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A Review of Prolate Spheroidal Wave Functions from the Perspective of Spectral Methods

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Abstract. This paper is devoted to a review of the prolate spheroidal wave functions (PSWFs) and their variants from the viewpoint of spectral/spectral-element approximations using such functions as basis functions. We demonstrate the pros and cons over their polynomial counterparts, and put the emphasis on the construction of essential building blocks for efficient spectral algorithms.

AMS subject classifications: 33C47, 33E30, 41A30, 42C05, 65D32, 65N35

Key words: Prolate spheroidal wave functions and their generalisations, time-frequency concentration problem, bandlimited functions, finite Fourier/Hankel transforms, quasi-uniform grids, well-conditioned prolate collocation scheme, prolate-Galerkin method, spectral accuracy.

Contents

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- *"Investigation of the problem of simultaneously concentrating a function and its Fourier transform differed from the other problems I have worked on in two fundamental ways. First, we solved it—completely, easily and quickly. Second, the answer was interesting—even elegant and beautiful."* — Slepian [85] (1983).

- *"The prolate spheroidal wave functions are likely to be a better tool for the design of spectral and pseudo-spectral techniques than the orthogonal polynomials and related functions."* — Xiao, Rokhlin and Yarvin [101] (2001).

- *"The prolate functions are the basis that is "plug-and-play" compatible with finite element or spectral element or other programs that employ Legendre polynomials. The claimed advantage of prolate functions is that they resolve wavy, bandlimited signals with only two points per wavelength, whereas Legendre polynomials and Chebyshev polynomials require a minimum of π degrees of freedom per wavelength."* — Boyd [9] (2013).

1 Introduction

Claude E. Shannon (1916–2001) once posed the question: *To what extent are functions, which are confined to a finite bandwidth, also concentrated in the time domain?* (cf. [60]). This open question was answered by David Slepian (1923-2007) *et al.* at Bell Laboratories in a series of seminal papers dated back to 1960s (see e.g., [53,83,86]). *"We found a second-order differential equation that commuted with an integral operator that was at the heart of the problem,"* as commented by D. Slepian in [85]. This statement best testifies to their findings: the prolate spheroidal wave functions of order zero (PSWFs), being the eigenfunctions of a second-order singular Sturm-Liouville equation, are coincidentally the spectrum of an integral operator related to the finite Fourier transform. It is also from the latter that a collection of remarkable properties of PSWFs was discovered. For example, PSWFs are bi-orthogonal in the sense that they are orthogonal over both a given finite interval and