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A Multigrid Block LU-SGS Algorithm for Euler Equations on Unstructured Grids

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Abstract. We propose an efficient and robust algorithm to solve the steady Euler equations on unstructured grids. The new algorithm is a Newton-iteration method in which each iteration step is a linear multigrid method using block lower-upper symmetric Gauss-Seidel (LU-SGS) iteration as its smoother. To regularize the Jacobian matrix of Newton-iteration, we adopted a local residual dependent regularization as the replacement of the standard time-stepping relaxation technique based on the local CFL number. The proposed method can be extended to high order approximations and three spatial dimensions in a nature way. The solver was tested on a sequence of benchmark problems on both quasi-uniform and local adaptive meshes. The numerical results illustrated the efficiency and robustness of our algorithm.

AMS subject classifications: 65N22, 65N50, 65N55

Key words: Multigrid, block LU-SGS, Euler equations, aerodynamics, airfoil.

1. Introduction

In the last decades, one of the most active research areas in computational aerodynamics has been concerned with the numerical simulation of the complex flow field of aircrafts with practical configuration. Nowadays its rapid development and daily improvement play an important part in accelerating the revolution of aerofoil designing strategies and methods. Because of the hyperbolic nature of Euler equations in the subsonic, transonic and supersonic regimes, many numerical schemes can be chosen to solve the unsteady Euler equations. The finite volume method [11] is one of the most widely used schemes. The nonlinear algebraic system obtained from the finite volume discretization of Euler equations was often solved by certain Newton-iteration. It is a main challenge to develop efficient and robust iterative algorithms for solving the nonlinear algebraic system, especially on unstructured grids. In spite of the difficulties of this problem, remarkable progress

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has been made. The solver developed by Jameson, with a series of customized numerical techniques including multigrid acceleration, local time stepping, implicit residual smoothing and enthalpy damping, on structured grids demonstrated great efficiency with a lot of impressive numerical examples [6–8, 10, 11]. Such highly nonlinear system can now be solved with residual convergent to machine accuracy on current desktop computers within minutes. Among these acceleration techniques, the local preconditioning can be quite effective [13, 14], too, with a judiciously chosen precondition matrix. For the system discretized on unstructured grids, the implicit LU-SGS iterative algorithm has been extensively adopted since it was first introduced by Jameson and Turkel [12]. The LU-SGS method was used as a relaxation method for solving the unfactorized implicit scheme by Yoon and Jameson [22–24]. It was further developed and applied to 3D viscous flow fields by Riger and Jameson [17]. Since then, many authors have applied the LU-SGS method to viscous flows on both structured and unstructured grids [2, 4, 18, 25]. Noticing the special formation of the equations, it is more appropriate to solve the linearized Jacobian matrix block by block. Therefore as a further development of the LU-SGS iteration, Wang [5] proposed a block LU-SGS method together with some numerical examples, converged at a satisfactory speed as expected.

In this paper, we developed a multigrid solver using the block LU-SGS iteration as its smoother. On the unstructured grids, we first discretized the steady Euler equations to obtain the nonlinear algebraic system. Then the nonlinear system was linearized with the standard Newton-iteration. It is popular to regularize the linearized system by adding a local artificial time relaxation. The weight of this time relaxation term was calculated dynamically using a local CFL number. This CFL number is different from the CFL number used in solving a time-dependent conservation law. For a time-dependent conservation law, the essential role of the CFL number is to keep the stability of the numerical schemes. Therefore, it is an $\mathcal{O}(1)$ number to make the time stepping length to be the ratio of the typical mesh size and the maximal wave propagation speed. As one of the basic differences in solving the steady Euler equations, the intermediate state of the solution is out of the main concerns. Only if the iterative algorithm can converge, the CFL number can be chosen as large as possible to achieve a faster convergence rate. Generally, the CFL number should be about $\mathcal{O}(1)$ at the beginning of the iteration as a bootstrap of the total algorithm, and then it can be dynamically increased for better efficiency. A balance between the magnitude of the CFL number and the convergence of the iteration is generally required by maximizing the total convergence rate. Based on such a understanding of the local CFL number choosing strategy, we will not use the standard regularization in which a local artificial time relaxation term is added into the Jacobian matrix of the Newton-iteration. Instead, we used a residual related regularization,i.e. $\alpha \|RHS\|_{l^1}$, where RHS is the residual of the linearized system on each grid cell. The magnitude of the cell residual can locally quantify how close to the steady state the flow field is. Therefore it is quite natural to require the local CFL number to be dependent on the local residual. Due to the same scaling between a norm of the local residual and local grid cell size, the regularization term can be simply set as a constant times of a norm of the local residual. With this local residual dependent regularization, our algorithm can automatically choose a moderate local CFL number so as