu}^{-1}A$ and $M_{MTFFD}^{-1}A$ protrude in opposite directions, which implies that a judicious combination of two preconditioners is able to improve the eigenvalue distribution. The eigenvalue distribution of the coefficient matrix A, the preconditioned matrix by using ILU(0), MTFFD, and the composite preconditioner (3.2) are displayed in Figs. 3.2 - 3.3. From these two figures we can see that eigenvalue distribution of the Dirichlet case can be well predicted by the periodic case. The composite preconditioner is able to make eigenvalues better clustered around 1. This implies that the composite preconditioner should have better preconditioning effect.

Remarks: We note that the ILU preconditioners is originated for M matrix [34], and later generalized to H matrix and other more general matrices. The theoretical assurances of the stability may be problematical for general matrices. For example, the preconditioner need to consider the occurrence of small pivots, need to balance between fill-in and accuracy, need to consider the instability of triangular solver [6, 27, 43]. As a comparisons, the IQR and MIQR [7–9] use Givens rotations to overcome the drawback of producing non-orthogonal factor Q in similar approaches and are able to produce a orthogonal matrix Q, and nonsingular upper triangular matrix R for invertible matrix A. Which bypasses the stability problems and therefore numerically more robust. It would be interesting to analyze the properties of the IQR and MIQR preconditioner by Fourier analysis. Moreover, in future development of preconditioning techniques for block tridiagonal matrices, the idea of IQR and MIQR may also be adopted. We leave these work for future research.

Table 4.1: Test results for non-Homogeneous problems, symmetric matrices.

	TFFD		$M_{TFFD+ILU}$		MTFFD		$M_{MTFFD+ILU}$	
$1/h$	iter	*cpu*	*iter*	*cpu*	*iter*	*cpu*	*iter*	*cpu*
100	58	3.1	26	1.1	26	1.2	19	0.9
200	84	26.3	38	10.0	33	10.5	23	6.5
300	105	118.0	47	51.1	37	42.3	26	28.8
400	124	360.9	54	154.7	44	119.8	28	82.0

Table 4.2: Test results for advection-diffusion problems, nonsymmetric matrices.

	TFFD		$M_{TFFD+ILU}$		MTFFD		$M_{MTFFD+ILU}$	
$1/h$	iter	*cpu*	*iter*	*cpu*	*iter*	*cpu*	*iter*	*cpu*
100	58	2.4	27	1.3	26	1.1	19	0.7
200	84	22.4	39	9.3	33	9.5	23	5.7
300	105	116.8	47	52.2	38	41.7	26	29.5
400	124	336.2	53	148.4	43	122.8	28	83.9

4. Numerical Examples

In this section, we compare performance of the composite preconditioner with some other related preconditioners. The comparison is performed on several examples arising from the discretization of partial differential equations. All tests are run on an AMD Athlon Dual-Core processor using MATLAB 7.5 on a XP system. The machine precision is $eps = 2.22 \times 10^{-16}$.

We use preconditioned GMRES(m) [43] with $m = 30$ as the iterative solver. The algorithm is stopped when the relative norm satisfies

$$\frac{||b - Ax_k||_2}{||b||_2} \leq 10^{-12}.$$

Both the exact solution and the initial approximate solution are chosen randomly. The vector $\mathbf{e} = [1, \ldots, 1]^T$ is always selected as the filtering vector in the construction of the MTFFD preconditioner. In the following tables, *iter* denotes the number of iterations, *cpu* denotes the cpu time in seconds. We use "†" to denote that the method fails to converge within 200 iterations. We use $M_{TFFD+ILU}$ ($M_{MTFFD+ILU}$) to denote the composite preconditioner constructed by combining TFFD and ILU(0) (MTFFD and ILU(0)).

Example 1. We consider the boundary value problem

$$\begin{array}{ll} div(\mathbf{a}(x)u) - div(\kappa(x)\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega_D, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega_N, \end{array} \tag{4.1}$$

where $\Omega = [0,1]^k$ with $k = 2$, $\partial\Omega_N = \partial\Omega \backslash \partial\Omega_D$, $\partial\Omega_D = [0,1] \times \{0,1\}$. We consider five different cases. The matrices are symmetric in the first case and nonsymmetric in the other four cases. As these problems no longer have constant coefficients, we choose the additional term as $c\Lambda_i h^{\frac{4}{3}}$, where $\Lambda_i = diag(D_i)$, i.e., diagonal matrix of the ith diagonal block of the coefficient matrix. Selection of the parameter c is described in each of following cases.

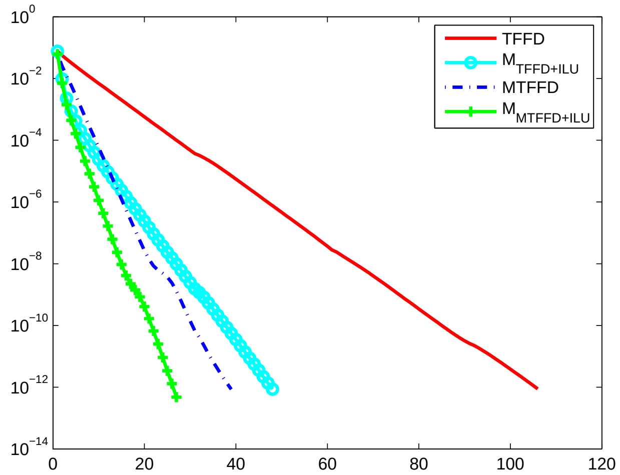

Fig. 4.1. Convergence behavior of preconditioned GMRES(30) with different preconditioners.

`Case I`: *Non-Homogeneous problems with large jumps in the coefficients in two dimensions:* The coefficient $\mathbf{a}(x)$ is zero. The tensor κ is isotropic and discontinuous. It jumps from the constant value 10^3 in the ring $\frac{1}{2\sqrt{2}} \leq |x-\mathbf{c}| \leq \frac{1}{2}$, $\mathbf{c} = (\frac{1}{2}, \frac{1}{2})^T$ to 1 outside. We tested uniform grids with $n \times n$ nodes, $n = 100, 200, 300, 400$. We set the parameter $c = 2.5$ for MTFFD, and $c = 0.8$ for $M_{MTFFD+ILU}$. Table 4.1 displays the results obtained by using the three preconditioners. For the case of $h = \frac{1}{300}$, the convergence curves are plotted in Fig. 4.1. We can see that $M_{MTFFD+ILU}$ leads to considerably faster convergence speed than that of $M_{TFFD+ILU}$.

`Case II`: *The advection-diffusion problem with a rotating velocity in two dimensions:* The tensor κ is the identity, and the velocity is $\mathbf{a}(x) = (2\pi(x_2 - 0.5), 2\pi(x_1 - 0.5))^T$. The uniform

Table 4.3: Test results for skyscraper problems, nonsymmetric matrices.

	TFFD		$M_{TFFD+ILU}$		MTFFD		$M_{MTFFD+ILU}$	
$1/h$	iter	*cpu*	*iter*	*cpu*	*iter*	*cpu*	*iter*	*cpu*
100	†	†	26	1.8	†	†	21	0.8
200	†	†	40	11.7	†	†	33	8.9
300	†	†	48	56.6	†	†	39	43.9
400	†	†	60	172.1	†	†	54	146.2

Table 4.4: Test results for convective skyscraper problems, nonsymmetric matrices.

	TFFD		$M_{TFFD+ILU}$		MTFFD		$M_{MTFFD+ILU}$	
$1/h$	iter	*cpu*	*iter*	*cpu*	*iter*	*cpu*	*iter*	*cpu*
100	†	†	19	0.73	68	2.7	18	0.7
200	†	†	26	8.1	97	26.8	25	7.4
300	†	†	28	31.2	85	91.8	27	28.3
400	†	†	40	106.9	129	354.0	38	102.2

Table 4.5: Test results for anisotropic layers problems, nonsymmetric matrix.

	TFFD		$M_{TFFD+ILU}$		MTFFD		$M_{MTFFD+ILU}$	
$1/h$	iter	*cpu*	*iter*	*cpu*	*iter*	*cpu*	*iter*	*cpu*
100	70	3.1	17	0.7	29	1.4	16	0.6
200	103	39.7	29	11.6	41	13.8	25	9.3
300	129	176.5	41	62.8	44	58.8	31	44.2
400	152	458.6	50	142.7	45	128.3	36	105.0

grid has $n \times n$ nodes. Four cases with $n = 100, 200, 300, 400$ are tested. The diagonal elements of A are close to 4. We use the same parameter c as the `case I`. The numerical results are illustrated in Table 4.2. From this table, we can see that the the composite preconditioner proposed in this paper is considerably better than other preconditioners.

`Case III`: *Skyscraper problems:* The tensor κ is isotropic and discontinuous. The domain contains many zones of high permeability which are isolated from each other. Let $[x]$ denote the integer value of x. In 2D, we have

$$\kappa(x) = \begin{cases} 10^3 * ([10 * x_2] + 1), & \text{if } [10 * x_i] = 0 \; mod(2) \;, \; i = 1, 2, \\ 1, & \text{otherwise.} \end{cases}$$

The diagonal elements of A jump between 4 and 36000. The parameter c is chosen as 10 for MTFFD and 0.001 for $M_{MTFFD+ILU}$. The numerical results are shown in Table 4.3.

`Case IV`: *Convective skyscraper problems:* The same as skyscraper problems, except that velocity field is changed to be $\mathbf{a}(x) = (1000, 1000)^T$. The diagonal elements of A jump between 24 and 36020. The parameter c is chosen as 1 for MTFFD and 0.001 for $M_{MTFFD+ILU}$. Numerical results are reported in Table 4.4.

The skyscraper problems and convective skyscraper problems are quite challenge for conventional preconditioners [2]. From Table 4.3 and Table 4.4 we can observe that TFFD preconditioned GMRES(30) failed to solve both of the problems and MTFFD failed in solving the skyscraper problems. The $M_{MTFFD+ILU}$ preconditioner leads to better performance than that of $M_{TFFD+ILU}$.

`Case V`: *Anisotropic layers:* The domain is made of 10 anisotropic layers with jumps of up to four orders of magnitude and an anisotropy ratio of up to 10^3 in each layer. The diagonal elements jump between 22 and 220000. The parameter c is chosen as 0.4 for MTFFD and 0.06 for $M_{MTFFD+ILU}$. Test results are displayed in Table 4.5. In this case, we can also notice the better performance of $M_{MTFFD+ILU}$.

5. Conclusions

In this paper, a composite preconditioner based on the classical ILU(0) and modified tangential frequency filtering preconditioner is analyzed. Theoretical analysis indicates that condition number of the preconditioned matrix by the composite preconditioner is $\mathcal{O}(h^{-\frac{2}{3}})$. Numerical results show that performance of the composite preconditioner is better than other related preconditioners on several difficult problems arising from the discretization of PDEs with discontinuous coefficients.

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References

[1] Y. Achdou and F. Nataf, An iterated tangential filtering decomposition, *Numer. Linear Algebra Appl.*, **10** (2003), 511-539.

[2] Y. Achdou and F. Nataf, Low frequency tangential filtering decomposition, *Numer. Linear Algebra Appl.*, **14** (2007), 129-147

[3] J. R. Appleyard and I. M. Cheshire, Nested factorization, SPE 12264, presented at *the seventh SPE symposium on reservoir simulation*, San Francisco, 1983.

[4] O. Axelsson and H. Lu, On the eigenvalue estimates for block incomplete factorization methods, *SIAM. J. Matrix Anal. Appl.*, **16** (1995), 1074-1085.

[5] O. Axelsson and G. Lindskog, On the eigenvalue distribution of a class of preconditioning methods, *Numer. Math.*, **48** (1986), 479-498.

[6] O. Axelsson, Iterative Solution Methods, Cambridge University Press, New York, 1994.

[7] Z.-Z. Bai, I.S. Duff and A.J. Wathen, A class of incomplete orthogonal factorization methods. I: methods and theories, *BIT.*, **41** (2001), 53-70.

[8] Z.-Z. Bai, I.S. Duff and J.-F. Yin, Numerical study on incomplete orthogonal factorization preconditioners, *J. Comput. Appl. Math.*, **226** (2009), 22-41.

[9] Z.-Z. Bai and J.-F. Yin, Modified incomplete orthogonal factorization methods using Givens rotations, *Computing.*, **86** (2009), 53-69.

[10] Z.-Z. Bai, J.-F. Yin and Y.-F. Su, A shift-splitting preconditioner for non-Hermitian positive definite matrices, *J. Comput. Math.*, **24** (2006), 539-552.

[11] Z.-Z. Bai and G.-Q. Li, Restrictively preconditioned conjugate gradient methods for systems of linear equations, *IMA J. Numer. Anal.*, **23** (2003), 561-580.

[12] Z.-Z. Bai and Z.-Q. Wang, Restrictive preconditioner for conjugate gradient methods for symmetric positive definite linear system, *J. Comput. Appl. Math.*, **187** (2006), 202-226.

[13] Z.-Z. Bai, G.-Q. Li and L.-Z. Lu, Combinative preconditioners of modified imcomplete Cholesky factorization and Sherman-Morrison-Woodbury update for self-adjoint elliptic Dirichlet-periodic boundary value problems, *J. Comput. Math.*, **22** (2004), 833-856.

[14] Z.-Z. Bai, G.H. Golub and M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *SIAM J. Matrix. Anal. Appl.*, **24** (2003), 603-626.

[15] Z.-Z. Bai, G.H. Golub, L.-Z. Lu and J.-F. Yin, Block triangular and skew-Hermitian splitting methods for positive definite linear systems, *SIAM J. Sci. Comput.*, **26** (2005), 844-863.

[16] Z.-Z. Bai, A class of parallel decomposition-type relaxation methods for large sparse systems of linear equations, *Linear Algebra Appl.*, **282** (1998), 1-24.

[17] Z.-Z. Bai and R. Nabben, Some properties of the block matrices in the parallel decomposition-type relaxation methods, *Appl. Numer. Math.*, **29** (1999), 167-170.

[18] Z.-Z. Bai, A class of new parallel hybrid algebraic multilevel iterations, *J. Comput. Math.*, **19** (2001), 651-672.

[19] M. Benzi, M.K. Ng, Q. Niu and Z. Wang, A relaxed dimensional factorization preconditioner for the incompressible Navier-Stokes equations, *J. Comput. Phys.*, **230** (2011), 6185-6202.

[20] A. Brandt, Multi-level adaptive solutions to boundary-value problems, *Math. Comput.*, **31** (1977), 333-390.

[21] A. Buzdin, Tangential decomposition, *Computing.*, **61** (1998) 257-276.
[22] A. Buzdin and G. Wittum, Two-frequency decomposition, *Numer. Math.*, **97** (2004), 269-295.
[23] B. Carpentieri, L. Giraud and S. Gratton, Additive and multiplicative composite spectral preconditioning for general linear systems, *SIAM J. Sci. Comput.*, **29** (2007), 1593-1612.
[24] T.F. Chan and H.C. Elman, Fourier analysis of iterative methods for elliptic problems, *SIAM Review.*, **31** (1989), 20-49.
[25] T.F. Chan, Fourier analysis of relaxed incomplete factorization preconditioners, *SIAM J. Sci. Comput.*, **12** (1991), 668-690.
[26] T.F. Chan and J.M. Donato, Fourier analysis of incomplete factorization preconditioners for three-dimensional anisotropic problems, *SIAM J. Sci. Statist. Comput.*, **13** (1992), 319-338.
[27] K. Chen, Matrix Preconditioning Techniques and Applications, Cambridge University Press, 2005.
[28] P. Concus, G.H. Golub and G. Meurant, Block preconditioning for the conjugate gradient method, *SIAM J. Sci. Statist. Comput.*, **6** (1985), 220-252.
[29] I. S. Duff, L. Giraud, J. Langou and E. Martin, Using spectral low rank preconditioners for large electromagnetic calculations, *Int. J. Numer. Methods Eng.*, **62** (2005), 416-434.
[30] I. Gustafsson, A class of first order factorization methods, *BIT.*, **18** (1978), 142-156.
[31] W. Hackbusch, Iterative Solution of Large Sparse Systems of Equations, Springer, New York, 1994.
[32] L. Grigori, F. Nataf and Q. Niu, Two sides tangential filtering decomposition, *J. Comput. Appl. Math.*, **235** (2010), 2647-2661.
[33] R.J. LeVeque and L.N. Trefethen, Fourier anlaysis of the SOR iterations, *IMA J. Numer. Anal.*, **8** (1988), 273-279.
[34] J.A. Meijerink and H.A. van der Vorst, An iterative solution method for linear systems of which the coefficient matrix is symmetric M-matrix, *Math. Comput.*, **137** (1977), 148-162
[35] C. Mense and R. Nabben, On algebraic multilevel methods for non-symmetric systems-comparison results, *Linear Algebra Appl.*, **429** (2008), 2567-2588.
[36] C. Mense and R. Nabben, On algebraic multilevel methods for non-symmetric systems-convergence results, *ETNA.*, **30** (2008), 323-345.
[37] G. Meurant, Computer Solution of Large Linear Systems, North-Holland, 1999.
[38] R. Nabben and C. Vuik, A comparison of abstract versions of deflation, balancing and additive coarse grid correction preconditioners, *Numer. Linear Algebra Appl.*, **15** (2008), 355-372.
[39] Q. Niu, L. Grigori, P. Kumar and F. Nataf, Modified tangential frequency filtering decomposition and its Fourier analysis, *Numer. Math.*, **116** (2010), 123-148.
[40] K. Otto, Analysis of preconditioners for hyperbolic partial differential equations, *SIAM J. Numer. Anal.*, **33** (1996), 2131-2165.
[41] D. Peaceman and H. Rachfor, The numerical solution of elliptic and parabolic differential equations, *SIAM J. Numer. Anal.*, **3** (1955), 28-41.
[42] R.D. Richtmyer and K.W. Morton, *Difference methods for initial-value problems*, Interscience, New York, 1967.
[43] Y. Saad, Iterative methods for sparse linear systems, PWS Publishing Company, Boston, MA, 1996.
[44] J.M. Tang, R. Nabben, C. Vuik and Y.A. Erlangga, comparison of composite preconditioners derived from deflation, domain decomposition and multigrid methods, *J. Scient. Comput.*, **39** (2009), 340-370.
[45] J.M. Tang, S.P. MacLachlan, R. Nabben and C. Vuik, A comparison of composite preconditioners based on multigrid and deflation, *SIAM. J. Matrix Anal. and Appl.*, **31** (2010), 1715-1739.
[46] R.S. Varga, Matrix Iterative Analysis, Springer-Verlag, Berlin Heigelberg, 2000.
[47] P.S. Vassilevski, Multilevel Block Factorization Preconditioners: Matrix-Based Analysis And Algorithms For Solving Finite Element Equations, Springer, 2008.
[48] C. Wagner, Tangential frequency filtering decompositions for symmetric matrices, *Numer. Math.*,

78 (1997), 119-142.

[49] C. Wagner, Tangential frequency filtering decompositions for unsymmetric matrices, *Numer. Math.*, **78** (1997), 143-163.

[50] C. Wagner and G. Wittum, Adaptive filtering, *Numer. Math.*, **78** (1997), 305-382.

[51] G. Wittum, Filternde Zerlegungen, Schnelle Löser für große Gleichungssysteme, Teubner Skripten zur Numerik, Band 1, Teubner-Verlag, Stuttgart, 1992.

[52] R. Wienands and W. Joppich, Practical Fourier Analysis for Multigrid Methods, CRC press, 2005.