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THE PRIMAL-DUAL POTENTIAL REDUCTION ALGORITHM FOR POSITIVE SEMI-DEFINITE PROGRAMMING*[1)]

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Abstract

In this paper we introduce a primal-dual potential reduction algorithm for positive semi-definite programming. Using the symetric preserving scalings for both primal and dual interior matrices, we can construct an algorithm which is very similar to the primal-dual potential reduction algorithm of Huang and Kortanek [6] for linear programming. The complexity of the algorithm is either $O(n\log(X^0 \bullet S^0/\epsilon)$ or $O(\sqrt{n}\log(X^0 \bullet S^0/\epsilon)$ depends on the value of ρ in the primal-dual potential function, where X^0 and S^0 is the initial interior matrices of the positive semi-definite programming.

Key words: Positive semi-definite programming, Potential reduction algorithms, Complexity.

1. Introduction

In this paper, we consider the following standard form of positive semi-definite programming:

(PSP) Minimize $C \bullet X$

Subject to $A_i X = b_i, \quad i = 1, ..., m, \quad X \succeq 0,$

where $C, X \in \mathcal{M}^n$, $A_i \in \mathcal{M}^n$, $i = 1, ..., m$, and $b \in R^m$. Here $\mathcal{M}^n$ denotes the set of symetric matrices in $R^{n\times n}$. Let $\mathcal{M}^n_+$ denotes the set of positive semi-definite matrices in $\mathcal{M}^n$ and $\mathcal{M}^n_{++}$ denotes the set of positive definite matrices in $\mathcal{M}^n$. We call $\mathcal{M}^n_{++}$ the interior of $\mathcal{M}^n$. The notation $X \succeq 0$ means that $X \in \mathcal{M}^n_+$, and $X \succ 0$ means that $X \in \mathcal{M}^n_{++}$. If $X \succ 0$ satisfies all equations in (PSP), it is called a primal interior feasible solution. The $\bullet$ operation is the matrix inner product

$$A \bullet B := trA^T B = \sum_{i,j} A_{ij} B_{ij}.$$

The dual problem to (PSP) can be written as:

(PSD) Maximize $b^T y$

Subject to $S = C - \sum_{i=1}^m y_i A_i, \qquad S \succeq 0,$

where $S \in \mathcal{M}^n$, $y \in R^m$. If a point $(y, S \succ 0)$ satisfies all equations in (PSD), it is called a dual interior feasible solution.

Define the Frobenius norm, or the l_2 norm, of the matrix $X \in \mathcal{M}^n$ by

$$\|X\| := \|X\|_f = \sqrt{X \bullet X} = \sqrt{\sum_{j=1}^n (\lambda_j(X))^2},$$

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where $\lambda_j(X)$ is the jth eigenvalue of X, and the l_∞ norm of X by

$$\|X\|_\infty := \max_{j\in\{1,\ldots,n\}}\{|\lambda_j(X)|\}.$$

Since semi-definite programming has many applications in combinatorial optimization, control theory, statistics, etc., it becomes a hot research topic in optimization over the last decade. Many interior point algorithms have been developed to solve the semi-definite programming. The primal potential reduction algorithms were developed by Alizadeh [1], Nesterov and Nemirovskii [7], Ye [13], etc.; the primal-dual potential reduction algorithms using symetric matrix scaling were proposed by Nesterov and Todd [8], Kojima, Shindoh and Hara [4], among others. In this paper, we introduce a primal-dual potential reduction algorithm, which uses seperate matrices scaling, for above positive semi-definite programming. This kind of scaling has been used extensively in interior point algorithms for linear programming (e.g.,Kojima et la' [4], Huang and Kortanek [7],[8], Gonzaga and Todd [6]). To the best of our knowledge we have not seen a paper on interior-point algorithms for semi-definite programming which uses such seperate matrices scaling.

To measure the progress of the algorithm, we will use the following primal-dual potential function

$$\phi(X,S) = \rho \log X \bullet S - \log det XS. \tag{1}$$

The reduction in potential function is controlled by the length of projection of the search directions. In this paper we show that the length of projection is bounded below by 1/4 if $\rho = n+\sqrt{n}$. Furthermore, we prove that the length is greater than or equal to one if $\rho \geq 2n+\sqrt{2n}$. These results are the extentions of the results in Huang and Kortanek [8] for linear programming to semi-definite programming.

2. The Search Directions

The gradient matrices of (1) are

$$\nabla\phi_X(X,S) = \frac{\rho}{X\bullet S}S - X^{-1}, \tag{2}$$

$$\nabla\phi_S(X,S) = \frac{\rho}{X\bullet S}X - S^{-1}. \tag{3}$$

Let $A = (a_1,\cdots,a_n)$ be any $n\times n$ matrix, where a_j $(j=1,\cdots,n)$ are columns of A, we define the vector of A as follows:

$$vec(A) = (a_1^T,\cdots,a_n^T)^T.$$

Then define

$$\mathcal{A} = \begin{pmatrix} vec(A_1)^T \\ vec(A_2)^T \\ \vdots \\ vec(A_m)^T \end{pmatrix}.$$

Also define the operator $\mathcal{A}:\mathcal{M}^n \to R^m$ as follows:

$$\mathcal{A}X = \mathcal{A}vec(X).$$

Furthermore

$$\mathcal{A}^T y = \sum_{i=1}^m y_i A_i.$$

Given a primal-dual interior feasible solution (X^0,y^0,S^0) such that $\mathcal{A}X^0 = b$ and $S^0 = C - \mathcal{A}^T y^0$, and a $\beta\in(0,1)$, we consider the following homogeneous minimization problem:

(HPSD) min $\nabla\phi_X(X^0,S^0)\bullet\triangle X + \nabla\phi_S(X^0,S^0)\bullet\triangle S$

s.t. $\mathcal{A}\triangle X = 0$

$\mathcal{A}^T\triangle y + \triangle S = 0$

$\|(X^0)^{-.5}\triangle X(X^0)^{-.5}\|^2 + \|(S^0)^{-.5}\triangle S(S^0)^{-.5}\|^2 \leq \beta^2 < 1.$