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Grothendieck Property for the Symmetric Projective Tensor Product

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Abstract. For a Banach space E, we give sufficient conditions for the Grothendieck property of $\hat{\otimes}_{n,s,\pi}E$, the symmetric projective tensor product of E. Moreover, if E^* has the bounded compact approximation property, then these sufficient conditions are also necessary.

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1 Results

Recall that a Banach space is said to have the *Grothendieck property* (GP in short) if every weak* convergent sequence in its dual is weakly convergent (see, e.g., [6, 10]). González and Gutiérrez in [8] showed that if $n\geqslant 2$ then $\hat{\otimes}_{n,s,\pi}E$, the symmetric projective tensor product of a Banach space E, has GP if and only if $\hat{\otimes}_{n,s,\pi}E$ is reflexive. In this short paper, we show that for any $n\geqslant 1$, if E has GP and every scalar-valued continuous n-homogeneous polynomial on E is weakly continuous on bounded sets, then $\hat{\otimes}_{n,s,\pi}E$ has GP. Moreover, if E^* has the bounded compact approximation property, then these sufficient conditions for $\hat{\otimes}_{n,s,\pi}E$ having GP are also necessary.

Let E and F be Banach spaces over $\mathbb{R}$ or $\mathbb{C}$ and let n be a positive integer. A map $P:E\to F$ is said to be an n-homogeneous polynomial if there is a symmetric n-linear operator T from $E\times\cdots\times E$ (a product of n copies of E) into F such that $P(x)=T(x,\ldots,x)$. Indeed, the symmetric n-linear operator $T_P:E\times\cdots\times E\to F$ associated to P can be given by the *Polarization Formula*:

$$T_P(x_1,\ldots,x_n)=\frac{1}{2^n n!}\sum_{\epsilon_i=\pm 1}\epsilon_1\cdots\epsilon_n P\Big(\sum_{i=1}^{n}\epsilon_i x_i\Big),\quad \forall x_1,\ldots,x_n\in E.$$

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Let $\mathcal{P}(^nE;F)$ denote the space of all continuous n-homogeneous polynomials from E into F with its norm

$$\|P\|=\sup\{\|P(x)\|:x\in E,\|x\|\leqslant 1\},$$

and let $\mathcal{P}_w(^nE;F)$ denote the subspace of all P in $\mathcal{P}(^nE;F)$ that are weakly continuous on bounded sets. In particular, if $F=\mathbb{R}$ or $\mathbb{C}$, then $\mathcal{P}(^nE;F)$ and $\mathcal{P}_w(^nE;F)$ are simply denoted by $\mathcal{P}(^nE)$ and $\mathcal{P}_w(^nE)$ respectively.

Let $\otimes_n E$ denote the *n-fold algebraic tensor product* of E. For $x_1\otimes\cdots\otimes x_n\in\otimes_n E$, let $x_1\otimes_s\cdots\otimes_s x_n$ denote its symmetrization, that is,

$$x_1\otimes_s\cdots\otimes_s x_n=\frac{1}{n!}\sum_{\sigma\in\pi(n)}x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(n)},$$

where $\pi(n)$ is the group of permutations of $\{1,\dots,n\}$. Let $\otimes_{n,s}E$ denote the *n-fold symmetric algebraic tensor product* of E, that is, the linear span of $\{x_1\otimes_s\cdots\otimes_s x_n:x_1,\dots,x_n\in E\}$ in $\otimes_n E$. It is known that each $u\in\otimes_{n,s}E$ has a representation $u=\sum_{k=1}^m\lambda_k x_k\otimes\cdots\otimes x_k$ where $\lambda_1,\dots,\lambda_m$ are scalars and $x_1,\dots,x_m$ are vectors in E. Let $\hat{\otimes}_{n,s,\pi}E$ denote the *n-fold symmetric projective tensor product* of E, that is, the completion of $\otimes_{n,s}E$ under the *symmetric projective tensor norm* on $\otimes_{n,s}E$ defined by

$$\|u\|=\inf\left\{\sum_{k=1}^m|\lambda_k|\cdot\|x_k\|^n:x_k\in E,u=\sum_{k=1}^m\lambda_k x_k\otimes\cdots\otimes x_k\right\},\quad u\in\otimes_{n,s}E.$$

For each n-homogeneous polynomial $P:E\to F$, let $A_P:\otimes_{n,s}E\to F$ denote its linearization, that is,

$$A_P(x\otimes\cdots\otimes x)=P(x),\quad\forall x\in E.$$

Then under the isometry: $P\to A_P$,

$$\mathcal{P}(^nE;F)=\mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F),$$

where $\mathcal{L}(\hat{\otimes}_{n,s,\pi}E;F)$ is the space of all continuous linear operators from $\hat{\otimes}_{n,s,\pi}E$ to F. In particular,

$$\mathcal{P}(^nE)=(\hat{\otimes}_{n,s,\pi}E)^*,$$

where $(\hat{\otimes}_{n,s,\pi}E)^*$ is the topological dual of $\hat{\otimes}_{n,s,\pi}E$.

For the basic knowledge about homogeneous polynomials and symmetric projective tensor products, we refer to [7,12,13].

For a Banach space E, let E^* denote its dual and E^{**} denote its second dual. For every $P\in\mathcal{P}(^nE)$, let $\widetilde{P}\in\mathcal{P}(^nE^{**})$ denote the *Aron-Berner extension* of P (see, e.g., [1,5]). To obtain $\hat{\otimes}_{n,s,\pi}E$ having GP, we first need the following lemma, which is a special case of [9, Corollary 5].

Lemma 1.1. ([9]) *Let $P_k,P\in\mathcal{P}_w(^nE)$ for each $k\in\mathbb{N}$. Then $\lim_k P_k=P$ weakly in $\mathcal{P}_w(^nE)$ if and only if $\lim_k\widetilde{P}_k(z)=\widetilde{P}(z)$ for every $z\in E^{**}$.*