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Existence and Orbital Stability of Solitary-Wave Solutions for Higher-Order BBM Equations

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Abstract. This paper discusses the existence and stability of solitary-wave solutions of a general higher-order Benjamin-Bona-Mahony (BBM) equation, which involves pseudo-differential operators for the linear part. One of such equations can be derived from water-wave problems as second-order approximate equations from fully nonlinear governing equations. Under some conditions on the symbols of pseudo-differential operators and the nonlinear terms, it is shown that the general higher-order BBM equation has solitary-wave solutions. Moreover, under slightly more restrictive conditions, the set of solitary-wave solutions is orbitally stable. Here, the equation has a nonlinear part involving the polynomials of solution and its derivatives with different degrees (not homogeneous), which has not been studied before. Numerical stability and instability of solitary-wave solutions for some special fifth-order BBM equations are also given.

AMS subject classifications: 35Q35, 76B15, 76B25, 35Q35

Key words: Higher-order BBM equations, solitary-wave solutions, orbital stability.

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1 Introduction

The theory of solitary waves on water of finite depth has a long history, which was started by a remarkable discovery of solitary waves in a canal by Scott Russell [28]. The formal asymptotic theory under long-wave assumption was obtained by Boussinesq [13] and Korteweg and de Vries [20] and the famous Korteweg-de Vries (KdV) equation was derived. Assume that dissipation and surface tension effects are ignored and water flow is

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considered as a two-dimensional flow with a coordinate system (x,y) chosen so that y is in the opposite direction of gravity and x-axis is the wave propagation direction. If h_0 is the depth of channel over a flat horizonal bottom at infinity, in terms of $\eta(x,t)=h(x,t)-h_0$, where $h(x,t)$ is the height of free surface at location x and time t, the KdV equation has a form

$$\eta_t+\eta_x+(3/2)\eta\eta_x+(1/6)\eta_{xxx}=0.$$

Here $x=(\tilde{x}/h_0),t=\sqrt{g/h}\tilde{t},\eta=(\tilde{\eta}/h_0)$ with g the gravitational acceleration constant and $\tilde{x},\tilde{t},\tilde{\eta}$ are non-dimensional variables. Peregrine [26] and Benjamin *et al.* [5] derived an equivalent equation, called the regularized long-wave (RLW) equation or Benjamin-Bona-Mahony (BBM) equation,

$$\eta_t+\eta_x+(3/2)\eta\eta_x-(1/6)\eta_{xxt}=0. \tag{1.1}$$

Those model equations provide the first-order approximations of the fully nonlinear governing equations under the small-amplitude and long-wave assumptions.

Mathematical theory of the KdV and BBM equations was initiated in a half century ago and has been a very active research topic in partial differential equations. Numerous literatures have been devoted to the well-posedness problems of those type of equations. Here, we only focus on the study of the BBM equations. Benjamin, *et al.* [5] first gave a rigorous discussion on the well-posedness of the BBM equations. They showed that the initial-value problem of (1.1) is well-posedness in certain Banach spaces and then compared the solutions of the KdV equation with those of the BBM equations. The generalized BBM equations were discussed in [1,2]. Existence and stability of solitary-wave solutions for some BBM-type of equations were discussed in [3,8,29,33] and references therein.

This paper mainly concentrates on general higher-order BBM-type equations based upon the model equations derived in [7]. If the parameters are in the Boussinesq region and the second-order terms in the formal asymptotic expansions of the solutions for the water-wave problems are kept in the approximation, the following non-dimensionalized and scaled higher-order BBM equation was derived [7],

$$\begin{aligned}\eta_t+\eta_x-(1/6)\eta_{xxt}+\delta_1\eta_{xxxxt}+\delta_2\eta_{xxxxx}+(3/4)(\eta^2)_x\\+\gamma(\eta^2)_{xxx}-(1/12)(\eta_x^2)_x-(1/4)(\eta^3)_x=0,\end{aligned} \tag{1.2}$$

where δ_1,δ_2 and γ are constants. (1.2) was derived from a general Boussinesq system obtained in [9,10] under the assumption of one-way wave propagation. It was pointed out in [7] that δ_1 has to be positive in order to obtain the local well-posedness of the corresponding initial-value problems, while the global well-posedness can be proved if $\gamma=1/12$. In this case, (1.2) has a Hamiltonian structure and some conserved quantities can be derived. Since we are interested in the existence of solitary-wave solutions of (1.2) and the stability of these solitary waves, the global existence of the solutions for (1.2) is needed, thereby $\delta_1>0$ and $\gamma=1/12$ are assumed in the paper. Although the existence of