J. Math. Study
doi: 10.4208/jms.v49n1.16.09

Vol. **49**, No. 1, pp. 72-81
March 2016

Spectra of Corona Based on the Total Graph

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Received 21 September 2015; Accepted 15 Febuary 2016

Abstract. For two simple connected graphs G_1 and G_2, we introduce a new graph operation called the total corona $G_1 \circledast G_2$ on G_1 and G_2 involving the total graph of G_1. Subsequently, the adjacency (respectively, Laplacian and signless Laplacian) spectra of $G_1 \circledast G_2$ are determined in terms of these of a regular graph G_1 and an arbitrary graph G_2. As applications, we construct infinitely many pairs of adjacency (respectively, Laplacian and signless Laplacian) cospectral graphs. Besides we also compute the number of spanning trees of $G_1 \circledast G_2$.

AMS subject classifications: 05C50, 05C90

Key words: Adjacency matrix, Laplacian matrix, signless Laplacian matrix, spectrum, total corona.

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1 Introduction

In this paper, all graphs considered are finite, simple connected graphs. Let $G=(V,E)$ be a graph with vertex set $V=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)$. The adjacency matrix of G is an $n\times n$ matrix whose (i,j)-entry is 1 if v_i and v_j are adjacent in G and 0 otherwise, denoted by $A(G)$. The degree of v_i in G is denoted by $d_i=d_G(v_i)$. Let $D(G)$ be the diagonal degree matrix of G which diagonal entries are $d_1,d_2,\ldots,d_n$. The Laplacian matrix $L(G)$ of G is defined as $D(G)-A(G)$. The signless Laplacian matrix of G is defined as $Q(G)=D(G)+A(G)$. For an $n\times n$ matrix M associated to G, the characteristic polynomial $\det(xI_n-M)$ of M is called the M-characteristic polynomial of G and is denoted by $\phi(M;x)$. I_n denotes the identity matrix. The roots of $\phi(M;x)$ are called the eigenvalues of matrix M. The set of all eigenvalues is called the spectrum of matrix M or graph G. In particular, if M is the adjacency matrix $A(G)$ of G, then the A-spectrum of G is denoted by

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$\sigma(A(G))=(\lambda_1(G),\lambda_2(G),\ldots,\lambda_n(G))$, where $\lambda_1(G)\geq\lambda_2(G)\geq\cdots\geq\lambda_n(G)$ are the eigenvalues of $A(G)$. If M is the Laplacian matrix $L(G)$ of G, then the L-spectrum of G is denoted by $\sigma(L(G))=(\mu_1(G),\mu_2(G),\ldots,\mu_n(G))$, where $\mu_1(G)\leq\mu_2(G)\leq\cdots\leq\mu_n(G)$ are the eigenvalues of $L(G)$. If M is the signless Laplacian matrix $Q(G)$ of G, then the Q-spectrum of G is denoted by $\sigma(Q(G))=(\nu_1(G),\nu_2(G),\ldots,\nu_n(G))$, where $\nu_1(G)\leq\nu_2(G)\leq\cdots\leq\nu_n(G)$ are the eigenvalues of $Q(G)$. For more review about the A-spectrum, L-spectrum and Q-spectrum of G, readers may refer to [4–7] and the references therein.

It is of interest to study some spectral properties of certain composite operations between two graphs such as the Cartesian product, the Kronecker product, the corona, the edge corona, the neighbourhood corona, the subdivision-vertex neighbourhood corona, the subdivision-edge neighbourhood corona. For example, the A-spectra, L-spectra and Q-spectra of the (edge) corona of two graphs can be expressed by these of the two factor graphs [1–3, 8, 9, 11, 13–17]. Recently, the R-vertex (neighbourhood) corona and R-edge (neighbourhood) corona of two graphs have been defined in [12] and the A-spectra, L-spectra and Q-spectra of these four operations of two graphs were computed in [12].

Motivated by the works above, we define a new graph operation based on the total graph as follows. The total graph [6] of a graph G, denoted by $T(G)$, is that graph whose set of vertices is the union of the set of vertices and the set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.

Definition 1.1. The total corona of G_1 and G_2, denoted by $G_1 \circledast G_2$, is obtained by taking one copy of $T(G_1)$ and $|V(G_1)|$ copies of G_2, and joining the ith vertex of G_1 to every vertex in the ith copy of G_2.

Let P_n be a path of order n. Figure 1 depicts the total corona $P_3 \circledast P_2$ of P_3 and P_2. Note that if G_1 is an r-regular graph on n_1 vertices and m_1 edges, and G_2 is an arbitrary graph on n_2 vertices and m_2 edges, then $G_1 \circledast G_2$ has $n_1+m_1+n_1n_2$ vertices and $n_1m_2+n_1n_2+3m_1+\frac{n_1r(r-1)}{2}$ edges.

In this paper, we focus on determining the A-spectra, L-spectra and the Q-spectra of $G_1 \circledast G_2$ in terms of the corresponding spectra of a regular graph G_1 and an arbitrary graph G_2. As applications of these results, we construct infinitely many pairs of adjacency (respectively, Laplacian and signless Laplacian) cospectral graphs. Moreover, we also compute the number of spanning trees of $G_1 \circledast G_2$ in terms of the L-spectra of two factor graphs G_1 and G_2.

2 Main results

In this section, we determine the spectra of total corona with the help of the coronal of a matrix. The M-coronal $\Gamma_M(x)$ of a matrix M of order n is defined [3, 16] to be the sum of

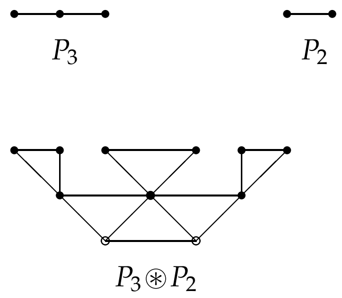

Figure 1: The total corona of $P_3 \circledast P_2$ of two paths P_3 and P_2.

the entries of the matrix $(xI_n - M)^{-1}$, that is, $\Gamma_M(x) = 1_n^T(xI_n - M)^{-1}1_n$, where 1_n denotes the column vector of size n with all the entries equal to one. The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. This is an associative operation with the property that $(A\otimes B)^T = A^T \otimes B^T$ and $(A\otimes B)(C\otimes D) = AC\otimes BD$. whenever the products AC and BD exist. The latter implies $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$ for no-singular matrices A and B. Moreover, if A and B are $n\times n$ and $p\times p$ matrices, then $det(A\otimes B) = (detA)^p(detB)^n$. The reader can refer to [10] for other properties of the Kronecker product.

Let $G = (V,E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The incident matrix of G is an $n\times m$ matrix whose (i,j)-entry is 1 if v_i and e_j are incident in G and 0, otherwise , denoted by $R(G)$. If the graph G is an r regular, then $R(G)R(G)^T = A(G) + rI_n$.

Let G_1 be an r-regular graph on n_1 vertices and m_1 edges, and G_2 is an arbitrary graph on n_2 vertices. We first label the vertices of $G_1 \circledast G_2$ as follows. Let $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$, and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $i = 1,2,\ldots,n_1$, let $V^i(G_2) = \{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ denote the vertex set of the ith copy of G_2. Then $V(G_1)\cup I(G_1)\cup [\bigcup_{i=1}^{n_1} V^i(G_2)]$ is a partition of $V(G_1\circledast G_2)$. It is clear that degrees of the vertices of $G_1\circledast G_2$ are

$$\begin{aligned}
&d_{G_1\circledast G_2}(v_i) = 2d_{G_1}(v_i) + n_2,\ i = 1,2,\ldots n_1,\\
&d_{G_1\circledast G_2}(e_i) = 2r,\ i = 1,2,\ldots m_1,\\
&d_{G_1\circledast G_2}(u_j^i) = d_{G_2}(u_j) + 1,\ i = 1,2,\ldots n_1, j = 1,2,\ldots,n_2.
\end{aligned}$$

In the following, we first consider the adjacency spectra of $G_1 \circledast G_2$.

Theorem 2.1. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 is an arbitrary* *graph on n_2 vertices. Then*

$$\phi(A(G_1\circledast G_2);x)=(x+2)^{m_1-n_1}(\phi(A(G_2)))^{n_1}\prod_{i=1}^{n_1}\big[x^2+(2-\Gamma_{A(G_2)}(x)-r_1-2\lambda_i)x \\ +\lambda_i^2+(r_1+\Gamma_{A(G_2)}(x)-3)\lambda_i+(r_1-2)\Gamma_{A(G_2)}(x)-r_1\big].$$

Proof. We label the vertices of $G_1\circledast G_2$ as above, the adjacency matrix of $G_1\circledast G_2$ can be written as

$$A(G_1\circledast G_2)=\begin{pmatrix} A(G_1) & R & I_{n_1}\otimes 1_{n_2}^T \\ R^T & B & 0 \\ I_{n_1}\otimes 1_{n_2} & 0 & I_{n_1}\otimes A(G_2)\end{pmatrix},$$

where R is the vertex-edge incidence matrix of G_1, $B=R^TR-2I_{m_1}$. Then the characteristic polynomial of $G_1\circledast G_2$ is

$$\begin{aligned}\phi(A(G_1\circledast G_2))&=\det\begin{pmatrix} xI_{n_1}-A(G_1) & -R & -I_{n_1}\otimes 1_{n_2}^T \\ -R^T & xI_{m_1}-B & 0 \\ -I_{n_1}\otimes 1_{n_2} & 0 & I_{n_1}\otimes(xI_{n_2}-A(G_2))\end{pmatrix}\\ &=\det(I_{n_1}\otimes(xI_{n_2}-A(G_2)))\det(S)\\ &=(\phi(A(G_2)))^{n_1}\det(S),\end{aligned}$$

where

$$\begin{aligned}S=&\begin{pmatrix} xI_{n_1}-A(G_1) & -R \\ -R^T & xI_{m_1}-B\end{pmatrix}-\begin{pmatrix} -I_{n_1}\otimes 1_{n_2}^T \\ 0\end{pmatrix}(I_{n_1}\otimes(xI_{n_2}-A(G_2))^{-1})\bullet\\ &\begin{pmatrix} -I_{n_1}\otimes 1_{n_2} & 0\end{pmatrix}\\ =&\begin{pmatrix} xI_{n_1}-A(G_1) & -R \\ -R^T & xI_{m_1}-B\end{pmatrix}-\begin{pmatrix} \Gamma_{A(G_2)}(x)I_{n_1} & 0 \\ 0 & 0\end{pmatrix}\\ =&\begin{pmatrix} (x-\Gamma_{A(G_2)}(x))I_{n_1}-A(G_1) & -R \\ -R^T & xI_{m_1}-B\end{pmatrix}\end{aligned}$$

is the Schur complement [18] of $I_{n_1}\otimes(xI_{n_2}-A(G_2))$ and

$$\begin{aligned}\det S=&\det\begin{pmatrix} (x-\Gamma_{A(G_2)}(x))I_{n_1}-RR^T+r_1I_{n_1} & -R \\ -R^T & (x+2)I_{m_1}-R^TR\end{pmatrix}\\ =&\det\begin{pmatrix} (x-\Gamma_{A(G_2)}(x)+r_1)I_{n_1}-RR^T & -R \\ -(1+x-\Gamma_{A(G_2)}(x)+r_1)R^T+R^TRR^T & (x+2)I_{m_1}\end{pmatrix}\\ =&\det\\ &\begin{pmatrix} (x-\Gamma_{A(G_2)}(x)+r_1)I_{n_1}-(1+\frac{1+x-\Gamma_{A(G_2)}(x)+r_1}{x+2})RR^T+\frac{1}{x+2}RR^TRR^T & 0 \\ -(1+x-\Gamma_{A(G_2)}(x)+r_1)R^T+R^TRR^T & (x+2)I_{m_1}\end{pmatrix}\end{aligned}$$

$$
\begin{aligned}
&=(x+2)^{m_1}\det((x-\Gamma_{A(G_2)}(x)+r_1)I_{n_1}-\frac{2x-\Gamma_{A(G_2)}(x)+r_1+3}{x+2}(A(G_1)+r_1I_{n_1})\\
&\quad+\frac{1}{x+2}(A(G_1)+r_1I_{n_1})^2)\\
&=(x+2)^{m_1}\det(\frac{1}{x+2}A^2(G_1)+\frac{r_1+\Gamma_{A(G_2)}(x)-2x-3}{x+2}A(G_1)\\
&\quad+\frac{x^2+(2-\Gamma_{A(G_2)}(x)-r_1)x-(r_1-2)\Gamma_{A(G_2)}(x)-r_1}{x+2}I_{n_1})\\
&=(x+2)^{m_1-n_1}\prod_{i=1}^{n_1}(x^2+(2-\Gamma_{A(G_2)}(x)-r_1-2\lambda_i(G_1))x+\lambda_i^2(G_1)\\
&\quad+(r_1+\Gamma_{A(G_2)}(x)-3)\lambda_i(G_1)+(r_1-2)\Gamma_{A(G_2)}(x)-r_1),
\end{aligned}
$$

where $RR^T=A(G_1)+r_1I_{n_1}$ and $\lambda_i(G_1)$ is the ith eigenvalue of matrix A. Hence the result follows. □

Corollary 2.1. Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the A-spectrum of $G_1\circledast G_2$ consists of:

(i) -2, repeated m_1-n_1 times;

(ii) $\lambda_i(G_2)$, repeated n_1 times for $i=2,3,\ldots,n_2$;

(iii) three roots of the equation, for $j=1,2,\ldots,n_1$,

$$
\begin{aligned}
&x^3+(2-2\lambda_j(G_1)-r_2-r_1)x^2+[\lambda_j^2(G_1)+(2r_2+r_1-3)\lambda_j(G_1)+r_2(r_1-2)\\
&-n_2-r_1]x+[-r_2\lambda_j^2(G_1)+(n_2-r_1r_2+3r_2)\lambda_j(G_1)+n_2(r_1-2)+r_1r_2]=0.
\end{aligned}
$$

Proof. Since G_2 is r_2-regular. Then Proposition 2 in [3] implies that

$$\Gamma_{A(G_2)}(x)=\frac{n_2}{x-r_2}.$$

Thus, by Theorem 2.1, $\lambda_i(G_2)$ is an eigenvalue of $G_1\circledast G_2$ repeated n_1 times, for each i=2,3,…,n_2 and -2 is also an eigenvalue of $G_1\circledast G_2$ repeated m_1-n_1 times. The remaining eigenvalues are obtained by solving the equation as above. □

Corollary 2.2. Let G be an r-regular graph on n vertices and m edges, where $r\geq 2$, and H be a complete bipartite graph $K_{p,q}$ with $p,q\geq 1$. Then the A-spectrum of $G\circledast H$ consists of:

(i) -2, repeated $m-n$ times;

(ii) 0, repeated $n(p+q-2)$ times;

(iii) four roots of the equation, for $j=1,2,\dots,n$,

$$x^4+(2-r-2\lambda_j(G))x^3+[\lambda_j^2(G)+(r-3)\lambda_j(G)-r-pq-p-q]x^2+[pq(r+2\lambda_j(G)-4)$$
$$+(p+q)(\lambda_j(G)+r-2)]x+pq[-\lambda_j^2(G)+(5-r)\lambda_j(G)+3r-4]=0.$$

Proof. It is well known [16] that the $A(K_{p,q})$-coronal of $K_{p,q}$ is

$$\Gamma_{A(K_{p,q})}(x)=\frac{(p+q)x+2pq}{x^2-pq}.$$

Simplifying the characteristic polynomial in Theorem 2.1, we can obtain 0 is the eigenvalue repeated $n(p+q-2)$ times. The remaining eigenvalues are obtained by solving the equation as above. □

Applying Theorem 2.1, we can obtain infinitely many pairs of A-cospectral graphs in the following corollary.

Corollary 2.3. (i) If G_1,G_2 are two A-cospectral regular graphs and H is an arbitrary graph, then $G_1\circledast H$ and $G_2\circledast H$ are A-cospectral;

(ii) If G is a regular graph and H_1,H_2 are two A-cospectral graphs with $\Gamma_{A(H_1)}(x)=\Gamma_{A(H_2)}(x)$, then $G\circledast H_1$ and $G\circledast H_2$ are A-cospectral;

(iii) If G_1,G_2 are two A-cospectral regular graphs and H_1,H_2 are two A-cospectral graphs with $\Gamma_{A(H_1)}(x)=\Gamma_{A(H_2)}(x)$, then $G_1\circledast H_1$ and $G_2\circledast H_2$ are A-cospectral.

Next, we consider the Laplacian spectra of $G_1\circledast G_2$.

Theorem 2.2. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 is an arbitrary graph on n_2 vertices. Then*

$$\begin{aligned}\phi(L(G_1\circledast G_2);x)=&(x-2r_1-2)^{m_1-n_1}\prod_{i=2}^{n_2}(x-1-\mu_i(G_2))^{n_1}\prod_{i=1}^{n_1}[x^3-(2\mu_i(G_1)+r_1+n_2+3)x^2\\&+((\mu_i(G_1)+1)r_1+\mu_i^2(G_1)+(n_2+5)\mu_i(G_1)+2n_2+2)x\\&-\mu_i^2(G_1)-\mu_i(G_1)(r_1+3)].\end{aligned}$$

Proof. We labele the vertices of G as above, the diagonal degree matrix of $G_1\circledast G_2$ can be written as

$$D(G_1\circledast G_2)=\begin{pmatrix} D(G_1)+(n_2+r_1)I_{n_1} & 0 & 0\\ 0 & 2r_1I_{m_1} & 0\\ 0 & 0 & I_{n_1}\otimes(D(G_2)+I_{n_2})\end{pmatrix}.$$

Since $L(G)=D(G)-A(G)$, the Laplacian matrix of $G_1\circledast G_2$ is

$$L(G_1\circledast G_2)=\begin{pmatrix} L(G_1)+(n_2+r_1)I_{n_1} & -R & -I_{n_1}\otimes 1_{n_2}^T\\ -R^T & 2r_1I_{m_1}-B & 0\\ -I_{n_1}\otimes 1_{n_2} & 0 & I_{n_1}\otimes(L(G_2)+I_{n_2})\end{pmatrix}.$$

Then the Laplacian characteristic polynomial of $G_1 \circledast G_2$ is

$$\begin{aligned}
\phi(L(G_1 \circledast G_2)) &= \det \begin{pmatrix} (x-r_1-n_2)I_{n_1} - L(G_1) & R & I_{n_1} \otimes 1_{n_2}^T \\ R^T & (x-2r_1)I_{m_1}+B & 0 \\ I_{n_1} \otimes 1_{n_2} & 0 & I_{n_1} \otimes ((x-1)I_{n_2} - L(G_2)) \end{pmatrix} \\
&= \det(I_{n_1} \otimes ((x-1)I_{n_2} - L(G_2))) \cdot \det(S) \\
&= \prod_{i=1}^{n_2} (x-1-\mu_i(G_2))^{n_1} \cdot \det(S),
\end{aligned}$$

where

$$S = \begin{pmatrix} (x-r_1-n_2-\Gamma_{L(G_2)}(x-1))I_{n_1} - L(G_1) & R \\ R^T & (x-2r_1)I_{m_1}+B \end{pmatrix}$$

is the Schur complement [18] of $I_{n_1} \otimes ((x-1)I_{n_2} - L(G_2))$. Since each row sum of $L(G_2)$ equals 0,

$$\Gamma_{L(G_2)}(x-1) = \frac{n_2}{x-1}.$$

Note that $RR^T = A(G_1) + r_1 I_{n_1}$, then

$$\begin{aligned}
\det S &= \det \begin{pmatrix} (x-r_1-n_2-\Gamma_{L(G_2)}(x-1))I_{n_1} - L(G_1) & R \\ R^T & (x-2r_1-2)I_{m_1}+R^TR \end{pmatrix} \\
&= \det \begin{pmatrix} (x-r_1-n_2-\Gamma_{L(G_2)}(x-1))I_{n_1} - L(G_1) & R \\ R^T - (x-r_1-n_2-\Gamma_{L(G_2)}(x-1))R^T + R^T L(G_1) & (x-2r_1-2)I_{m_1} \end{pmatrix} \\
&= (x-2r_1-2)^{m_1-n_1} \prod_{i=1}^{n_1} (\lambda_i^2(G_1) + (2x-3r_1-n_2-\frac{n_2}{x-1}-3)\lambda_i(G_1) + x^2 \\
&\quad -(3r_1+n_2+2+\frac{n_2}{x-1})x + 2r_1^2 + (3+n_2+\frac{n_2}{x-1})r_1 + 2n_2 + \frac{2n_2}{x-1}).
\end{aligned}$$

Note that $\mu_1(G_2) = 0$. Now the result follows easily. $\square$

Let G be a connected graph of order n with Laplacian eigenvalues $0 = \mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$. It is well known [6] that the number of spanning trees of G is

$$t(G) = \frac{\mu_2(G)\mu_3(G)\cdots\mu_n(G)}{n}.$$

Thus, by Theorem 2.2, we can obtain the following results.

Corollary 2.4. Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 is an arbitrary graph on n_2 vertices. Then the number of spanning trees of $G_1 \circledast G_2$ is

$$t(G_1 \circledast G_2) = \frac{(2r_1+2)^{m_1-n_1} \prod_{i=2}^{n_2} (1+\mu_i(G_2))^{n_1} \prod_{i=1}^{n_1} (\mu_i^2(G_1) + \mu_i(G_1)(r_1+3))}{n_1+m_1+n_1 n_2}.$$

Corollary 2.5. (i) If G_1,G_2 are two L-cospectral regular graphs and H is an arbitrary graph, then $G_1 \circledast H$ and $G_2 \circledast H$ are L-cospectral;

(ii) If G is a regular graph and H_1,H_2 two are L-cospectral graphs, then $G\circledast H_1$ and $G\circledast H_2$ are L-cospectral;

(iii) If G_1,G_2 are L-cospectral regular graphs and H_1,H_2 are L-cospectral graphs, then $G_1\circledast H_1$ and $G_2\circledast H_2$ are L-cospectral.

Finally, we consider the signless Laplacian spectra of $G_1\circledast G_2$.

Theorem 2.3. *Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 is an arbitrary graph on n_2 vertices. Then*

$$\begin{aligned}\phi(Q(G_1\circledast G_2);x)=&(x-2r_1+2)^{m_1-n_1}\prod_{i=1}^{n_2}(x-1-\nu_i(G_2))^{n_1}\prod_{i=1}^{n_1}\big[x^2+(2-2\nu_i(G_1)-3r_1-n_2\\&-\Gamma_{Q(G_2)}(x-1))x+2r_1^2+(3\nu_i(G_1)+2n_2+2\Gamma_{Q(G_2)}(x-1)-2)r_1\\&+\nu_i^2(G_1)+(n_2+\Gamma_{Q(G_2)}(x-1)-3)\nu_i(G_1)-2n_2-2\Gamma_{Q(G_2)}(x-1)\big].\end{aligned}$$

Proof. With respect to the partition as above, we have

$$Q(G_1\circledast G_2)=\begin{pmatrix} Q(G_1)+(n_2+r_1)I_{n_1} & R & I_{n_1}\otimes 1_{n_2}^T\\ R^T & B+2r_1I_{m_1} & 0\\ I_{n_1}\otimes 1_{n_2} & 0 & I_{n_1}\otimes(Q(G_2)+I_{n_2})\end{pmatrix}.$$

Then the signless Laplacian characteristic polynomial of $G_1\circledast G_2$ is

$$\begin{aligned}\phi(Q(G_1\circledast G_2))&=\det\begin{pmatrix}(x-r_1-n_2)I_{n_1}-Q(G_1) & -R & -I_{n_1}\otimes 1_{n_2}^T\\ -R^T & -B+(x-2r_1)I_{m_1} & 0\\ -I_{n_1}\otimes 1_{n_2} & 0 & I_{n_1}\otimes((x-1)I_{n_2}-Q(G_2))\end{pmatrix}\\&=\det(I_{n_1}\otimes((x-1)I_{n_2}-Q(G_2)))\cdot\det(S)\\&=\prod_{i=1}^{n_2}(x-1-\nu_i(G_2))^{n_1}\cdot\det(S),\end{aligned}$$

where

$$S=\begin{pmatrix}(x-r_1-n_2-\Gamma_{Q(G_2)}(x-1))I_{n_1}-Q(G_1) & -R\\ -R^T & (x-2r_1)I_{m_1}-B\end{pmatrix}$$

is the Schur complement [18] of $I_{n_1}\otimes((x-1)I_{n_2}-Q(G_2))$ and

$$\begin{aligned}\det S&=\det\begin{pmatrix}(x-r_1-n_2-\Gamma_{Q(G_2)}(x-1))I_{n_1}-Q(G_1) & -R\\ -R^T & (x-2r_1+2)I_{m_1}-R^TR\end{pmatrix}\\&=\det\begin{pmatrix}(x-r_1-n_2-\Gamma_{Q(G_2)}(x-1))I_{n_1}-Q(G_1) & -R\\ -R^T-(x-r_1-n_2-\Gamma_{Q(G_2)}(x-1))R^T+R^TQ & (x-2r_1+2)I_{m_1}\end{pmatrix}\\&=(x-2r_1+2)^{m_1-n_1}\prod_{i=1}^{n_1}(x^2+(2-2\nu_i(G_1)-3r_1-n_2-\Gamma_{Q(G_2)}(x-1))x+2r_1^2+(3\nu_i(G_1)+2n_2\\&\quad+2\Gamma_{Q(G_2)}(x-1)-2)r_1+\nu_i^2(G_1)+(n_2+\Gamma_{Q(G_2)}(x-1)-3)\nu_i(G_1)-2n_2-2\Gamma_{Q(G_2)}(x-1)).\end{aligned}$$

Now the result follows easily. □

Corollary 2.6. Let G_1 be an r_1-regular graph on n_1 vertices and m_1 edges, and G_2 be an r_2-regular graph on n_2 vertices. Then the Q-spectrum of $G_1 \circledast G_2$ consists of:

(i) $2r_1-2$, repeated m_1-n_1 times;

(ii) $1+\nu_i(G_2)$, repeated n_1 times for $i=2,3,\ldots,n_2$;

(iii) three roots of the equation, for $j=1,2,\ldots,n_1$,

$$\begin{aligned}&x^3+(1-2\nu_i(G_1)-n_2-3r_1-2r_2)x^2+[2r_1^2+(3\nu_i(G_1)+2n_2+6r_2+1)r_1+\nu_i(G_1)^2\\&+(4r_2+n_2-1)\nu_i(G_1)+(2r_2-2)n_2-4r_2-2]x-(4r_2+2)r_1^2+(2r_2+1)[(2-3\nu_i(G_1))r_1\\&-\nu_i^2(G_1)]+4n_2r_2(1-r_1)+[(3-n_2)(2r_2+1)+n_2]\nu_i(G_1)=0.\end{aligned}$$

Proof. Since each row sum of $Q(G_2)$ equals $2r_2$,

$$\Gamma_{Q(G_2)}(x-1)=\frac{n_2}{x-1-2r_2}.$$

Thus, by Theorem 2.3, $1+\nu_i(G_2)$ is a signless Laplacian eigenvalue of $G_1 \circledast G_2$ repeated n_1 times, for $i=2,\ldots,n_2$, and $2r_1-2$ is also a signless Laplacian eigenvalue of $G_1 \circledast G_2$, repeated m_1-n_1 times. The remaining signless Laplacian eigenvalues are obtained by solving the equation as above. □

Applying Theorem 2.3, we can obtain infinitely many pairs of Q-cospectral graphs in the following corollary.

Corollary 2.7. (i) If G_1,G_2 are Q-cospectral r-regular graphs and H is an arbitrary graph, then $G_1 \circledast H$ and $G_2 \circledast H$ are Q-cospectral.

(ii) If G is a regular graph, H_1 and H_2 are Q-cospectral graphs with $\Gamma_{Q(H_1)}(x)=\Gamma_{Q(H_2)}(x)$, then $G \circledast H_1$ and $G \circledast H_2$ are Q-cospectral.

(iii)If G_1,G_2 are Q-cospectral regular graphs and H_1,H_2 are Q-cospectral graphs with $\Gamma_{Q(H_1)}(x)=\Gamma_{Q(H_2)}(x)$, then $G_1 \circledast H_1$ and $G_2 \circledast H_2$ are Q-cospectral.

Acknowledgments

This work was in part supported by NSFC (Grant No. 11371328, No. 11271334) and by the Natural Science Foundation of Zhejiang Province, China (Grant No. LY15A010011).

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