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# OPTIMAL DELAUNAY TRIANGULATIONS *[1)]

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**Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday**

**Abstract**

The Delaunay triangulation, in both classic and more generalized sense, is studied in this paper for minimizing the linear interpolation error (measure in $L^p$-norm) for a given function. The classic Delaunay triangulation can then be characterized as an optimal triangulation that minimizes the interpolation error for the isotropic function $\|\mathbf{x}\|^2$ among all the triangulations with a given set of vertices. For a more general function, a function-dependent Delaunay triangulation is then defined to be an optimal triangulation that minimizes the interpolation error for this function and its construction can be obtained by a simple lifting and projection procedure.

The optimal Delaunay triangulation is the one that minimizes the interpolation error among all triangulations with the same number of vertices, i.e. the distribution of vertices are optimized in order to minimize the interpolation error. Such a function-dependent optimal Delaunay triangulation is proved to exist for any given convex continuous function. On an optimal Delaunay triangulation associated with $f$, it is proved that $\nabla f$ at the interior vertices can be exactly recovered by the function values on its neighboring vertices. Since the optimal Delaunay triangulation is difficult to obtain in practice, the concept of nearly optimal triangulation is introduced and two sufficient conditions are presented for a triangulation to be nearly optimal.

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## 1. Introduction

In this paper, we shall consider optimal triangulations from a function approximation point of view. Here "triangulation" is extended from the planar usage to arbitrary dimension: a triangulation $\mathcal{T}$ decomposes a bounded domain $\Omega \subset \mathbb{R}^n$ into $n$-simplices such that the intersection of any two simplices in $\mathcal{T}$ either consists of a common lower dimensional simplex or is empty.

The Delaunay triangulation (DT) of a finite point set $V$, the most commonly used unstructured triangulation, can be defined by the empty sphere property: no vertices in $V$ are inside the circumsphere of any simplex in the triangulation. There are many optimality characterizations for Delaunay triangulation [7], in which the most well known is that in two dimensions it maximizes the minimum angle of triangles in the triangulation [15, 20]. We, however, would like to characterize the Delaunay triangulation from a function approximation point of view.

Let us denote $Q(\mathcal{T}, f, p) = \|f - f_{I,\mathcal{T}}\|_{L^p(\Omega)}$, where $f_{I,\mathcal{T}}(\mathbf{x})$ is the linear interpolation of $f$ based the triangulation $\mathcal{T}$ of a domain $\Omega \subset \mathbb{R}^n$. We shall prove that

$$Q(DT, \|\mathbf{x}\|^2, p) = \min_{\mathcal{T} \in \mathcal{P}_V} Q(\mathcal{T}, \|\mathbf{x}\|^2, p),\ 1 \le p \le \infty, \tag{1.1}$$

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where $\mathcal{P}_V$ is the set of all triangulations that have a given set $V$ of vertices and $\Omega$ is chosen as the convex hull of $V$. This type of result was first proved in $\mathbb{R}^2$ by D'Azevedo and Simpson [4] for $p = \infty$ and then Rippa [19] for $1 \le p < \infty$ in two dimensions. Our result is a generalization of their work to higher dimensions.

A Delaunay triangulation is therefore characterized as the optimal triangulation for piecewise linear interpolation to isotropic function $\|\mathbf{x}\|^2$ for a given point set in the sense of minimizing the interpolation error in $L^p(1 \le p \le \infty)$ norm. Based on this characterization, we will introduce the concept of function-dependent Delaunay triangulation $(DT)_f$ for a given convex function with $f$ in place of $\|\mathbf{x}\|^2$ in (1.1).

We further let the set $V$ vary all sets of triangulations at most $N$ points and consider the optimization problem corresponding to the error-based mesh quality $Q(\mathcal{T}, f, p)$. That is to find a triangulation $\mathcal{T}^*$ such that

$$Q(\mathcal{T}^*, f, p) = \inf_{\mathcal{T} \in \mathcal{P}_N} Q(\mathcal{T}, f, p), \ 1 \le p \le \infty, \tag{1.2}$$

where $\mathcal{P}_N$ stands for the set of all triangulations with at most $N$ vertices. Any minimizer of (1.2) is called an optimal Delaunay triangulation associated with $f$.

We shall prove the existence of the optimal Delaunay triangulation for a convex function. With the formulation of $Q(\mathcal{T}, f, 1)$, we obtain a necessary condition. More precisely, if triangulation is optimal in the sense of minimizing $Q(\mathcal{T}, f, 1)$ for a convex function $f$ in $\mathcal{C}^1(\Omega)$, then for an interior vertex $\mathbf{x}_i$, we have

$$\nabla f(\mathbf{x}_i) = -\frac{1}{|\Omega_i|} \sum_{\tau_j \in \Omega_i} \Big[ \nabla |\tau_j|(\mathbf{x}) \sum_{\mathbf{x}_k \in \tau_j, \mathbf{x}_k \neq \mathbf{x}_i} f(\mathbf{x}_k) \Big]. \tag{1.3}$$

Here $\Omega_i$ is the patch of $\mathbf{x}_i$ which consists of all simplices using $\mathbf{x}_i$ as a vertex and $|A|$ is the Lebesgue measure of set $A$ in $\mathbb{R}^n$. We free the vertex $\mathbf{x}_i$ to be a variable $\mathbf{x} \in \Omega_i$ and treat $|\tau_j|(\mathbf{x})$ as a linear function of $\mathbf{x}$ whose gradient is a combination of other vertices in $\tau_j$; See Fig. 2.

The identity (1.3) states that the gradient of $f$ can be recovered exactly at a grid point on an optimal Delaunay triangulation by taking a special linear combination of function values on its neighboring nodes. If the triangulation is not optimal, (1.3) will guide us to move the vertex $\mathbf{x}_i$ in its local patch to optimize the interpolation error and thus can be used as a mesh smoothing scheme.

While an optimal Delaunay triangulation is desired, but it is difficult to obtain in practice. We therefore introduce the concept of nearly optimal triangulation. We call a triangulation is nearly optimal if $Q(\mathcal{T}, f, p) \le CQ(\mathcal{T}^*, f, p)$ with a constant $C$ independent of the number of vertices $N$. For the practical propose, we present two sufficient conditions for a triangulation to be nearly optimal. One such a condition is that the triangulation should be quasi-uniform under a new metric obtained by a modification of the Hessian matrix of object function $f$.

By choosing $f$ of interest, we develop an unified approach to generalize the main concepts and techniques used in isotropic mesh generation, for example the Delaunay triangulation and edge swapping algorithm, to anisotropic and high dimensional cases which become a challenging and active research in the last decade [19, 21, 10, 1, 9, 5, 13, 16]

The rest of this paper is organized as follows. In Section 2, we discuss the Delaunay triangulation and present the characterization in terms of linear interpolation error. In Section 3, we introduce optimal Delaunay triangulations, prove the existence and present a necessary condition for gradient recovery. In Section 4, sufficient conditions for a nearly optimal Delaunay are presented. The last section is the concluding remark.

## 2. Function-Dependent Delaunay Triangulation

The Delaunay triangulation (DT) is the most commonly used triangulation for the gen-