Journal of Computational Mathematics, Vol.12, No.4, 1994, 298–311.

CONVERGENCE OF LEGENDRE METHODS FOR NAVIER-STOKES EQUATIONS*

Chen Zheng-ting
(Luoyang Educational Institute, Luo Yang, Henan, China)
E Wei-nan
(Department of Mathematics, University of California, Los Angeles, USA)

Abstract

This paper is concerned with spectral type of methods using Legendre polynomials. Both Galerkin and collocation approximations for the Navier-Stokes equations are considered and their rates of convergence are obtained. As a consequence, it is shown that these methods achieve spectral accuracy if the solutions to the Navier-Stokes equations are smooth.

1. Introduction

In this paper we study spectral type of methods based on Legendre polynomials. We prove stability and convergence results for these methods using energy estimates. The convergence results we obtained are nearly optimal in the sense that the error estimates for the numerical solution is of the same order as the error estimates in approximation theory [7, 14]. A trivial consequence is that these methods are indeed spectrally accurate.

Spectral methods have been used quite extensively in the past two decades. Because of their high resolution power, these methods receive particular attention in simulating incompressible flows in high Reynolds number. We refer to [6] for a review of applications of the spectral methods in the computation of fluid flows and for the computational issues involved in these applications.

There has also been quite extensive work on the analysis of these methods. The basic stability and convergence results are summarized in [10] for linear hyperbolic problems. The approximation theory in the setting of Sobolev spaces for projections and interpolations using Fourier, Legendre and Chebyshev polynomials are presented in [7]. For the steady Navier-Stokes equations on simple geometries, a complete theory has been established by Canuto, Maday, Quarteroni and their co-workers [3, 4, 5, 6]. For the time-dependent Navier-Stokes equations, previous work has been restricted to Fourier methods with periodic boundary conditions [8, 11, 12]. The present paper extends these results to Legendre methods.

* Received May 12, 1994.

The key to convergence results and error estimates is stability. Roughly speaking, for the steady Navier-Stokes equations, the stability condition amounts to the inf-sup condition; whereas for the unsteady problem, the stability condition amounts to some unform (independent of the discretization parameter) a priori estimates for the numerical solutions. If we have uniform a priori estimates under sufficiently strong norms, e.g. the $W^{1,\infty}$ norm, convergence and error estimates follow as a consequence of the Gronwall inequality. If the numerical method is linearly stable, then we have a uniform L^2 estimate. If the method is sufficiently accurate, then we also get uniform control of higher norms and L^∞ norms using inverse inequality. If the method is not accurate enough, we may still obtain these estimates by applying Strang's trick. Spectral methods are high order methods and Strang's argument can be replaced by standard smoothness assumptions on the exact solutions.

There is a link between the stability estimates for the steady and unsteady problems. This is explored in [9] for general parabolic equations. For the Navier-Stokes equations, the idea can be summarized as follows. The inf-sup condition for the Stokes equations usually implies some estimates for the resolvent of the Stokes equations. These estimates can then be used to prove stability for the time-dependent Stokes equations using semi-group formulations. Finally stability for the nonlinear time-dependent Navier-Stokes equations can be obtained using the ideas outlined in the last paragraph.

In this paper we more or less follow the argument indicated above, although we will not use semi-group formulations explicitly. The important difference between this work and the work of Canuto, Maday and Quarteroni is that we are concerned only with the approximation of velocity, not pressure. For this purpose it is not necessary to identify all the spurious modes in pressure, whereas in their work spurious modes in pressure are directly linked to the inf-sup condition.

This paper is organized as follows. In the next section we study the Legendre-Galerkin method. After a brief introduction of the method, we summarize the results on the Stokes problem. We then use these results to prove the error estimates for the full time-dependent Navier-Stokes equations. Similar results are proved for the Legendre-collocation method in section 3.

2. The Legendre-Galerkin method

2.1. Preliminaries on the Legendre-Galerkin method

We will use standard notations for Sobolev spaces (see [1]). We use $\|\cdot\|_{m,p}$ to denote the $W^{m,p}$ norm and $\|\cdot\|_m$ to denote the H^m norm. A generic point in the plane R^2 will be denoted by $x = (x_1, x_2)$. $\Omega = (-1,1) \times (-1,1)$. For $u(x,t) \in C([0,T], H^s(\Omega))$, we let

$$|||u|||_s = \max_{0\le t\le T}\{||u(\cdot,t)||_s\}. \tag{2.1}$$