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A TWO-GRID METHOD FOR THE STEADY PENALIZED NAVIER-STOKES EQUATIONS *1)

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Abstract

A two-grid method for the steady penalized incompressible Navier-Stokes equations is presented. Convergence results are proved. If $h = O(H^{3-s})$ and $\epsilon = O(H^{5-2s})$ ($s = 0$ ($n = 2$); $s = \frac{1}{2}$ ($n = 3$)) are chosen, the convergence order of this two-grid method is the same as that of the usual finite element method. Numerical results show that this method is efficient and can save a lot of computation time.

Key words: Penalized Navier-Stokes equations, Two-grid method, Error estimate, Numerical test.

Introduction

It is well known that numerically solving the incompressible Navier-Stokes equations has two difficulties: the nonlinear term and the incompressibility condition. Firstly, we use the penalized Navier-Stokes equations to conquer the second difficulty, and a two-grid method presented in [1-2] to save a lot of computation time. Secondly, we analyze the convergence of the numerical solution, and derive that if $h = O(H^{3-s})$ and $\epsilon = O(H^{5-2s})$ ($s = 0$ ($n = 2$); $s = \frac{1}{2}$ ($n = 3$)) are chosen, the convergence order of this method is the same as that of the usual finite element method. However, the computational attraction of this two-grid method is that it finds a solution for a small nonlinear problem on a coarse mesh finite element space X^H, and a solution for a linear problem on a fine mesh finite element space X^h ($h \ll H$), compared with the usual finite element method finding a solution for the same large nonlinear problem on X^h. Thus, this method can save a lot of computation time. Finally, numerical tests are given to support our theoretic results.

It is noticeable that this method is totally different from the Nonlinear Galerkin method presented in [3-4]. The Nonlinear Galerkin method is a numerical method for dissipative evolution partial differential equations where the spatial discretization relies on a nonlinear manifold instead of a linear space as in the classical Galerkin method. More precisely, one considers a finite dimensional space X^h which is split as $X^h = X^H + W^h$, where $H \gg h$ and W^h is a convenient supplementary of X^H in X^h. One looks for an approximate solution u^h lying in a manifold Σ= graphϕ of X^h; u^h takes the form $u^h = v^H + \phi(v^H)$ where v^H lies in X^H and ϕ is a mapping from X^H to W^h. The method reduces to an evolution equation for X^H, obtained by projecting the equations under consideration on the manifold Σ=graphϕ. In the usual finite element method, typically, we have $\phi = 0$. The two-grid method is based on a coarse grid finite element space X^H and a fine grid finite element space X^h ($X^H \subset X^h$, $H \gg h$). This method consists of finding a solution v^H for a nonlinear problem on X^H by the usual finite element method, a solution v^h for a linear problem on X^h by one-step Newton method, and a solution w^H for a linear correctness problem on X^H, where an approximate solution $u^h = w^H + v^h$ is defined as in the following step 1-step 4 in §2.

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1. Notations and Mathematical Preliminaries

Consider the incompressible Navier-Stokes problem

$$\begin{cases} -\lambda\Delta u + (u\cdot\nabla)u + \nabla p = f & in \quad \Omega, \\ \nabla\cdot u = 0 & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega, \end{cases} \tag{1.1}$$

in a convex polygon or polyhedron domain Ω of R^n, with $n = 2$ or 3. Here, $\lambda = Re^{-1}$, Re is the Reynolds number, $u : \Omega \to R^n$ the velocity, $p : \Omega \to R$ the pressure and f the prescribed external force.

Hereafter, we will need the following functional spaces:

$$X = H_0^1(\Omega)^n \equiv \{u \in (H^1(\Omega))^n : \quad u = 0 \quad on \quad \partial\Omega\},$$
$$V \equiv \{u \in H_0^1(\Omega)^n : \nabla\cdot u = 0 \quad in \quad \Omega\}$$

with scalar product $(u, v)_1 := (\nabla u, \nabla v), \quad u, v \in H_0^1(\Omega)^n$, and

$$M = L_0^2(\Omega) \equiv \{p \in L^2(\Omega) : \quad \int_\Omega p dx = 0\}.$$

Let $H^{-1}(\Omega)^n$ be a dual space of $H_0^1(\Omega)^n$ with the corresponding norm:

$$\|f\|_{-1} \equiv \sup_{0\neq u\in H_0^1(\Omega)^n} \frac{\langle f, u\rangle}{|u|_1}, \quad f \in H^{-1}(\Omega)^n.$$

We will use the standard notations $L^2(\Omega)^n$, $H^k(\Omega)^n$ and $H_0^k(\Omega)^n$ to denote the usual Sobolev spaces over Ω. The norm and seminorm corresponding to $H^k(\Omega)^n$ will be denoted by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. In particular, we will use $\|\cdot\|_0$ and $(\cdot,\cdot)$ to denote the norm and the scalar product in $L^2(\Omega)^n$, respectively.

With the above notations, the weak form of problem (1.1) reads: find $(u, p) \in (X, M)$, such that

$$\begin{cases} a(u, v) + N(u, u, v) - b(p, v) &= \langle f, v\rangle, \quad \forall v \in X, \\ b(q, u) &= 0, \quad \forall q \in M, \end{cases} \tag{1.2}$$

where

$$a(u, v) = (u, v)_1, \quad b(p, v) = (p, \nabla\cdot v),$$
$$\text{and} \quad N(u, v, w) = \tfrac{1}{2}[((u\cdot\nabla)v, w) - ((u\cdot\nabla)w, v)].$$

Because of pressure not being in the second equation of problem (1.2), the algebraic equations generated by a finite dimensional approximation are not positive definite, which results in difficulty in solving the numerical solution of problem (1.2). If the positive definite form connected with the pressure is obtained in problem (1.2), then this difficulty will be conquered. For brevity, $\epsilon(p, q)$ is introduced, where ϵ is a penalty parameter. Then, introduce the following penalized problem: find $u_\epsilon \in X$ satisfying

$$\begin{cases} a_\epsilon(u_\epsilon, v) + N(u_\epsilon, u_\epsilon, v) = \langle f, v\rangle, \quad \forall v \in X, \\ p_\epsilon = -\dfrac{1}{\epsilon}\nabla\cdot u_\epsilon, \end{cases} \tag{1.3}$$

where

$$a_\epsilon(u, v) \equiv a(u, v) + \epsilon^{-1}(\nabla\cdot u, \nabla\cdot v).$$