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STABILITY ANALYSIS AND APPLICATION OF THE EXPONENTIAL TIME DIFFERENCING SCHEMES [*1)]

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

Exponential time differencing schemes are time integration methods that can be efficiently combined with spatial spectral approximations to provide very high resolution to the smooth solutions of some linear and nonlinear partial differential equations. We study in this paper the stability properties of some exponential time differencing schemes. We also present their application to the numerical solution of the scalar Allen-Cahn equation in two and three dimensional spaces.

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1. Introduction

For many time dependent partial differential equations with smooth or regularized solutions, spectral and pseudo–spectral methods have been shown to provide remarkably effective spatial discretization. The application of these discretization methods often results in systems of stiff ordinary differential equations (ODEs) in time and thus make the efficient and stable time integration scheme very essential.

The subject of solving stiff systems has been well studied in the literature [11], including, for instance, linearly implicit methods [1], semi-implicit methods [4], time-splitting methods, projection methods [8], multiscale methods [7, 9], integrating factors (IF), and the exponential time differencing (ETD) methods [6, 12]. In this paper, of particular interests to us are the ETD schemes and their modifications which have been shown to perform extremely well in solving various one dimensional diffusion type problems [12]. ETD schemes have also been used by other authors under different names [2, 15]. In essence, for nonlinear time dependent equations, the ETD schemes provide a systematic coupling of the explicit treatment of nonlinearities and the implicit *and possibly exact* integration of the stiff linear parts of the equations, while achieving high accuracy and maintaining good stability.

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It is natural to expect that ETD like schemes can be very useful in phase field computations that involve more complex physics [4, 16]. In fact, some preliminary numerical studies conducted jointly by us and by a group of material scientists at the Penn State University have indicated that the higher order ETD based schemes can be several orders of magnitude faster than the low-order semi-implicit methods in some simulations of microstructure evolution [17]. This further motivates our theoretical investigation on the various properties of the ETD schemes while carrying out more extensive computational studies for real application problems. As an initial attempt, the work reported here mainly contains an analytical examination of the stability and monotonicity of the ETD type schemes for some model parabolic equations. Until now, the stability of ETD schemes is only studied in [6] for ODEs, while the stability of the modified ETD scheme has not been touched upon in [12]. Using techniques ranging from spectral decomposition, energy estimates, and maximum principles, our study here provides a more detailed theoretical analysis of the ETD schemes. Due to the page limit, we do not include any discussion on the modified ETD schemes proposed in [12], though similar analysis can also be performed. As one of the main applications we are working on, we conduct numerical studies of the the ETD scheme for the solution of a time dependent scalar Ginzburg-Landau equation (also named as the Allen-Cahn equation) in the two and three dimensional spaces. Such a study is often the first step towards a more realistic model for many phase transition problems [3]. Again, only some preliminary two and three dimensional numerical simulation results are provided here to conserve space. Our short presentation merely serves as a hint to the techniques and applications of more detailed theoretical and numerical investigations on the ETD schemes to be carried out in the subsequent study [5].

The rest of the paper is organized as follows. In Section 2, we introduce the original and modified ETD schemes. We then turn to discuss the asymptotic L^2–stability and L^∞–stability of those schemes for some model problems in Sections 3 and 4. Some numerical experiments will be given in Section 5.

2. The Exponential Time Differencing Schemes

Given a linear elliptic operator $\mathcal{L}$, we consider the partial differential equation (PDE) for a scalar function u defined in a spatial domain $\Omega = [0, 2\pi]^d \subset \mathbb{R}^d$ and for time $t > 0$:

$$u_t = \mathcal{L}u + \mathcal{N}(u, t, x), \tag{1}$$

along with suitable initial and boundary conditions. Here, $\mathcal{N}$ denotes a generic nonlinear term. A particular case of our interests in this paper is the dimensionless time-dependent Ginzburg-Landau (Allen-Cahn) equation where $\mathcal{N}(u, t, x) = u(1 - u^2)/\epsilon^2$ for some interfacial parameter ϵ, and Δ being the Laplace operator. It is also convenient for us to introduce a related linear equation of the type

$$u_t = \Delta u + \lambda u \ . \tag{2}$$

where λ can either be a constant or a function of the time and the spatial variables. In the latter case, λu can represent an approximation of a nonlinear term through either linearization or by frozen coefficients techniques. For most of our discussion, initial boundary value problems with either periodic or homogeneous Dirichlet boundary conditions are considered for the equation (2).

Discretizing the PDE (1) in the spatial variables, for instance, by spectral approximations or by finite element approximations, a system of ordinary differential equations (ODEs) is often obtained

$$u_t = \boldsymbol{L}u + \boldsymbol{N}(u, t) \ . \tag{3}$$

The exponential time differencing (ETD) methods can be described in the context of solving