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# A FAST NUMERICAL METHOD FOR INTEGRAL EQUATIONS OF THE FIRST KIND WITH LOGARITHMIC KERNEL USING MESH GRADING [*1)]

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**Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday**

**Abstract**

The aim of this paper is to develop a fast numerical method for two-dimensional boundary integral equations of the first kind with logarithm kernels when the boundary of the domain is smooth and closed. In this case, the use of the conventional boundary element methods gives linear systems with dense matrix. In this paper, we demonstrate that the dense matrix can be replaced by a sparse one if appropriate graded meshes are used in the quadrature rules. It will be demonstrated that this technique can increase the numerical efficiency significantly.

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## 1. Introduction

Consider the first kind integral equation with logarithmic kernel:

$$-\int_{\Gamma} \log|x-y| u(y) d\nu_y = f(x), \quad x=(x_1,x_2)\in\Gamma \tag{1.1}$$

where $\Gamma \subset \mathbf{R}^2$ is a smooth and closed curve in the plane, $u$ is a unknown function, $f(x)$ is a given function, $|x-y|$ is the Euclidean distance and $d\nu_y$ is the element of arc length. It should be pointed out that the well-posedness of (1.1) may be subject to some additional conditions, see, e.g., [8]. Hsiao and Wendland [8] were the first to give a rigorous error analysis for the Galerkin method applied to this boundary integral equation where $\Gamma$ is a smooth and closed curve. Later Costabel and Stephan [7] extended this analysis to treat the more difficult case where $\Gamma$ is a polygon. More generally, Sloan and Spence [11] have investigated (1.1) for $\Gamma$ either a closed contour or an open arc. In this case, singularities in the unknown function $u(x)$ are produced at corners and ends (see, [7, 11]).

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In this work we assume that the boundary $\Gamma$ is a simple closed smooth curve with a twicely continuously differentiable parametrization. More precisely, let $\Gamma$ be parameterized by the arclength,

$$\nu : [0, L] \to \Gamma,$$

where $L$ is the length of $\Gamma$, $|d\nu/ds| = 1$ and $\nu(\sigma)$ is a periodic function with period of $L$. Then the integral equation (1.1) is transformed into the following form

$$-\int_{-L/2}^{L/2} \log|\nu(s) - \nu(\sigma)| u(\nu(\sigma)) d\sigma = f(\nu(s)), \quad s \in [-L/2, L/2]. \tag{1.2}$$

The conventional way in solving the equation (1.2) is to use $n$ collocation points to obtain $n$ collocation equations. Then for each fixed $s$ the integral in (1.2) will be approximated by an appropriate quadrature rule using the information on the $n$ collocation points. This approach leads to a linear system whose matrix is a full matrix. In this work, we will approximate the integral term by using a subset of the $n$ collocation points. More precisely, we consider the case when the unknown function $u$ is reasonably smooth and the curve $\Gamma$ is smooth and closed. In this case, some suitable graded-meshes can be used as the quadrature points to handle the logarithmic kernel, which yields a linear system whose matrix is sparse. The graded-mesh concept was proposed by Rice [10]. It was then used to improve the formal order of convergence when solutions have weak singularity, see, e.g., [5, 14] for boundary integral equations and [2, 12] for weakly singular Volterra integral equations. In [2], the analysis of graded mesh convergence was specifically based on the fact that the solution was non-smooth (i.e. having an unbounded derivative at $t = 0$). A much more general analysis of graded mesh collocation for weakly singular Volterra integral equations (including logarithmic kernels) can be found in [3]. It is noticed that some other types of numerical methods have been proposed to handle the singularities in integrals equations [4, 9].

In this work the solution of the integral equation is assumed to be regular. In the earlier works for handling solution singularity such as [2, 13] a graded mesh is used as the collocation points. However, with a smooth solution we just need to use a uniform mesh for the collocation points. A graded mesh which is a *subset* of the uniform mesh will be employed to efficiently evaluate the integrals. We will show that the proposed approach can not only preserve the formal rate of convergence but also save a significant amount of CPU time. We point out that similar efforts have been made by Chen et al. [6] who developed a fast collocation method for integral equations with weakly singular kernels by constructing multiscale interpolating functions and collocation functionals that have vanishing moments.

A related integral equation to (1.1) is

$$-\int_{\Gamma} \log|x - y| u(y) d\nu_y + \omega = f(x), \;\; x \in \Gamma, \quad \int_{\Gamma} u(y)\, d\nu_y = b, \tag{1.3}$$

where the function $u$ and the scalar $\omega$ are unknowns, and $b$ is a given real number. It is known that the solution of interior or exterior Dirichlet problem for Laplace equation in two-dimension can be reduced to the integral equations (1.3).

## 2. Numerical Approximations for Singular Integrals

First we describe several numerical quadrature methods for the weakly singular integration

$$I = \int_{-L/2}^{L/2} G(\sigma) g(\sigma) d\sigma = Kg, \tag{2.1}$$

where $g(\sigma)$ is a smooth function, $G(\sigma)$ is a singular kernel satisfying

$$|G^{(i)}(\sigma)| \le \begin{cases} C|\sigma|^{-i}, & i = 1, 2, \cdots \\ C|\log|\sigma||, & i = 0. \end{cases} \tag{2.2}$$