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# THE SOLVABILITY CONDITIONS FOR THE INVERSE PROBLEM OF MATRICES POSITIVE SEMIDEFINITE ON A SUBSPACE*[1)]

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**Abstract**

This paper studies the following two problems:

**Problem I.** Given $X, B \in R^{n\times m}$, find $A \in P_{s,n}$, such that $AX = B$, where $Ps, n = \{A \in SR^{n\times n} | x^T Ax \geq 0, \forall S^T x = 0$, for given $S \in R_p^{n\times p}\}$.

**Problem II.** Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$, such that $\|A^* - \hat{A}\| = \inf_{A\in S_E} \|A^* - A\|$ where $S_E$ denotes the solution set of Problem I.

The necessary and sufficient conditions for the solvability of Problem I, the expression of the general solution of Problem I and the solution of Problem II are given for two cases. For the general case, the equivalent form of conditions for the solvability of Problem I is given.

Inverse problems for real symmetric matrices and symmetric nonnegative definite matrices have been studied in [1], [2]. The conditions for the existence of a solution the expression of the general solution and optimal approximate solution have been given. This paper studies the inverse problem of one kind of matrices between the above two kinds of matrices — matrices positive semidefinite on a subspace. The conditions for the existence of a solution, the expression of the general solution and the optimal approximate solution are given.

In this paper, $R^{n\times m}$ denotes the set of all real $n \times m$ matrices, $R_r^{n\times m}$ its subset whose elements have rank $r$, $SR^{n\times n}$ the set of all real $n \times n$ symmetric matrices, and $SR_0^{n\times n}$ the set of all $n \times n$ symmetric nonnegative definite matrices. $I_k$ denotes the $k \times k$ unit matrix, $R(A), N(A), A^+$ denote column space, null space and Moore-Penrose generalized inverse matrix of matrix $A$ respectively. $\|\cdot\|$ is Frobenius norm, and $A \geq 0$ represents that $A$ is a symmetric nonnegative definite matrix.

Let $P_{s,n} = \{A \in SR^{n\times n} | x^T Ax \geq 0, \forall S^T x = 0$, for given $S \in R_p^{n\times p}\}$.

**Problem I.** Given $X, B \in R^{n\times m}$, find $A \in P_{s,n}$ such that $AX = B$.

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**Problem II.** Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$, such that $\|A^* - \hat{A}\| = \inf_{A\in S_E} \|A^* - A\|$, where $S_E$ denotes the solution set of Problem I.

We will introduce some Lemmas in **1**, and give the conditions for the solvability, the expression of the general solution of Problem I and optimal approximate solution of Problem II for two cases respectively in **2** and **3**. We will give an equivalent form of conditions for the solvability of Problem I for the general case in **4** and suggest a problem which is worth investigating.

## 1. Some Lemmas

Suppose $S \in R_p^{n\times n}$. We construct the orthogonal triangular decomposition for $S$:

$$S = Q^T \begin{pmatrix} 0 \\ L \end{pmatrix} = Q_2^T L, \tag{1.1}$$

where $Q^T = (Q_1^T, Q_2^T)$ is an $n \times n$ orthogonal matrix, and $Q_1^T \in R^{n\times(n-p)}$ $L$ is a $p \times p$ nonsingular lower triangular matrix. Then

$$R(S) = R(Q_2^T), N(S^T) = R(Q_1^T). \tag{1.2}$$

**Lemma 1.** *Suppose $S$ has factorization in the form* (1.1). *Let*

$$QAQ^T = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, A_{11} \in R^{(n-p)\times(n-p)}. \tag{1.3}$$

*Then $A \in Ps, n \Leftrightarrow A_{11} \in SR_0^{(n-p)\times(n-p)}, A_{21} = A_{12}^T, A_{22} = A_{22}^T$.*

*Proof.* **Sufficiency.** Because $A \in P_{s,n}$ it is evident that $A_{22} = A_{22}^T, A_{12} = A_{21}^T, A_{11} = A_{11}^T, \forall y \in R^{(n-p)}$. Then $Q_1^T y \in N(S^T)$. From $A \in P_{s,n}$, we get $y^T A_{11} y = (Q_1^T y)^T A (Q_1^T y) \geq 0$, i.e. $A_{11} \in SR_0^{(n-p)\times(n-p)}$.

**Necessity.** It is evident that $A = A^T$, and for any given $x \in N(S^T)$, there exists $y \in R^{n-p}$ satisfying $x = Q_1^T y$, i.e. $Qx = \binom{y}{0}$.

Thus we have $x^T A x = x^T Q^T Q A Q^T Q x = y^T A_{11} y \geq 0$, where $A \in P_{s,n}$.

**Lemma 2.** *Suppose $X, B \in R^{n\times m}$, and $S$ has factorization in the form* (1.1). *Write*

$$QX = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{1.4}$$

$X_1 = Q_1 X$, $B_1 = Q_1 B \in R^{(n-p)\times m}$, $X_2 = Q_2 X$, $B_2 = Q_2 B \in R^{p\times m}$.

*If $R(X) \subseteq R(S)$, then we have* (i) $X_2^T B_2 = B_2^T X_2 \Leftrightarrow X^T B = B^T X$; (ii) $B_1 X_2^+ X_2 = B_1$ *and* $B_2 X_2^+ X_2 = B_2 \Leftrightarrow BX^+X = B$.

*If $R(X) \subseteq N(S^T)$, then we have* (iii) $X_1^T B_1 = B_1^T X_1 \geq 0 \Leftrightarrow X^T B = B^T X \geq 0$; (iv) $\operatorname{rank}(X_1^T B_1) = \operatorname{rank}(B_1) \Leftrightarrow \operatorname{rank}(B_1) = \operatorname{rank}(X^T B)$; (v) $B_1 X_1^+ X_1 = B_1, B_2 X_1^+ X_1 = B_2 \leftrightarrow BX^+X = B$.

*Proof.* Because $X^T B = X^T Q^T Q B = X_1^T B_1 + X_2^T B_2$, if $R(X) \subseteq R(S) = R(Q_2^T)$, then we have $X_1 = 0$. Therefore (i) holds. Furthermore, because $X^T X = X^+ Q^T Q X =$