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THE INVERSE PROBLEM OF CENTROSYMMETRIC MATRICES WITH A SUBMATRIX CONSTRAINT [*1)]

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Abstract

By using Moore-Penrose generalized inverse and the general singular value decomposition of matrices, this paper establishes the necessary and sufficient conditions for the existence of and the expressions for the centrosymmetric solutions with a submatrix constraint of matrix inverse problem $AX = B$. In addition, in the solution set of corresponding problem, the expression of the optimal approximation solution to a given matrix is derived.

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Key words: Matrix norm, Centrosymmetric matrix, Inverse problem, Optimal approximation.

1. Introduction

Inverse eigenvalue problem has widely used in control theory [1, 2], vibration theory [3, 4], structural design [5], molecular spectroscopy [6]. In recent years, many authors have been devoted to the study of this kind of problem and a serial of good results have been made [7, 8, 9]. Centrosymmetric matrices have practical application in information theory, linear system theory and numerical analysis theory. However, inverse problems of centrosymmetric matrices, specifically centrosymmetric matrices with a submartix constraint, have not be concerned with. In this paper, we will discuss this problem.

Denote the set of all n-by-m real matrices by $R^{n\times m}$ and the set of all n-by-n orthogonal matrices in $R^{n\times n}$ by $OR^{n\times n}$. Denote the column space, the null space, the Moore-Penrose generalized inverse and the Frobenius norm of a matrix A by $R(A), N(A), A^+$ and $\|A\|$, respectively. I_n denotes the $n\times n$ unit matrix and S_n denotes the $n\times n$ reverse unit matrix. We define the inner product in space $R^{n\times m}$ by

$$\langle A, B\rangle = trace(B^H A),\ \forall A, B \in R^{n\times m}.$$

Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix generated by this inner product space is the Frobenius norm. For $A = (a_{ij}), B = (b_{ij}) \in R^{n\times m}$, we using the notation $A * B = (a_{ij}b_{ij}) \in R^{n\times n}$ denotes the Hadmard product of matrices A and B.

Definition 1 [10,15]**.** *$A = (a_{ij}) \in R^{n\times n}$ is termed a Centrosymmetric matrix if*

$$a_{ij} = a_{n+1-j,n+1-i} \quad i, j = 1, 2, \ldots, n.$$

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The set of $n \times n$ centrosymmetric matrices denoted by $CSR^{n\times n}$.

In this paper, we consider the following two problems:

Problem 1. Given $X, B \in R^{n\times m}, A_0 \in R^{r\times r}$, find $A \in CSR^{n\times n}$ such that

$$AX = B, \quad A_0 = A([1, r]),$$

where $A([1, r])$ is a leading $r \times r$ principal submatrix of matrix A.

Problem 2. Given $A^* \in R^{n\times n}$, find $\hat{A} \in S_E$ such that

$$\|A^* - \hat{A}\| = \min_{A\in S_E} \|A^* - A\|,$$

where S_E is the solution set of Problem 1.

In Section 2, we first discuss the structure of the set $CSR^{n\times n}$, and then present the solvability conditions and provide the general solution formula for Problem 1. In Section 3, we first show the existence and uniqueness of the solution for Problem 2, and then derive an expression of the solution when the solution set S_E is nonempty. Finally, in section 4, we first give an algorithm to compute the solution to Problem 2, and then give a numerical example to illustrate the results obtained in this paper are correction.

2. Solving Problem 1

We first characterize the set of all centrosymmetric matrices. For all positive integers k, let

$$D_{2k} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix}, D_{2k+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix}. \tag{1}$$

Clearly, D_n is orthogonal for all n.

Lemma 1 [10]**.** *$A \in CSR^{n\times n}$ if and only if $A = S_n A S_n$.*

Lemma 2. *$A \in CSR^{n\times n}$ if and only if A can be expressed as*

$$A = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T, \tag{2}$$

where $A_1 \in R^{(n-k)\times(n-k)}, A_2 \in R^{k\times k}$.

Proof. We only prove the case for $n = 2k$, the case for $n = 2k+1$ can be discussed similarly. Partition the matrix A into the following form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, A_{11}, A_{22} \in R^{k\times k}.$$

If $A \in CSR^{2k\times 2k}$, then we have from Lemma 1 that

$$\begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} 0 & S_k \\ S_k & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$