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On a New Class of Projectively Flat Finsler Metrics

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Abstract. A class of Finsler metrics with three parameters is constructed. Moreover, a sufficient and necessary condition for this Finsler metrics to be projectively flat was obtained.

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1 Introduction

Finsler geometry is more colorful than Riemannian geometry because there are several non-Riemannian quantities on a Finsler manifold besides the Riemannian quantities. One of the important problems in Finsler geometry is to study and characterize the projectively flat metrics on an open domain $U \subset \mathbb{R}^n$. Projectively flat metrics on U are Finsler metrics whose geodesics are straight lines. This is the Hilbert's 4th problem in the regular case [5]. In 1903, Hamel [4] found a system of partial differential equations

$$F_{x^k y^l} y^k = F_{x^l}, \tag{1.1}$$

which can characterize the projectively flat metrics $F = F(x,y)$ on an open subset $U \subset \mathbb{R}^n$. And we know that Riemannian metrics form a special and important class in Finsler geometry. Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature [10]. The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. Besides, every locally projectively flat Finsler metric F on a manifold M is of scalar flag curvature, i.e., the flag curvature $K = K(x,y)$ is a scalar function on $TM\backslash\{0\}$. Many projectively flat Finsler metrics with constant flag curvature are obtained in [8], [1], [12], [2]. Besides, there are a lot of locally projectively flat Finsler metrics which are not of constant flag curvature [9], [13], [6]. Thus, the Beltrami's theorem is no longer true for Finsler metrics.

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Recently, Huang and Mo discussed a class of interesting Finsler metrics [13], [6] satisfying

$$F(Ax,Ay)=F(x,y), \tag{1.2}$$

for all $A\in O(n)$. A Finsler metric F is said to be spherically symmetric if F satisfies (1.2) for all $A\in O(n)$. Besides, it was pointed out in [7] that a Finsler metric F on $\mathbb{B}^n(r)$ is a spherically symmetric if and only if there is a function $\phi:[0,r)\times\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$F(x,y)=|y|\phi\left(|x|,\frac{\langle x,y\rangle}{|y|}\right), \tag{1.3}$$

where$(x,y)\in T\mathbb{R}^n(r)\backslash\{0\}$.

In this paper, we construct a new class of Finsler metrics with three parameters and obtain the formula of the flag curvature of this kind of metrics.

Let ζ be an arbitrary constant and $\Omega=\mathbb{B}^n(r)\subset\mathbb{R}^n$ where $r=\frac{1}{\sqrt{-\zeta}}$ if $\zeta<0$ and $r=+\infty$ if $\zeta\geqslant 0$, $|\cdot|$ and $\langle,\rangle$ be the standard Euclidean norm and inner product in $\mathbb{R}^n$, respectively.

Define $F:T\Omega\to[0,+\infty)$ by

$$F=\frac{\sqrt{\kappa^2\langle x,y\rangle^2+\epsilon|y|^2(1+\zeta|x|^2)}}{1+\zeta|x|^2}+\frac{\kappa\langle x,y\rangle}{(1+\zeta|x|^2)^{\frac{3}{2}}}, \tag{1.4}$$

where ϵ is an arbitrary positive constant, κ is an arbitrary constant.

As a natural prolongation, we obtain the following results

Theorem 1.1. *Let $F:T\Omega\to[0,+\infty)$ be a function given by* (1.4)*. Then, it has the following properties.*

(1) F is a Finsler metric.

(2) F is projectively flat Finsler metric if and only if $\kappa^2+\epsilon\zeta=0$.

(3) When $\kappa^2+\epsilon\zeta=0$, the flag curvature of the Finsler metrics (1.4) *is given by*

$$K=\frac{\kappa^2}{\epsilon^2F^2}\left[\frac{\Delta\kappa\langle x,y\rangle}{F(1+\zeta|x|^2)^{\frac{7}{2}}}-\frac{\Delta}{(1+\zeta|x|^2)^2}-\frac{\Delta^2+6\Delta\kappa^2\langle x,y\rangle^2+6\Delta^{\frac{3}{2}}\kappa\langle x,y\rangle(1+\zeta|x|^2)^{\frac{1}{2}}}{4F^2(1+\zeta|x|^2)^5}\right],$$

where $\Delta=\epsilon|y|^2(1+\zeta|x|^2)+\kappa^2\langle x,y\rangle^2$.

2 Preliminaries

A Minkowski norm $\Psi(y)$ on a vector space V is a C^∞ function on $V\backslash\{0\}$ with the following properties:

(1) $\Psi(y)\geqslant 0$ and $\Psi(y)=0$ if and only if $y=0$;

(2) $\Psi(y)$ is positively homogeneous function of degree one, i.e., $\Psi(ty)=t\Psi(y), t\geqslant 0$;

(3) $\Psi(y)$ is strongly convex, i.e., for any $y\neq 0$, the matrix $g_{ij}(x,y):=\frac{1}{2}[F^2]_{y^iy^j}(x,y)$ is positive definite.

A Finsler metrict F on a manifold M is C^∞ function on $TM\backslash\{0\}$ such that $F_x:=F|_{T_xM}$ is a Minkowski norm on T_xM for any $x\in M$. The fundamental tensor $g_{ij}(x,y):=\frac{1}{2}[F^2]_{y^iy^j}(x,y)$ is positive definite. If $g_{ij}(x,y)=g_{ij}(x)$, F is a Riemannian metric. If $g_{ij}(x,y)=g_{ij}(y)$, F is a locally Minkowski metric. If all geodesics are straight lines, F is projectively flat [3], [11], [14]. This is equivalent to $G^i=P(x,y)y^i$ are geodesic coefficients of F, and G^i are given by

$$G^i=\frac{g^{il}}{4}\{[F^2]_{x^my^l}y^m-[F^2]_{x^l}\}.$$

For each tangent plane $\Pi\subset T_xM$ and $y\in\Pi$, the flag curvature of (Π,y) is defined by

$$K(\Pi,y)=\frac{g_{im}R^i_ku^ku^m}{F^2g_{ij}u^iu^j-[g_{ij}y^iu^j]^2},$$

where Π= span$\{y,u\}$, and

$$R^i_k=2\frac{\partial G^i}{\partial x^k}-y^i\frac{\partial^2G^i}{\partial y^i\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

We need the following lemmas for later use.

Lemma 2.1. ([15]) *Let M be an $n-$dimensional mainfold. $F=\alpha\phi(b,\frac{\beta}{\alpha})$ is a Finsler metric on M for any Riemannian metric α and 1-form β with$\|\beta\|_\alpha<b_0$ if and only if $\phi=\phi(b,s)$ is a positive C^∞ function satisfying*

$$\phi-s\phi_2>0,\phi-s\phi_2+(b^2-s^2)\phi_{22}>0, \tag{2.1}$$

when $n\geqslant 3$ or

$$\phi-s\phi_2+(b^2-s^2)\phi_{22}>0,$$

when $n=2$, where s and b are arbitrary numbers with $|s|\leqslant b<b_0$. ϕ_2 means derivation of ϕ with respect to the second variable s.

Lemma 2.2. ([7]) *Let $F=|y|\phi(|x|,\frac{\langle x,y\rangle}{|y|})$ be a spherically symmetric Finlser metric on an $\mathbb{B}^n(r)$. Then $F=F(x,y)$ is projectively flat if and only if $\phi=\phi(b,s)$ satisfies*

$$s\phi_{bs}+b\phi_{ss}-\phi_b=0, \tag{2.2}$$

where $b=\|\beta\|_\alpha, s=\frac{\beta}{\alpha}, \phi=\phi(|x|,\frac{\langle x,y\rangle}{|y|})$. ϕ_b means derivation of ϕ with respect to the first variable b.

Lemma 2.3. ([4]) *Let F be a Finlser metric on an open domain $U\subset\mathbb{R}^n$, F is projectively flat on this domain if and only if*

$$F_{x^k y^l}y^k = F_{x^l}.$$

In this case, the flag curvature K of F is given by

$$K=\frac{P^2-P_{x^k}y^k}{F^2},$$

where the projective factor can be expressed as

$$P=\frac{F_{x^m}y^m}{2F}.$$

3 Proof of Theorem 1.1

Proof. (1) Firstly, we prove that F is a Finsler metric.

By (1.4), we have

$$F=\frac{\sqrt{\kappa^2\langle x,y\rangle^2+\epsilon|y|^2(1+\zeta|x|^2)}}{1+\zeta|x|^2}+\frac{\kappa\langle x,y\rangle}{(1+\zeta|x|^2)^{\frac{3}{2}}}.$$

Let $\alpha=|y|$, $\beta=\langle x,y\rangle$, $s=\frac{\langle x,y\rangle}{|y|}$, $b=\|\beta\|_\alpha=|x|$, so F can be expressed as

$$\begin{aligned}F&=|y|\left(\frac{\sqrt{\epsilon(1+\zeta|x|^2)+\frac{\kappa^2\langle x,y\rangle^2}{|y|^2}}}{1+\zeta|x|^2}+\frac{\frac{\kappa\langle x,y\rangle}{|y|}}{(1+\zeta|x|^2)^{\frac{3}{2}}}\right)\\&=|y|\left(\frac{\sqrt{\epsilon(1+\zeta b^2)+\kappa^2s^2}}{1+\zeta b^2}+\frac{\kappa s}{(1+\zeta b^2)^{\frac{3}{2}}}\right)\\&=\alpha\phi(b,s).\end{aligned}\tag{3.1}$$

We set

$$\Upsilon=\kappa^2s^2+\epsilon(1+\zeta b^2),\tag{3.2}$$

where Υ is non-negative.

After substituting (3.2) into (3.1), we have

$$\begin{aligned}\phi&=\phi(b,s)\\&=\frac{\sqrt{\Upsilon}}{1+\zeta b^2}+\frac{\kappa s}{(1+\zeta b^2)^{\frac{3}{2}}}.\end{aligned}\tag{3.3}$$

Differentiating ϕ with respect to s, we have

$$\phi_s=\frac{\Upsilon^{-\frac{1}{2}}\kappa^2 s}{1+\zeta b^2}+\frac{\kappa}{(1+\zeta b^2)^{\frac{3}{2}}},\tag{3.4}$$

it follows that

$$\phi_{ss}=\frac{\Upsilon^{-\frac{1}{2}}\kappa^2-\Upsilon^{-\frac{3}{2}}\kappa^4 s^2}{1+\zeta b^2}. \tag{3.5}$$

Using (3.3) and (3.4) , we have

$$\begin{aligned}\phi-s\phi_s&=\frac{\sqrt{\Upsilon}}{1+\zeta b^2}+\frac{\kappa s}{(1+\zeta b^2)^{\frac{3}{2}}}-s\left[\frac{\Upsilon^{-\frac{1}{2}}\kappa^2 s}{1+\zeta b^2}+\frac{\kappa}{(1+\zeta b^2)^{\frac{3}{2}}}\right]\\&=\epsilon\Upsilon^{-\frac{1}{2}}>0.\end{aligned} \tag{3.6}$$

Combing (3.5) and (3.6) gives

$$\begin{aligned}\phi-s\phi_s+(b^2-s^2)\phi_{ss}&=\epsilon\Upsilon^{-\frac{1}{2}}+(b^2-s^2)\frac{\Upsilon^{-\frac{1}{2}}\kappa^2-\Upsilon^{-\frac{3}{2}}\kappa^4 s^2}{1+\zeta b^2}\\&=\epsilon\Upsilon^{-\frac{1}{2}}+(b^2-s^2)\epsilon\kappa^2\Upsilon^{-\frac{3}{2}}>0.\end{aligned} \tag{3.7}$$

Then, according to Lemma 2.1, we know F is a Finsler metric.

(2) In this part, we will prove that F is projectively flat Finsler metric if and only if $\kappa^2+\epsilon\zeta=0$.

From (3.3), we have

$$\phi_b=\frac{\Upsilon^{-\frac{1}{2}}\epsilon\zeta(1+\zeta b^2)-2\sqrt{\Upsilon\zeta b}}{1+\zeta b^2}-\frac{3\kappa s\zeta b}{(1+\zeta b^2)^{\frac{3}{2}}}, \tag{3.8}$$

$$\phi_{bs}=\frac{-\Upsilon^{-\frac{3}{2}}\epsilon\zeta b\kappa^2 s}{(1+\zeta b^2)}-\frac{2\Upsilon^{-\frac{1}{2}}\zeta b\kappa^2 s}{(1+\zeta b^2)^2}-\frac{3\kappa\zeta b}{(1+\zeta b^2)^{\frac{5}{2}}}. \tag{3.9}$$

Using (3.5), (3.8) and (3.9), we have

$$\begin{aligned}s\phi_{bs}+b\phi_{ss}-\phi_b=&\,s\left[\frac{-\Upsilon^{-\frac{3}{2}}\epsilon\zeta b\kappa^2 s}{(1+\zeta b^2)}-\frac{2\Upsilon^{-\frac{1}{2}}\zeta b\kappa^2 s}{(1+\zeta b^2)^2}-\frac{3\kappa\zeta b}{(1+\zeta b^2)^{\frac{5}{2}}}\right]+b\left[\frac{\Upsilon^{-\frac{1}{2}}\kappa^2-\Upsilon^{-\frac{3}{2}}\kappa^4 s^2}{1+\zeta b^2}\right]\\&-\left[\frac{\Upsilon^{-\frac{1}{2}}\epsilon\zeta(1+\zeta b^2)-2\sqrt{\Upsilon\zeta b}}{1+\zeta b^2}-\frac{3\kappa s\zeta b}{(1+\zeta b^2)^{\frac{3}{2}}}\right]\\=&\,\Upsilon^{-\frac{3}{2}}b\epsilon(\kappa^2+\epsilon\zeta).\end{aligned}$$

It's easy to see that $s\phi_{bs}+b\phi_{ss}-\phi_b=0$ is equivalent to $\kappa^2+\epsilon\zeta=0$. Then from Lemma 2.2, we know that F is projectively flat Finsler metric if and only if $\kappa^2+\epsilon\zeta=0$.

(3) From Lemma 2.3, we know that F is projectively flat Finsler metric and its projective factor and flag curvature are given by

$$P=\frac{F_{x^m}y^m}{2F},\quad K=\frac{P^2-P_{x^k}y^k}{F^2}.$$

From (1.4), we obtain

$$F_{x^k}y^k=\frac{\Delta^{-\frac{1}{2}(\kappa^2+\epsilon\zeta)|y|^2\langle x,y\rangle}}{1+\zeta|x|^2}-\frac{2\sqrt{\Delta}\zeta\langle x,y\rangle}{(1+\zeta|x|^2)^2}+\frac{\kappa|y|^2(1+\zeta|x|^2)-3\kappa\zeta\langle x,y\rangle^2}{(\langle x,y\rangle)^{\frac{5}{2}}},$$

when $\kappa^2+\epsilon\zeta=0$, F can be expressed as the following form.

$$F_{x^k}y^k=\frac{\frac{\kappa}{\epsilon}\Delta+2\frac{\kappa^3\langle x,y\rangle^2}{\epsilon}}{(1+\zeta|x|^2)^{\frac{5}{2}}}+\frac{2\sqrt{\Delta}\frac{\kappa^2}{\epsilon}\langle x,y\rangle}{(1+\zeta|x|^2)^2}. \tag{3.10}$$

By (3.10), we get

$$\begin{aligned} P&=\frac{F_{x^k}y^k}{2F}\\ &=\frac{1}{2F}\left[\frac{\frac{\kappa}{\epsilon}\Delta+2\frac{\kappa^3\langle x,y\rangle^2}{\epsilon}}{(1+\zeta|x|^2)^{\frac{5}{2}}}+\frac{2\sqrt{\Delta}\frac{\kappa^2}{\epsilon}\langle x,y\rangle}{(1+\zeta|x|^2)^2}\right]. \end{aligned} \tag{3.11}$$

Using (3.11), we have

$$P^2=\frac{1}{4F^2}\left[\frac{\frac{\kappa}{\epsilon}\Delta+2\frac{\kappa^3\langle x,y\rangle^2}{\epsilon}}{(1+\zeta|x|^2)^{\frac{5}{2}}}+\frac{2\sqrt{\Delta}\frac{\kappa^2}{\epsilon}\langle x,y\rangle}{(1+\zeta|x|^2)^2}\right]^2, \tag{3.12}$$

$$P_{x^k}y^k=\frac{\kappa^2}{\epsilon^2}\left[\frac{\Delta+\kappa^2\langle x,y\rangle^2}{(1+\zeta|x|^2)^2}-\frac{\Delta(\Delta-3\kappa^2\langle x,y\rangle^2-3\sqrt{\Delta}\kappa\langle x,y\rangle(1+\zeta|x|^2)^{\frac{1}{2}})}{2F^2(1+\zeta|x|^2)^5}\right]. \tag{3.13}$$

Using(3.12) and (3.13), we get

$$\begin{aligned} K&=\frac{P^2-P_{x^k}y^k}{F^2}\\ &=\frac{1}{4F^4}\left[\frac{\frac{\kappa}{\epsilon}\Delta+2\frac{\kappa^3\langle x,y\rangle^2}{\epsilon}}{(1+\zeta|x|^2)^{\frac{5}{2}}}+\frac{2\sqrt{\Delta}\frac{\kappa^2}{\epsilon}\langle x,y\rangle}{(1+\zeta|x|^2)^2}\right]^2\\ &\quad-\frac{\kappa^2}{F^2\epsilon^2}\left[\frac{\Delta+\kappa^2\langle x,y\rangle^2}{(1+\zeta|x|^2)^2}-\frac{\Delta(\Delta-3\kappa^2\langle x,y\rangle^2-3\sqrt{\Delta}\kappa\langle x,y\rangle(1+\zeta|x|^2)^{\frac{1}{2}})}{2F^2(1+\zeta|x|^2)^5}\right]\\ &=\frac{\kappa^2}{\epsilon^2F^2}\left[\frac{\Delta\kappa\langle x,y\rangle}{F(1+\zeta|x|^2)^{\frac{7}{2}}}-\frac{\Delta}{(1+\zeta|x|^2)^2}-\frac{\Delta^2+6\Delta\kappa^2\langle x,y\rangle^2+6\Delta^{\frac{3}{2}}\kappa\langle x,y\rangle(1+\zeta|x|^2)^{\frac{1}{2}}}{4F^2(1+\zeta|x|^2)^5}\right], \end{aligned}$$

where $\Delta=\epsilon|y|^2(1+\zeta|x|^2)+\kappa^2\langle x,y\rangle^2$.

□

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