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THE LOWER APPROXIMATION OF EIGENVALUE BY LUMPED MASS FINITE ELEMENT METHOD *1)

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Abstract

In the present paper, we investigate properties of lumped mass finite element method (LFEM hereinafter) eigenvalues of elliptic problems. We propose an equivalent formulation of LFEM and prove that LFEM eigenvalues are smaller than the standard finite element method (SFEM hereinafter) eigenvalues. It is shown, for model eigenvalue problems with uniform meshes, that LFEM eigenvalues are not greater than exact solutions and that they are increasing functions of the number of elements of the triangulation, and numerical examples show that this result equally holds for general problems.

Mathematics subject classification: 65N30.
Key words: Lumped mass, Eigenvalue, Min-max principle, Finite element.

1. Introduction

The finite element method has been widely and successfully applied to both boundary value and eigenvalue problems for a solid continuum. In the boundary value problem, it has been shown that if the interpolation functions satisfy certain criteria [7],the finite element solution will converge to the exact solution as the size of the element is diminished. The convergence likewise occurs for the eigenvalue problems (Ref. [7, 3, 5, 8, 1, 2] and references therein).

There exist two finite element methods for solving eigenvalue problems, one is SFEM (Ref.[1, 2]), the other is LFEM. LFEM has been extensively applied to science and engineering computations because of its simplicity. LFEM in particular can largely simplify the computation of generalized eigenvalue problems (Ref.[3, 5]). The convergence of LFEM for eigenvalue problems was established by Tong.et al [8] and Strang and Fix [7]. Strang and Fix in [7] gave an error expansion of LFEM eigenvalue for one dimensional Neumann problem, the error expansion of LFEM eigenvalue for one dimensional Dirichlet problem was presented in [1], where some comments on the asymptotic lower bound when h tends to zero for the problem therein were also given. Tong.et al in [8] proved that LFEM didn't lose the accuracy of approximation compared with SFEM as long as proper lumped mass method was chosen. The concept of lower approximations of eigenvalues was first introduced in [6]. Numerical experiments therein indicated that LFEM eigenvalues are lower approximations to the exact ones, however the analysis therein is not rigorous.

In the present paper, we investigate properties of LFEM eigenvalues. It is well known that SFEM eigenvalues approximate exact solutions from above [1, 2] and that they are in some sense decreasing functions of the number of elements of the partition of the domain considered, on the contrary, what we are interested in is to show that LFEM eigenvalues approximate exact solutions from below and that they are increasing functions of the number of elements. For model eigenvalue problems with uniform meshes, we provide a rigorous analysis for these properties. For general problems, we propose an equivalent form for LFEM and show that

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LFEM eigenvalues are smaller than SFEM eigenvalues, and the final numerical experiments demonstrate that LFEM eigenvalues are exactly increasing functions of the number of elements, then we can safely assert LFEM eigenvalues are not greater than exact ones. The paper is organized as following. In section 2, we recall the weak formulation of the elliptic eigenvalue problem. LFEM and its equivalent formulation will be described in section 3, and in section 4 we shall show that LFEM eigenvalues are not greater than SFEM eigenvalues, as applications, we shall also prove, for model problems with uniform meshes, that LFEM eigenvalues are lower approximations in the same section. Numerical results are illustrated in section 5. This paper ends with section 6, which brings our final remark.

2. Variational Formulation of Eigenvalue Problem

We shall consider the eigenvalue problem in the divergence form which is read as

$$\begin{cases} Lu = -\frac{\partial}{\partial x_j}(a_{ij}\frac{\partial}{\partial x_i}u) + c(x)u = \lambda\rho u \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset R^d$ is a bounded open domain with smooth enough boundary $\partial\Omega$, $a_{ij}(x)$ have local integrable derivatives, $c(x) \in L^\infty(\Omega)$ and $c(x) \geq 0$. We assume L is a strict elliptic operator.

For the eigenvalue problem $Lu = \lambda\rho u$, there are two variational formulation forms: Rayleigh quotient and weak form, which are expressed as, respectively

$$R(v) = \frac{a(v,v)}{(\rho v, v)}$$

$$a(u,v) = \lambda(\rho u, v) \qquad \forall v \in H_0^1$$

where

$$a(u,v) = \int_\Omega [a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + c(x)uv]dx \text{ and } (\rho u, v) = \int_\Omega \rho uv dx$$

G.Strang and G.J.Fix in [7] show that the two forms are equivalent, in particular, one has

Lemma 2.1 (min-max principle). *Let λ_l be l-th eigenvalue of problem (2.1), it holds that*

$$\lambda_l = \min_{s_l} \max_{v \in s_l} R(v) \tag{2.2}$$

where s_l is any l-dimension subspace of $H_0^1(\Omega)$.

Let A and B be $n \times n$ real symmetric and positive definite matrixes, discrete counterpart of Rayleigh quotient with respect to A and B can be stated as

$$R(x) = \frac{x^T A x}{x^T B x} \quad (x \in R^n, x \neq 0)$$

where x^T denotes the transpose of n-dimensional vector x, then one has

Lemma 2.2. *Let λ_k be k-th generalized eigenvalue of A with respect to B, then*

$$\lambda_k = \min_{V_k} \max_{x \in V_k, x \neq 0} R(x) \tag{2.3}$$

where V_k is any k-dimension subspace of R^n.