J. Math. Study
doi: 10.4208/jms.v48n4.15.04

Vol. **48**, No. 4, pp. 362-374
December 2015

Ball Convergence for Higher Order Methods under Weak Conditions

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Received 6 May, 2015; Accepted 29 September, 2015

Abstract. We present a local convergence analysis for higher order methods in order to approximate a locally unique solution of an equation in a Banach space setting. In earlier studies, Taylor expansions and hypotheses on higher order Fréchet-derivatives are used. We expand the applicability of these methods using only hypotheses on the first Fréchet derivative. Moreover, we obtain a radius of convergence and computable error bounds using Lipschitz constants not given before. Numerical examples are also presented in this study.

AMS subject classifications: 65G99, 65D99, 65G99, 47J25, 45J05

Key words: Higher order method, Banach space, Fréchet derivative, local convergence.

1 Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x)=0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Denote by $L(X,Y)$ the space of bounded linear operators from X into Y.

A lot of problems from Computational Sciences and other disciplines can be brought in the form of Eq. (1.1) using Mathematical Modeling [2, 5, 10, 17, 22]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [1–22].

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The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1–22]. In order to obtain a higher order of convergence, Newton-like methods have been studied such as Potra-Ptak, Chebyshev, Cauchy, Halley and Ostrowski, methods [2,5,16,22]. The number of function evaluations per step increases with the order of convergence. In the scalar case the efficiency index [16,22] $EI = p^{\frac{1}{m}}$ provides a measure of balance where p is the order of the methods and m is the number of function evaluations.

We study the local convergence of the two-step methods defined for each $n = 0,1,2...$ by

$$\begin{aligned} y_n &= x_n - \Theta F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n) + (F'(x_n) - 3F'(y_n))^{-1}F(x_n) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} y_n &= x_n - \Theta F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - H(x_n,y_n)F'(x_n)^{-1}F(x_n) \end{aligned} \tag{1.3}$$

where x_0 is an initial point, $\Theta \in \mathbb{R}$ a parameter and $H: X^2 \to L(X,Y)$ a given continuous operator. Method (1.2) was studied by Basu in [7], when $X = Y = \mathbb{R}^m$ and $\Theta = \frac{2}{3}$. The method (1.2) was shown to be of order four using Taylor expansions and hypotheses reaching up to the sixth derivative of F. Notice that method (1.2) is really a particular case of Jarratt's method [2,5,16,22]. Moreover, method (1.3) was studied by Chun *et al.* in [8] the same way. This method is also of order four assuming that function H satisfies certain initial conditions [8]. The case $H(x_n,y_n) = F'(x_n)^{-1}F(y_n)$ was also studied in [8] (see also our Remark 2.3 (6)). Method (1.2) uses two inverses and one function evaluation. Two-step Newton methods comparable to method (1.2) are given by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(x_n) \end{aligned} \tag{1.4}$$

or

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n). \end{aligned} \tag{1.5}$$

Method (1.4) uses two inverses and one function evaluation (so does method (1.2)) but it is of order three. Moreover, method (1.5) uses two inverses and two function evaluations and it is of order four. However, method (1.2) is cheaper, since it uses one less function evaluation than method (1.5). The single step Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n)$$

uses one inverse and one function evaluation but it is of order two. Hence, method (1.2) is more robust than the preceding Newton methods. The hypotheses on the derivatives limit the applicability of these methods. As a motivational example, let us define function F on $X = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} &F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, F'(1) = 3, \\ &F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ &F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously function F does not have bounded third derivative in X. Notice that, there is a plethora of iterative methods for approximating solutions of nonlinear equations [1–22]. These results show that if the initial point x_0 is sufficiently close to the solution x^*, then the sequence $\{x_n\}$ converges to x^*. But how close to the solution x^* the initial guess x_0 should be? These local results give no information on the radius of the convergence ball for the corresponding method. We address this question for method (1.2) and method (1.3) in Section 2. The same technique can be used to other methods.

The paper is organized as follows. In Section 2 we present the local convergence analysis. We also provide a radius of convergence, computable error bounds and uniqueness result not given in the earlier studies using Taylor expansions. Special cases and numerical examples are presented in the concluding Section 3.

2 Local convergence analysis

We present the local convergence analysis of the method (1.2) and method (1.3) in this section. Let $L_0 > 0$, $L > 0$, $M \geq 1$ and $\Theta \in \mathbb{R}$ be given parameters. It is convenient for the local convergence analysis that follows to define some scalar functions and parameters. Define functions g_1, p and h_p on the interval $[0, \frac{1}{L_0})$ by

$$\begin{aligned} &g_1(t) = \frac{1}{2(1 - L_0 t)} (Lt + 2|1 - \Theta| M), \\ &p(t) = \frac{L_0}{2} (1 + 3g_1(t)) t, h_p(t) = p(t) - 1 \end{aligned}$$

and parameters r_1 and r_A by

$$r_1 = \frac{2(1-M|1-\Theta|)}{2L_0+L}, \quad r_A = \frac{2}{2L_0+L}.$$

Suppose that

$$M|1-\Theta| < 1. \tag{2.1}$$

Then, we have that $0 < r_1 < r_A$. We get by the the definition of functions p and h_p that $h_p(0) = -1 < 0$ and $h_p(t) \to +\infty$ as $t \to \frac{1}{L_0}^-$. It then, follows from the intermediate value theorem that function h_p has zeros in the interval $(0, \frac{1}{L_0})$. Denote by r_p the smallest zero of function h_p in the interval $(0, \frac{1}{L_0})$. Moreover, define functions g_2 and h_2 on the interval $[0, r_p)$ by

$$g_2(t) = \frac{1}{2(1-L_0 t)} \left(Lt + \frac{3L_0 M(1+g_1(t))}{2(1-p(t))} \right) t$$

and

$$h_2(t) = g_2(t) - 1.$$

We get that $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to r_p^-$. Denote by r_2 the smallest zero of function h_2 in the interval $(0, r_p)$. Set

$$r = \min\{r_1, r_2\}. \tag{2.2}$$

Then, we have that

$$0 < r \le r_A < \frac{1}{L_0}, \tag{2.3}$$

and for each $t \in [0, r)$

$$0 \le g_1(t) < 1, \tag{2.4}$$
$$0 \le p(t) < 1 \tag{2.5}$$

and

$$0 \le g_2(t) < 1. \tag{2.6}$$

Next, we present the local convergence analysis of the method (1.2), using the preceding notation.

Theorem 2.1. *Let $F: D \subset X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M \ge 1$ and $\Theta \in \mathbb{R}$ such that* (2.1) *and for each $x, y \in D$*

$$F(x^*) = 0, F'(x^*)^{-1} \in L(Y, X) \tag{2.7}$$
$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le L_0 \|x - x^*\|, \tag{2.8}$$
$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L \|x - x^*\|, \tag{2.9}$$
$$\|F'(x^*)^{-1} F'(x)\| \le M \tag{2.10}$$

and

$$\bar{U}(x^*,r)\subseteq D, \tag{2.11}$$

hold, where the radius r is given by (2.2). *Then, the sequence* $\{x_n\}$ *generated for* $x_0\in U(x^*,r)-\{x^*\}$ *by method* (1.2) *is well defined , remains in* $U(x^*,r)$ *for each* $n=0,1,2.....$ *and converges to* x^*. *Moreover, the following estimates hold*

$$\|y_n-x^*\|\le g_1(\|x_n-x^*\|)\|x_n-x^*\|<\|x_n-x^*\|<r, \tag{2.12}$$

and

$$\|x_{n+1}-x^*\|\le g_2(\|x_n-x^*\|)\|x_n-x^*\|<\|x_n-x^*\| \tag{2.13}$$

where the "g" functions are defined above Theorem 2.1. *Furthermore, for* $T\in[r,\frac{2}{L_0})$ *the limit point* x^* *is the only solution of the equation* $F(x)=0$ *in* $\bar{U}(x^*,T)\cap D$.

Proof. We shall show estimates (2.12) and (2.13) using mathematical induction. By hypothesis $x_0\in U(x^*,r)-\{x^*\}$, (2.3) and (2.8), we get that

$$\|F'(x^*)^{-1}(F'(x_0)-F'(x^*))\|\le L_0\|x_0-x^*\|<L_0r<1. \tag{2.14}$$

It follows from (2.14) and Banach Lemma on invertible operators [2,5,12,22] that $F'(x_0)^{-1}\in L(Y,X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\|\le\frac{1}{1-L_0\|x_0-x^*\|}. \tag{2.15}$$

Hence, y_0 is well defined by the first sub-step of method (1.2) for $n=0$. We can write by (2.7) that

$$F(x_0)=F(x_0)-F(x^*)=\int_0^1 F'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta. \tag{2.16}$$

Notice that $\|x^*+\theta(x_0-x^*)-x^*\|=\theta\|x_0-x^*\|<r$, so $x^*+\theta(x_0-x^*)\in U(x^*,r)$. Then, using (2.10) and (2.16), we get that

$$\begin{aligned}\|F'(x^*)^{-1}F(x_0)\|&=\left\|\int_0^1 F'(x^*)^{-1}F'(x^*+\theta(x_0-x^*))(x_0-x^*)d\theta\right\|\\&\le\|x_0-x^*\|.\end{aligned} \tag{2.17}$$

In view of (2.2), (2.4), (2.9), (2.15) and (2.17), we obtain in turn that

$$y_0-x^*=(x_0-x^*-F'(x_0)^{-1}F(x_0))+(1-\Theta)F'(x_0)^{-1}F(x_0),$$

so

$$\begin{aligned}\|y_0-x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\left\|\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*))\right.\\ &\quad \left.-F'(x_0))(x_0-x^*)d\theta\right\|\\ &\quad +|1-\Theta|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F'(x_0)\|\\ &\leq \frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)}+\frac{|1-\Theta|M\|x_0-x^*\|}{1-L_0\|x_0-x^*\|}\\ &= g_1(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r, \end{aligned} \tag{2.18}$$

which shows (2.12) for $n=0$ and $y_0\in U(x^*,r)$. Next, we shall show that $(F'(x_0)-3F'(y_0))^{-1}\in L(Y,X)$. Using (2.3), (2.5), (2.15) and (2.18), we get that

$$\begin{aligned}&\|(-2F'(x^*))^{-1}[(F'(x_0)-F'(x^*))-3(F'(y_0)-F'(x^*))]\|\\ &\leq \frac{1}{2}[\|F'(x^*)^{-1}(F'(x_0)-F'(x^*))\|+3\|F'(x^*)^{-1}(F'(y_0)-F'(x^*))\|]\\ &\leq \frac{L_0}{2}(\|x_0-x^*\|+3\|y_0-x^*\|)\\ &\leq \frac{L_0}{2}(1+3g_1(\|x_0-x^*\|)\|x_0-x^*\|)\\ &= p(\|x_0-x^*\|)<p(r)<1. \end{aligned} \tag{2.19}$$

We get from (2.19) that

$$\|(F'(x_0)-3F'(y_0))^{-1}F'(x^*)\|\leq \frac{1}{2(1-p(\|x_0-x^*\|))}. \tag{2.20}$$

That is x_1 is well defined by the second sub-step of method (1.2) for $n=0$. Then, we can write

$$\begin{aligned}x_1-x^* &= x_0-x^*-F'(x_0)^{-1}F(x_0)\\ &\quad +\frac{3}{2}(F'(x_0)-3F'(y_0))^{-1}(F'(x_0)-F'(x^*))+(F'(x^*)-F'(y_0))\\ &\quad (F'(x_0)^{-1}F'(x^*))(F'(x^*)^{-1}F(x_0)). \end{aligned} \tag{2.21}$$

Using (2.3), (2.6), (2.15), (2.17), (2.18) and (2.21), we have in turn that

$$\begin{aligned}\|x_1-x^*\| &\leq \frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)}+\frac{3L_0M(1+g_1(\|x_0-x^*\|))\|x_0-x^*\|^2}{4(1-L_0\|x_0-x^*\|)(1-p(\|x_0-x^*\|))}\\ &= g_2(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r, \end{aligned} \tag{2.22}$$

which shows (2.13) for $n=0$ and $x_1\in U(x^*,r)$. By simply replacing x_0,y_0,x_1 by x_k,y_k,x_{k+1} in the preceding estimates, we arrive at estimates (2.12) and (2.13). Then, from the estimate

$\|x_{k+1}-x^*\| < \|x_k-x^*\| < r$, we deduce that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1}\in U(x^*,r)$. To show the uniqueness part, let $Q=\int_0^1 F'(y^*+\theta(x^*-y^*)d\theta$ for some $y^*\in \bar{U}(x^*,T)$ with $F(y^*)=0$. Using (2.8) we get that

$$\begin{aligned}\|F'(x^*)^{-1}(Q-F'(x^*))\| &\le \int_0^1 L_0\|y^*+\theta(x^*-y^*)-x^*\|d\theta\\ &\le \int_0^1 L_0(1-\theta)\|x^*-y^*\|d\theta \le \frac{L_0}{2}T<1. \end{aligned} \tag{2.23}$$

It follows from (2.23) and the Banach Lemma on invertible functions that Q is invertible. Finally, from the identity $0=F(x^*)-F(y^*)=Q(x^*-y^*)$, we conclude that $x^*=y^*$.

$\square$

In order for us to study the local convergence analysis of method (1.3), we need to define some additional functions and parameters. Let $\psi:[0,\frac{1}{L_0})\to[0,+\infty)$ be a continuous nondecreasing function. Define functions $\bar{g}_2$ and $\bar{h}_2$ on the interval $[0,r_p)$ by

$$\bar{g}_2(t)=\frac{1}{2(1-L_0t)}(Lt+2\psi(t)M)$$

and

$$\bar{h}_2(t)=\bar{g}_2(t)-1. \tag{2.24}$$

Suppose that

$$M\psi(0)<1.$$

Then, we have that $\bar{h}_2(0)=M\psi(0)-1<0$ and $\bar{h}_2(t)\to+\infty$ as $t\to\frac{1}{L_0}^-$. Denote by R_2 the smallest zero of function $\bar{h}_2$ in the interval $(0,\frac{1}{L_0})$. Set

$$R=\min\{r_1,R_2\}. \tag{2.25}$$

Then, we have that

$$0<R\le r_A<\frac{1}{L_0} \tag{2.26}$$

and for each $t\in[0,r)$

$$0\le g_1(t)<1 \tag{2.27}$$

and

$$0\le \bar{g}_2(t)<1. \tag{2.28}$$

Next, we present the local convergence analysis of the method (1.3), using the preceding notation.

Theorem 2.2. *Let $F:D\subset X\to Y$ be a Fréchet-differentiable operator. Let also $H:X^2\to L(X,Y)$ be an operator and $\psi:[0,\frac{1}{L_0})\to[0,+\infty)$ be a continuous and nondecreasing function. Suppose that hypotheses* (2.7)-(2.10),

$$\|I-H(x,y)\|\le\psi(\|x-x^*\|)\ \text{ for each }\ x,y\in D, \tag{2.29}$$

$$M\psi(0)<1 \tag{2.30}$$

and

$$\bar{U}(x^*,R)\subseteq D, \tag{2.31}$$

hold, where the radius R is given by (2.25)*. Then, the sequence $\{x_n\}$ generated for $x_0\in U(x^*,r)-\{x^*\}$ by method* (1.3) *is well defined, remains in $U(x^*,R)$ for each $n=0,1,2.....$ and converges to x^*. Moreover, the following estimates hold*

$$\|y_n-x^*\|\le g_1(\|x_n-x^*\|)\|x_n-x^*\|<\|x_n-x^*\|<R, \tag{2.32}$$

and

$$\|x_{n+1}-x^*\|\le\bar{g}_2(\|x_n-x^*\|)\|x_n-x^*\|<\|x_n-x^*\|. \tag{2.33}$$

Furthermore, for $T\in[R,\frac{2}{L_0})$ the limit point x^* is the only solution of the equation $F(x)=0$ in $\bar{U}(x^*,R)\cap D$.

Proof. According to the proof of Theorem 2.1 we only need to show estimate (2.33). Using the second substep of method (1.3) for $n=0$, we can write

$$x_1-x^*=x_0-x^*-F'(x_0)^{-1}F(x_0)+(I-H(x_0,y_0))F'(x_0)^{-1}F(x_0). \tag{2.34}$$

Then, using (2.15), (2.17), (2.18), (2.25), (2.28), (2.29) and (2.34) (for $x=x_0$), we get in turn that

$$\begin{aligned}\|x_1-x^*\|&\le\frac{L\|x_0-x^*\|^2}{2(1-L_0\|x_0-x^*\|)}+\frac{\psi(\|x_0-x^*\|)M\|x_0-x^*\|}{1-L_0\|x_0-x^*\|}\\&=\bar{g}_2(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<R,\end{aligned}$$

which shows (2.33) for $n=0$ and $x_1\in U(x^*,R)$. The rest follows by induction and the uniqueness part follows by replacing r by R in the proof of Theorem 2.1. □

Remark 2.1. 1. In view of (2.8) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\|&=\|F'(x^*)^{-1}(F'(x)-F'(x^*))+I\|\\&\le1+\|F'(x^*)^{-1}(F'(x)-F'(x^*))\|\le1+L_0\|x-x^*\|\end{aligned}$$

condition (2.10) can be dropped and be replaced by

$$M(t)=1+L_0t,$$

or

$$M = M(t) = 2,$$

since $t \in [0, \frac{1}{L_0})$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3,6,17] of the form

$$F'(x) = G(F(x))$$

where T is a continuous operator. Then, since $F'(x^*) = G(F(x^*)) = G(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $G(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [2–7].

4. The parameter r_1 was shown by us to be the convergence radius of Newton's method [3,6]

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \text{ for each } n = 0,1,2,\cdots \tag{2.35}$$

under the conditions (2.8)-(2.10). It follows from the definitions of the radii that the convergence radius r of the preceding methods cannot be larger than the convergence radius r_1 of the second order Newton's method (2.35). As already noted in [2,5] r_1 is at least as large as the convergence ball given by Rheinboldt [19]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_1$$

and

$$\frac{r_R}{r_1} \to \frac{1}{3} \, as \, \frac{L_0}{L} \to 0.$$

That is our convergence ball r_1 is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [22].

5. It is worth noticing that the studied methods are not changing when we use the conditions of the preceding Theorems instead of the stronger conditions used in [7,16]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right) / \ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence.

6. Let us show how to chose function ψ in the case when $H(x_n,y_n) = F'(x_n)^{-1}F'(y_n)$. We have that

$$\begin{aligned} \|I-H(x_n,y_n)\| &\leq \|F'(x_n)^{-1}F'(x^*)\|(\|F'(x^*)^{-1}(F'(x_n)-F'(x^*))\| \\ &\quad +\|F'(x^*)^{-1}(F'(y_n)-F'(x^*))\|) \\ &\leq \frac{L_0(1+g_1(\|x_n-x^*\|)\|x_n-x^*\|}{1-L_0\|x_n-x^*\|}. \end{aligned}$$

So, we can define function ψ on the interval $[0,\frac{1}{L_0})$ by

$$\psi(t) = \frac{L_0(1+g_1(t))t}{1-L_0 t}.$$

3 Numerical examples

The numerical examples are presented in this section with $\psi(t) = \frac{L_0(1+g_1(t))t}{1-L_0 t}$.

Example 3.1. Let $D = (-\infty,+\infty)$. Define function f of D by

$$f(x) = \sin(x). \tag{3.1}$$

Then we have for $x^* = 0$ that $L_0 = L = M = 1$. The parameters for methods (1.2) and (1.3) are given in Table 1.

Table 1: Parameters for methods (1.2) and (1.3) of Example 3.1.

Θ	r_A	r_1	r_p	r_2	r	R_2	R
$\frac{2}{3}$	0.6667	0.4444	0.3508	0.2494	0.2494	0.2676	0.2676
$\frac{3}{4}$	0.6667	0.5000	0.3802	0.2687	0.2687	0.2761	0.2761
$\frac{4}{5}$	0.6667	0.5333	0.4000	0.2815	0.2815	0.2815	0.2815
$\frac{5}{6}$	0.6667	0.5556	0.4142	0.2908	0.2908	0.2853	0.2853
$\frac{8}{7}$	0.6667	0.5714	0.4249	0.2977	0.2977	0.2881	0.2881

Example 3.2. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v)=\begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.8)-(2.10) conditions, we get $L_0=e-1$, $L=e$, $M=2$. The parameters for methods (1.2) and (1.3) are given in Table 2.

Table 2: Parameters for methods (1.2) and (1.3) of Example 3.2.

Θ	r_A	r_1	r_p	r_2	r	R_2	R
$\frac{2}{3}$	0.3249	0.1083	0.1465	0.0869	0.0869	0.0911	0.0911
$\frac{3}{4}$	0.3249	0.1625	0.1648	0.0971	0.0971	0.0969	0.0969
$\frac{4}{5}$	0.3249	0.1950	0.1778	0.1044	0.1044	0.1007	0.1007
$\frac{5}{6}$	0.3249	0.2166	0.1875	0.1099	0.1099	0.1035	0.1035
$\frac{8}{7}$	0.3249	0.2321	0.1950	0.1141	0.1141	0.1055	0.1055

Example 3.3. Let $X=Y=C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $D=\overline{U}(0,1)$ and $B(x)=F''(x)$ for each $x\in D$. Define function F on D by

$$F(\varphi)(x)=\varphi(x)-5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.2}$$

We have that

$$F'(\varphi(\xi))(x)=\xi(x)-15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi\in D.$$

Then, we get that $x^*=0$, $L_0=7.5$, $L=15$, $M=2$. The parameters for methods (1.2) and (1.3) are given in Table 3.

Table 3: Parameters for methods (1.2) and (1.3) of Example 3.2.

Θ	r_A	r_1	r_p	r_2	r	R_2	R
$\frac{2}{3}$	0.0667	0.0222	0.0325	0.0196	0.0196	0.0201	0.0201
$\frac{3}{4}$	0.0667	0.0333	0.0362	0.0218	0.0218	0.0213	0.0213
$\frac{4}{5}$	0.0667	0.0400	0.0389	0.0234	0.0234	0.0221	0.0221
$\frac{5}{6}$	0.0667	0.0444	0.0408	0.0246	0.0246	0.0227	0.0227
$\frac{8}{7}$	0.0667	0.0476	0.0423	0.0255	0.0255	0.0231	0.0231

Example 3.4. Returning back to the motivational example at the introduction of this study, we have $L_0=L=146.6629073$, $M=2$. The parameters for methods (1.2) and (1.3) are given in Table 4.

Table 4: Parameters for methods (1.2) and (1.3) of Example 3.2.

Θ	r_A	r_1	r_p	r_2	r	R_2	R
$\frac{2}{3}$	0.0045	0.0045	0.0018	0.0011	0.0011	0.0011	0.0011
$\frac{3}{4}$	0.0045	0.0023	0.0021	0.0012	0.0012	0.0012	0.0012
$\frac{4}{5}$	0.0045	0.0027	0.0023	0.0013	0.0013	0.0013	0.0013
$\frac{5}{6}$	0.0045	0.0030	0.0024	0.0014	0.0014	0.0013	0.0013
$\frac{8}{7}$	0.0045	0.0032	0.0025	0.0014	0.0014	0.0013	0.0013

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